ADVANCED PRACTICAL ARITHMETIC

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THE ADVANCED

PRACTICAL ARITHMETIC

BY

FLETCHER DURELL, Ph.D.

Mathematical Master in the Lawrenceville School

AND

EDWARD R. ROBBINS, A.B.

Mathematical Master in the William Penn Charter School

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PREFACE

THE object in preparing this arithmetic has been the same which the authors had in view when writing their school algebra, viz.: "to show more plainly, if possible, than has been done heretofore, the practical or common sense reason for each step or process." It is believed that this treatment is not only "adapted to the practical American spirit, but also gives the study of the subject a larger educational value." It is also believed that the scholarly possibilities and value of the treatment, instead of being diminished, are increased thereby. For instance, it is hoped that the matter presented in the chapters on the Applications of Percentage and Interest, and on Arithmetical History, will have a new value.

The main principle which has governed the authors in writing the book has also been the controlling factor in the treatment of many matters of detail which are still in debate among teachers of arithmetic and writers on the subject.

Almost all are agreed that the study of arithmetic should begin with the study of *concrete objects*; and that the use of *geometric diagrams* is a great help in presenting certain parts of the subject, as fractions. But the authors believe that it is a mistake to have the concrete objects, or even pictures of them, constantly before the pupil. Nor should diagrams be printed as part of the text. The pure arithmetical processes should be made easy and natural as soon as possible, on account of their brevity and simplicity. Concrete objects or diagrams, if kept before the eye constantly, tend to clog and hamper the mind; hence they should be recalled only so often and at such places as may be necessary in order to make the subject vivid and real.

In the same way *algebraic symbols and methods* have been introduced only when they give a clear and pronounced advantage, and thus arouse in the pupil the desire to know more of them.

It is believed that *rules* for processes are useful in many ways when they are arrived at after the proper preliminary work and when they are used with discretion.

In order to cultivate habits of *analysis* and exact statement, and yet to prevent these from becoming mere mechanical rote processes, different forms of analysis have been used adapted to different kinds of work. These are indicated by the use of different words as "Operation," "Explanation," "Solution," etc.

In like manner, *oral exercises* are sometimes put before written exercises, sometimes after them.

The subject-matter is not spirally arranged, but is adapted to spiral study. The subject is presented as an organic whole, yet one which can be learned by successive steps (see p. 6).

A large number of examples adapted to the theory of the book has been made and carefully graded. Especial attention is also called to the chapter on the Metric System.

The authors will be glad to receive any corrections or suggestions from teachers using this book.

FLETCHER DURELL,
EDWARD R. ROBBINS.

LAWRENCEVILLE, N. J.,
PHILADELPHIA, PA.,
May 1, 1901.

TO THE TEACHER.

1. The teacher should make sure at different times that the pupil carries in mind the *concrete object for which a symbol stands.* Now show the pupil, now have him show the concrete object. Show him diagrams illustrating the properties of fractions. Have him make these diagrams. But do this only *occasionally,* and always for some good reason.

2. In *oral work* and *explanations* insist on *careful and accurate statements.* For instance, in oral work do not allow a pupil to give the answer merely without a formal statement of the analysis or steps by which the result was obtained.

3. In *written problems* in which the *analysis is difficult* (as in Exercise 23) elicit the analysis from the pupil by oral questions, and afterward have the pupil write out the analysis and solution.

4. Insist on the use of *cancellation* wherever possible. Train the pupil to combine all the operations required in the solution of a problem in a *comprehensive plan* or scheme; then to factor and cancel wherever possible; never to multiply till it is necessary to do so.

5. Train the pupil also to make a *rough estimate* or forecast of the answer before beginning the exact numerical work. This not only tends to eliminate large errors, but is also a valuable habit, since, in practical life, fully one-half the applications of arithmetic are made in this way.

6. Impress upon the minds of pupils in various ways the *local value of digits* and the *limitations in the accuracy* of all arithmetical work based on measurements (see Arts. 19, 77, 78).

7. Study to *vary methods* to suit the needs of different pupils, both in presenting topics and in meeting difficulties. It is to be remembered that pupils as they come from different homes probably have more varied capacities with respect to the subject of arithmetic than to any other.

8. Do not be satisfied till by long practice and working innumerable examples, if necessary, the pupil has become a rapid and accurate computer. The power of handling figures with facility and accuracy is of the first importance both in practical life and in its influence on the further educational development of the pupil.

9. Review constantly.

SCHEDULE RECOMMENDED IN USING DURELL AND ROBBINS' ARITHMETIC.

YEAR.	FIRST HALF-YEAR.	SECOND HALF-YEAR.
1	*Oral work without text-book.*	*Elementary Pract. Arith.*, pp. 1–32 (much of it read to pupil by teacher; supplemented by other oral work).
2	*Elementary Pract. Arith.*, pp. 1–32. Review and second course.	*Elementary Pract. Arith.*, pp. 33–77 (more difficult parts of some lessons omitted).
3	*Elementary Pract. Arith.*, pp. 33–77. Review and second course.	*Elementary Pract. Arith.*, pp. 78–121 (more difficult parts of some lessons omitted).
4	*Elementary Pract. Arith.*, pp. 55–121. Reviewed and completed in details.	*Elementary Pract. Arith.*, pp. 122–194 (leading principles and easier exercises).
5	*Elementary Pract. Arith.*, pp. 91–194. Reviewed and completed in details.	*Advanced Pract. Arith.* to p. 158 (leading principles and easier exercises).
6	*Advanced Pract. Arith.* to p. 158 (fuller course).	*Advanced Pract. Arith.*, pp. 158–255 (leading principles and easier exercises).
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8	*Advanced Pract. Arith.* Review. Rapid review to p. 256. Fill in details, pp. 256 to end.	*Elementary Algebra.* *Geometrical Drawing.*

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ARITHMETIC.

CHAPTER I.

NUMBER. NUMERATION. NOTATION.

1. Units.—For many purposes the most convenient way of dealing with quantity (as, for instance, with the length of a given line) is to take a certain definite part of the given quantity as a unit, and determine the number of times the unit must be used in order to make up the given quantity (or line).

Thus, in determining the length of a given linear object, as a rope, we do not depend merely on general impressions of its magnitude (formed by the eye or by moving the hand over it), but by taking a unit, as one inch or one foot, and determining the number of times the unit must be used in order to make up the line.

A boy dealing with a quantity of marbles in his possession does not do so merely by means of the aggregate impression which they make in his pocket, but by taking a single marble as a unit, and counting the number of marbles which he has.

This method of regarding quantity as made up of units gives greater ease and precision in all the ordinary uses made of an aggregate of material.

A unit is a certain quantity taken as a standard of reference when dealing with quantity of the same kind.

2. Kinds of Units.—Units are of different kinds.

Natural units are those which occur in the world about us, as one apple, one man, one year, one day.

Artificial units do not occur naturally, but are devised

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This method of regarding quantity as made up of units gives greater ease and precision in all the ordinary uses made of an aggregate of material.

A unit is a certain quantity taken as a standard of reference when dealing with quantity of the same kind.

2. Kinds of Units.—Units are of different kinds.

Natural units are those which occur in the world about us, as one apple, one man, one year, one day.

Artificial units do not occur naturally, but are devised by man so as to extend the advantages arising from the use of units as widely as possible, as one foot, one-third of an apple, etc.

A **primary** unit is a single unit of a given kind, as one dollar.

A **derived** unit is an aggregate of single units, as five dollars (a "V"); or a part of a unit, regarded as a new unit, as one-third of a dollar.

A unit of one kind may become, in certain relations, a unit of another kind. Thus, an artificial unit may become, in some senses, a natural unit, as one dollar. Also, a derived unit may come to be regarded as a primary unit, as one week, one quarter (of a dollar).

EXERCISE 1.

1. What unit of length is used in measuring the length of a room? The length of a pencil? Of a quantity of cloth?
2. What unit of length is used in measuring the distance between two cities? The diameter of the earth?
3. With what unit of capacity is milk measured? Grain? Strawberries? Potatoes?
4. Which unit of area is used in stating the size of a farm? Of a county?
5. Which of the following units are natural and which are artificial: year, second, week, foot, quart, yard, peck, mile, month, degree?

3. Number is a unit, or collection of like units.

When quantity is regarded as made up of like units, it becomes a number. Thus, when an aggregate of apples is regarded as made up of distinct apples, it becomes a number of apples.

Thus, also, when a line is regarded as made up of inches, it becomes a number of inches.

4. Arithmetic is the science which treats of number. It investigates the most advantageous ways of expressing quantities as numbers, and of using numbers when formed.

5. Number Words.—When we have determined a quantity as made up of units, and ascertained the number of the units in a given quantity, it is often useful to transfer

the number idea thus formed, to other persons, and thus give them a definite conception of the quantity dealt with, without labor on their part. Hence, words are useful by which to designate different aggregates of units, or numbers.

Number words are useful also to the person using them, in calling up the precise ideas connected with each aggregate of units.

The words used for the different aggregates of units (beginning with a single unit) are—

one, two, three, four, five, six, seven, eight, nine, ten.

For larger aggregates of units a system of grouping units and naming the groups formed is used, which is explained later.

6. Counting is the process of affixing to any group of units the number word belonging to that group, beginning with unity, and affixing its number word to each group, till the last unit of the entire group dealt with is reached.

7. Number Symbols.—Further economies and additional power in dealing with numbers are obtained by using a distinct *symbol* for each number apart from its number word. Thus, for the number words

	one,	two,	three,	four,	five,	six,	seven,	eight,	nine,
we use	1,	2,	3,	4,	5,	6,	7,	8,	9.

These number symbols, or **figures**, have advantages as compared with number words, in that they are easier to write, and to recognize when written. They have many other derived advantages, when used in combinations, both in denoting and in operating with, numbers larger than nine.

The number symbols 1, 2, 3, 4, 5, 6, 7, 8, 9 are called the nine **digits**. The absence of number is denoted by a symbol, 0, called **zero, naught,** or **cipher.**

Zero is sometimes regarded as a number.

8. Large Numbers.—In order to denote large numbers by words and symbols, it is necessary to devise a plan of so grouping units that a few words or symbols systematically used will represent any number, however large. It is plainly impracticable to denote each different number by an entirely new and distinct word or symbol.

NUMERATION.

9. Numeration is the process of grouping an aggregate of units according to a convenient, systematic plan, and of naming the groups so formed; or briefly, numeration is the expression of numbers in words.

10. Decimal System of Numeration.—Let us suppose a heap of like objects, as silver dollars, and let us suppose that we desire to determine the number of these objects, and to express the number of them in words in a convenient, systematic way. We first count ten of the dollars and set them aside as a single group (equivalent to a ten-dollar bill), then count ten more dollars and set them aside, and continue making like groups until the number of dollars left is less than ten. Suppose eight tens are formed and six dollars are left. By thus forming groups of ten each, and regarding each such group as a new unit of a higher order, we can express the given group of units (or number of dollars) in words without employing any new number word beside those already given (Art. 5), except a word to denote the new unit group of higher order—viz., *ten*. For the number of dollars in the original heap is expressed in words as eight tens and six units of (or eighty-six) dollars.

Similarly, if there are ten or more groups of the new unit groups of higher order (*i. e.*, of groups of ten dollars each) in

NOTE.—The number ten is used because most of our savage ancestors counted by aid of their ten fingers. Hence the number ten became the primary group in numeration, and has been so used ever since. Any other number (except unity) might be used as the primary group in numeration. Two, six, eight, and twelve are among those which have been suggested, of which, twelve, perhaps, would be the best.

the original heap, we regard ten ten-units taken together as a new unit group of still higher order, and call it one *hundred*. Similarly ten hundreds are regarded as a new unit group of higher order and called a *thousand*.

11. Numbers Larger than One Thousand.—Similarly we may form other new unit groups, each ten times as great as the preceding, and called one *ten thousand*, one *hundred thousand*, one *million*, one *ten million*, etc. But in denoting these groups (greater than one thousand) entirely new number words are used only for those groups which are one thousand times as great as the group denoted by the last preceding new number word, as *million* (one thousand times as great as one thousand), *billion* (one thousand times as great as one million), *trillion*, etc. The intermediate unit groups are denoted by using "ten" and "hundred" as modifiers to other number words.

12. Number Words Actually Used.—Beside the number words already given, it is found convenient to use a few others, though these are not actually necessary. Thus, some number words are formed by using two primary numbers and fusing them into a single word.

Thus, for "ten" and "one" we have "eleven" (formed by fusing the Gothic words for one and ten, *ain lif*); for "ten" and "two" we have "twelve" (the Gothic words for two and ten, *twa lif*, fused); for "ten" and "three" we have "thirteen" (for "three" and "ten" fused). Similarly we obtain "fourteen," "fifteen," "sixteen," "seventeen," "eighteen," "nineteen." Also, for "two tens" we have "twenty," by fusion of the words "two" and "ten." Similarly are obtained "thirty," "forty," etc.

Hence the number words in actual use are *one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, hundred, thousand, million, billion, trillion, quadrillion, quintillion, sextillion*, etc.

By the systematic use of these few words any number may be expressed in words.

13. Orders of Units.—Thus, in the decimal numeration, we use a *unit*, and a series of *derived units*, each ten times as great as the preceding—viz., *one*, *ten*, *hundred*, *thousand*, *ten thousand*, etc.

These units are of different orders.

One is called the unit of the *first order;*
ten is called the unit of the *second order;*
a *hundred* is called the unit of the *third order*, etc.

In naming any number we begin with the highest order, and state the number of units of each order which the given number contains. We speak, for example, of the number "three thousand, six hundred, seventy-two."

NOTATION.

14. Notation is the process of expressing a number in symbols according to a convenient, systematic plan.

Having grouped an aggregate (or number) of units according to a scale (the decimal scale, for instance), and given names to the number groups so as to express the number in words to other persons, we need also to express these groups in simple symbols so as to facilitate the extended use of the number.

15. Positional System of Notation.—The first nine numbers are denoted by the nine digits (Art. 7). A simple method of expressing larger numbers in symbols is illustrated if we express the number "three thousand, six hundred, seventy-two" as follows: 3672.

Here the number of units of each order is denoted by the appropriate digit (the thousands by 3, hundreds by 6, etc.), and the size of the unit for which each digit stands is indicated by writing in a row the digits employed, the highest order to the left, each successive lower unit group being one place to the right. The simplicity and power of this system

of notation should be carefully noted by the pupil and frequently recalled.

The simplicity is due to the fact that in denoting a number by figures, as 3672, each of the digits 6, 7, 2 has not only its own value, but is also employed to determine the order of the unit group denoted by 3—viz., thousands; similarly, 7 and 2 define the order of the unit denoted by 6—viz., hundreds, etc. Hence, when for 3 thousand, 6 hundred, 7 tens, and 2 units, we write 3672, the word "thousand" is replaced by 672, "hundred" by 72, "tens" by 2, units by the absence of another digit after 2. Hence, for instance, the symbol 2 as here used has four uses; it has its own value, and it helps determine the value of 3, 6, and 7. It is because of this manifold use of each symbol that we are able to substitute the four symbols of 3672 for the thirty-three symbols which compose the expression "three thousand, six hundred, seventy-two."

It is also to be noted that the symbolism 3672 is uniform in arrangement and spacing, while the expression of the number in number words is irregular in form and spacing.

These great advantages in expressing numbers in symbols give ease and power in the extended use of numbers and make a thorough science of numbers possible.

The student is aided to a full appreciation of the advantages of the positional decimal system of notation by comparing it with others that have been used to some extent, as the Roman notation (Art. 23 *et seq.*).

16. Zero Symbol in the Positional Notation.—When units of one or more orders do not occur in a given number, the absence is indicated by the use of the zero symbol in each place where such a unit is missing.

Thus, 5042 represents a number containing 5 thousands, 4 tens, and 2 units, but no hundreds.

17. Number vs. Number Symbols.—The student should carefully discriminate between a *number* and the *symbols* or words which represent a number.

Thus, a number (which is an aggregate or collection of units, as a heap of apples), may exist long before any words or symbols are used to represent it. It may also be represented by different sets of symbols, as by "twelve," or 12, or xii. These are not different numbers, but only different symbols for the same number. However, for the sake of brevity, the expression "number denoted by the figures 3276" is shortened into "the number 3276," but the student is not to be misled into regarding the number and 3276 as identical.

18. The **place** of a figure (in a given number) is the position which the figure occupies with reference to the other figures in the number. Thus, in the number 3672, the figure in the right-hand place, 2, is said to occupy the *first* place; 7, the *second* place; 6, the *third* place, etc.

Hence, moving a figure one place to the left increases its value tenfold; but moving a figure one place to the right divides its value by ten.

19. Absolute and Local Value.—The value of each figure in a number is determined by two things:

First, the value of the figure without regard to its position, called its **absolute** (or digit) value;

Second, the value given the figure by the place it occupies in the number, called its **local** value.

Thus, in 3672, the figure 6, for instance, has an absolute value, in that it represents 6 units, and a local value, in that each of the units represented by it is the hundred unit.

The student should not, as often happens, unconsciously form the habit of regarding the digits which form a given number as of equal importance and significance in a numerical result. This habit often arises perhaps from the fact that the digits as written are of equal size, and local value apparently neglected. He should frequently substitute (mentally) for 2, in 3672, a figure only one-tenth as large as 2, leave 7 unchanged, substitute for 6 a figure ten times as long as it is, for 3 one a hundred times as long. Or he may picture, back of 7, seven bundles of ten strokes each, back of 6, six bundles, each composed of ten ten-bundles, etc.

NUMERATION AND NOTATION.

20. Periods.—In order to write and read large numbers with facility, it is customary to separate the different orders of units used into sets of three each, called **periods**. Periods

are formed by beginning at the right and marking off three figures in each period by the use of a comma. In reading numbers it is customary to express the aggregate of each period in terms of the lowest unit in that period.

Number of Period.	6th.	5th.	4th.	3d.	2nd.	1st.
Name of Period.	Quadrillions.	Trillions.	Billions.	Millions.	Thousands.	Units.
Order of Units.	Hundreds of, Tens of, Units of	Hundreds of, Tens of, Units of	Hundreds of, Tens of, Units of	Hundreds of, Tens of, Units of	Hundreds of, Tens of, Units of	Hundreds of, Tens of, Units.
Number =	5 4,	2 0 3,	6 7 5,	4 0 0,	0 7 6,	5 4 2.

The number expressed in symbols is

54,203,675,400,076,542,

and is read,

Fifty-four quadrillion, two hundred three trillion, six hundred seventy-five billion, four hundred million, seventy-six thousand, five hundred forty-two.

The names of the periods above quadrillions are quintillions, sextillions, septillions, octillions, nonillions, decillions, undecillions, etc.

In actual practice, however, periods of the higher orders are little used.

21. I. **To express in figures a number given in words.**

Write the proper figure for the number of units of each order, putting a zero in each vacant place. Mark the figures off into periods of three figures each, beginning at the right.

Ex. Express in figures the number three billion, five hundred six million, seven thousand, twenty-two. We obtain 3,506,007,022.

22. II. To express in words (*i. e.*, to read) a number given in figures.

By use of commas and beginning at the right, separate the figures given into periods of three figures each. Beginning at the left, read each group, giving it the name of the period to which it belongs.

Omit the name of the units period in reading.

Ex. Read 5062380749.

We have 5,062,380,749, which is read, five billion, sixty-two million, three hundred eighty thousand, seven hundred forty-nine.

EXERCISE 2.

Read:

1. 28.	*8.* 107.	*15.* 1352.	*22.* 13456.
2. 27.	*9.* 705.	*16.* 3128.	*23.* 74901.
3. 63.	*10.* 450.	*17.* 4201.	*24.* 28074.
4. 92.	*11.* 910.	*18.* 3700.	*25.* 30212.
5. 125.	*12.* 711.	*19.* 4025.	*26.* 30077.
6. 378.	*13.* 818.	*20.* 7030.	*27.* 60103.
7. 554.	*14.* 666.	*21.* 8004.	*28.* 70007.

29. 63360 inches.	*32.* 25003 days.
30. 97056 men.	*33.* 86400 seconds.
31. 38020 miles.	*34.* 10101 tons.

35. 129345.	*41.* 3564320.	*47.* 250341702.
36. 704508.	*42.* 13705028.	*48.* 402000271.
37. 201009.	*43.* 37564005.	*49.* 300070005.
38. 300102.	*44.* 20024106.	*50.* 777505003.
39. 295004.	*45.* 10902070.	*51.* 909090909.
40. 300071.	*46.* 703201001.	*52.* 65004030.

53. 1305217456.	*56.* 435710302456081.
54. 17271005301.	*57.* 300310070004255.
55. 298012003819.	*58.* 8000500123005760.

Write in words:

59. 750.	*65.* 30201.	*71.* 214008.	*77.* $5071.
60. 342.	*66.* 65311.	*72.* 703307.	*78.* 9003 ft.
61. 1500.	*67.* 82005.	*73.* 1575014.	*79.* $40267.
62. 3027.	*68.* 90102.	*74.* 20501310.	*80.* $21006.
63. 2006.	*69.* 88217.	*75.* 42001025.	*81.* 4001 days.
64. 7102.	*70.* 57008.	*76.* 120320020.	*82.* $250405005.

83. Write the largest number that can be expressed by three figures; by six figures. Read each of them.

84. Write the smallest number that can be expressed by five figures; by eight figures. Read each of them.

Express in figures:

85. Thirty-seven.	*89.* Nine hundred eleven.
86. Eighty-three.	*90.* One thousand six.
87. One hundred forty.	*91.* Four hundred seventeen.
88. Two hundred ten.	*92.* Six hundred ninety-three.

93. Eight hundred twenty-five.
94. Two thousand four hundred sixty-one.
95. Five thousand two hundred eight.
96. Seven thousand three hundred twenty.
97. Nine thousand five hundred.
98. Twelve thousand two hundred sixty.
99. Seventeen thousand six hundred one.
100. Twenty-three thousand ninety-seven.
101. Forty thousand three hundred nineteen.
102. Seventy-one thousand three.
103. Eighty thousand eleven.
104. One hundred two thousand four hundred twelve.
105. Three hundred twenty-seven thousand seventeen.
106. Four hundred thousand two hundred five.
107. Seven hundred seven thousand seventy-seven.
108. Seventy-seven thousand seven hundred seven.

109. Six million one hundred seven thousand four hundred sixty-nine.

110. Twelve million two hundred nineteen thousand eighty-one.

111. Three hundred eleven million seven hundred sixteen thousand four hundred forty-four.

112. Six hundred million two thousand fifteen.

113. Eleven million eleven thousand eleven.

114. Seven billion twenty million fourteen thousand sixty.

ROMAN NOTATION.

23. Number Symbols of Roman Notation.—Beside the system of numeration and notation already explained (commonly called the Arabic system, owing to the fact that the peoples of Europe first learned it through the Arabs), there is another system still used to some extent, called the Roman system, because of its origin among the Romans.

The Roman system of notation uses seven capital letters of the alphabet as number symbols—viz.,

I, V, X, L, C, D, M.

To these, in order, the following values are assigned:

1, 5, 10, 50, 100, 500, 1000.

24. Combination of Number Symbols in the Roman Notation.—When the above symbols are used in combination, the value of each symbol in a combination is determined by the following laws:

1. Each repetition of a letter repeats its value.

Thus, XXX denotes 30, CC denotes 200, etc.

2. When a letter is placed after another letter of greater value, its value is to be added to that of the greater letter.

Thus, VI represents 5 + 1, or 6; XVI denotes 16; LXXXI denotes 81; DCC = 700.

3. When a letter is placed before another letter of greater value, its value is taken from that of the greater letter.

Thus, IV denotes 4; XL denotes 40; XC denotes 90.

A letter *between* two letters, each of which is of greater value than itself, is regarded as preceding the last letter.

Thus, XIV denotes 14; XIX denotes 19.

4. A bar (or dash) placed over a letter increases its value one thousand fold. Hence we have

Thousands.	*Hundreds.*	*Tens.*	*Units.*	
M	C	X	I	(=1)
MM	CC	XX	II	(=2)
MMM	CCC	XXX	III	(=3)
IV	CD	XL	IV	(=4)
V	D	L	V	(=5)
VI	DC	LX	VI	(=6)
VII	DCC	LXX	VII	(=7)
VIII	DCCC	LXXX	VIII	(=8)
IX	CM	XC	IX	(=9)

25. Uses of the Roman System of Notation.—The Roman system of notation is used at times in connection with other systems to prevent confusion when several different groupings of an aggregate of material are made. Thus, Arabic numerals are used in numbering the articles of this book, and the Roman numerals in numbering the chapters.

Roman numerals are also used on monuments and formal documents to give variety and distinction.

The Roman system of numeration also has an educational value. It is useful during the study of arithmetic to compare processes in the Arabic notation with what they would be in the clumsy Roman notation, in order to appreciate the simplicity and power of the former.

EXERCISE 3.

Express in Arabic notation—

1. XV.	*11.* XCI.	*21.* MDXC.
2. XX.	*12.* XCIV.	*22.* MDCXLIII.
3. XXIV.	*13.* CXVI.	*23.* MDCCCXCVIII.
4. XXXII.	*14.* CXLIX.	*24.* MMDCXLIX.
5. XIX.	*15.* CLXXXIV.	*25.* IVCDXLIV.
6. XXIX.	*16.* CCXCIX.	*26.* XDCCXXVI.
7. XLIV.	*17.* CDLVI.	*27.* XLVCCLXVI.
8. LVI.	*18.* DCIX.	*28.* DXCVIII.
9. LXVIII.	*19.* MCXLVII.	*29.* MCCLCXLV.
10. LXXIX.	*20.* MCCXLIX.	*30.* $\overline{\mathrm{MMDCCXV}}\mathrm{DCXXI}$.

Express in Roman notation—

31. 18.	*37.* 93.	*43.* 421.	*49.* 1492.
32. 27.	*38.* 98.	*44.* 490.	*50.* 1776.
33. 39.	*39.* 111.	*45.* 567.	*51.* 1865.
34. 46.	*40.* 120.	*46.* 719.	*52.* 2674.
35. 58.	*41.* 147.	*47.* 984.	*53.* 200468.
36. 72.	*42.* 375.	*48.* 1302.	*54.* 1321894.

CHAPTER II.

ADDITION.

26. Illustration.—If James has 5 apples and John has 4 apples, how many apples have they together?

If we take the 5 apples belonging to James and count on to them the 4 apples which John has, we get 6, 7, 8, 9 apples; that is, as final result, 9 apples. Or, if we are familiar with the results of counting together small groups, we may simply recall the result of a former counting together and say 5 apples and 4 apples are 9 apples.

In the latter case we substitute the less labor of recollection for the greater labor of counting the groups together. By the use of the memory we utilize the work which we have done at some former time, to obtain the number of units in two groups when taken together.

This process is called *addition.*

27. Definitions.—Addition is the process of obtaining in the simplest way a single number which shall contain as many units as there are units in two or more given numbers taken together.

The **sum** is the number obtained as the result of an addition.

The **addends** are the numbers added.

28. Symbols.—The symbol or sign used to denote addition is the erect cross, +, which reads "*plus.*" It means that the numbers between which it is placed are to be added.

The symbol, =, reads "equals," and is placed between two numbers to indicate that they are equal. Hence, it may be employed to denote the *equality* between a sum and the numbers added.

Thus, $5 + 4 = 9$, reads "5 plus 4 equals 9."

EXERCISE 3.

Express in Arabic notation—

1. XV.	*11.* XCI.	*21.* MDXC.
2. XX.	*12.* XCIV.	*22.* MDCXLIII.
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5. XIX.	*15.* CLXXXIV.	*25.* IVCDXLIV.
6. XXIX.	*16.* CCXCIX.	*26.* XDCCXXVI.
7. XLIV.	*17.* CDLVI.	*27.* XLVCCLXVI.
8. LVI.	*18.* DCIX.	*28.* DXCVIII.
9. LXVIII.	*19.* MCXLVII.	*29.* MCCLCXLV.
10. LXXIX.	*20.* MCCXLIX.	*30.* MMDCCXVDCXXI.

Express in Roman notation—

31. 18.	*37.* 93.	*43.* 421.	*49.* 1492.
32. 27.	*38.* 98.	*44.* 490.	*50.* 1776.
33. 39.	*39.* 111.	*45.* 567.	*51.* 1865.
34. 46.	*40.* 120.	*46.* 719.	*52.* 2674.
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Thus, $5 + 4 = 9$, reads "5 plus 4 equals 9."

29. Addition Table.—So convenient is the system of numeration and notation used for representing numbers, that all numbers, however large, may be resolved into digits, and the sum of any numbers obtained by taking the sums of pairs of digits. Hence, if the sum of each pair of digits be obtained and committed to memory, the addition of all larger numbers may be performed by their use. We have 1 unit + 1 like unit = 2 units (of the same kind), or, briefly, $1+1=2$; also, $1+2=3$, $1+3=4$, etc. Or, putting the pairs of digits in the position in which the pupil will need to use them, and leaving the sum in each case to be supplied by him, we have—

ADDITION TABLE.

1 1	1 2	1 3	1 4	1 5	1 6	1 7	1 8	1 9
2 1	2 2	2 3	2 4	2 5	2 6	2 7	2 8	2 9
3 1	3 2	3 3	3 4	3 5	3 6	3 7	3 8	3 9
4 1	4 2	4 3	4 4	4 5	4 6	4 7	4 8	4 9
5 1	5 2	5 3	5 4	5 5	5 6	5 7	5 8	5 9
6 1	6 2	6 3	6 4	6 5	6 6	6 7	6 8	6 9
7 1	7 2	7 3	7 4	7 5	7 6	7 7	7 8	7 9
8 1	8 2	8 3	8 4	8 5	8 6	8 7	8 8	8 9
9 1	9 2	9 3	9 4	9 5	9 6	9 7	9 8	9 9

30. Addition Independent of Order (Commutative).—If a group of units (as a group of 8 boys) be counted, it is evident that the same numerical result (or number) will be obtained, in whatever order the units be counted. Since addition is but a short way of counting different groups together, it follows that two or more given groups may be added together in any order. Hence, $8+7$ gives the same result as $7+8$, and the Addition Table in Art. 29 gives the sum of each pair of digits, in whatever order the digits occur. To be able to add in either way frequently saves labor.

31. Abstract and Concrete Number.—The work of dealing with numbers is further facilitated by the use of the idea of abstract numbers. For if we dealt with concrete units only, as marbles, apples, men, etc., as we find them in the world about us, we should, for instance, need to verify the addition table for each particular kind of concrete quantity before using the table in adding numbers composed of units of that sort.

Or, to put it in another way, in order to be sure that $7+5$ makes 12 under all circumstances, it would be necessary to take 7 apples and 5 apples and count them together and obtain 12 apples; to take 7 oranges and 5 oranges and count them together and obtain 12 oranges; and to proceed in like manner with marbles, bushels, men, and every kind of concrete units. Instead of this, we take 7 units of any kind and represent them, say, by 7 strokes, and 5 units of the same kind represented by 5 other strokes, and count them together and obtain 12 units of the same kind, as the sum—a result true for like units of any particular kind.

Hence, if we learn the addition table for units in general, we may then use it in adding numbers made up of like units of any particular kind.

A **concrete number** is a number made up of like units of any one particular kind.

An **abstract number** is a number used without reference to any particular thing or unit;

as when we say,

7 units + 9 like units = 16 units of the same kind.

Hence, while the addition table is given for abstract numbers, it applies only to concrete numbers of the same kind.

Thus, it is not possible to add 5 days and 6 apples.

32. Arrangement of Numbers to be Added.—The great value of the positional system of denoting numbers is forcibly illustrated in the process of adding large numbers.

Thus, if a merchant has $623 in one bank, $9024 in another, and $151 in another, and it is required to determine how many dollars he has in all the banks together, we can arrange the three numbers one over the other, putting units of the same order in the same column, and perform the addition by adding each column of units separately. This could not be done if the Roman notation were used in denoting the above numbers.

33. I. Addition when the sum of each column is less than 10 may be illustrated by the use of the example stated in the preceding article.

Setting down the numbers with like units in the same column, we have—

Operation.	
623 dollars.	
9024 "	
151 "	
9798 dollars, *Sum.*	

Explanation.

Adding the units column first, we have 1 and 4 are 5, and 5 and 3 are 8; we set down the sum 8 under the units column. Adding the tens column, we obtain the sum 9, which we set down in the tens place. The sums of the other columns are obtained and set down similarly.

Abstract numbers are added in the same manner.

Thus,
5762 units.
2125 like units.
10112 " "
17999 like units, *sum.*

The enormous saving of labor obtained by the use of the addition table is realized if we conceive of trying to obtain this result by counting merely—that is, by taking 5762 units and counting on 2125 units, and then counting on 10112 units to the result obtained.

34. II. Addition when the sum of any one column is greater than 9 is illustrated by the following:

Ex. A farmer owns three tracts of land, of which the first contains 598 acres, the second 1236 acres, and the third 8759 acres. How many acres does the farmer own?

Operation.	
598 acres.	
1236 "	
8759 "	
10593 acres, *Sum.*	

Explanation.

Arranging the numbers as before, and adding the units column, we obtain the sum 23. The 3 is put under the column of units, and the 2 tens are carried and added with the other tens. The sum of tens column (together with the 2 tens that are carried) is then obtained, and found to be 19. The 9 is set down and 1 (or 10 tens, *i. e.*, 1 hundred) carried to the hundreds column. Proceeding in like manner with the other columns, the entire sum is found to be 10593 acres.

35. Verification.—To prevent error in the work, it is best to perform each process of addition in at least two ways, and observe whether the results are identical. The second process is called a *verification* of the first. There are several different methods of verifying an addition, but the best for ordinary use is to *add the given number by columns in an opposite direction from that first used;*

As, first from the bottom upward, and then from the top downward. If a column be added the second time in an opposite direction from the first, the computer is much more likely to discover any mistake that may have been made, than by simply adding a column twice in the same direction, since in the latter case a mistake made in the first addition is likely to be repeated.

An addition may also be verified by separating into groups the numbers to be added, adding each group separately, and then taking the sum of the partial sums obtained.

36. General Rule for Addition.—*Write the numbers to be added so that figures of the same order shall stand in the same column; begin at the right and add each column separately, placing the sum underneath if it is less than ten; if the sum of any column exceeds nine, set down the right-hand figure only, and add the other figure to the next column to the left.*

Add:

EXERCISE 4.

1.	2.	3.	4.	5.
24 men.	71 miles.	132 hours.	375 boys.	$7356
52 men.	26 miles.	427 hours.	603 boys.	$1422

6.	7.	8.	9.	10.
34659	281367	546923	475013	73 men.
40140	707532	440013	503876	18 men.

11.	12.	13.	14.	15.
95 pens.	58 pages.	77 balls.	876 books.	$5734
47 pens.	65 pages.	58 balls.	309 books.	$6189

16.	17.	18.	19.	20.
35914	558093	307697	797512	98765
85377	483307	985457	123389	56789

21.	22.	23.	24.	25.
351 ft.	453 lines.	710 rods.	628 marks.	$14313
173 ft.	536 lines.	349 rods.	359 marks.	$76823
428 ft.	792 lines.	594 rods.	482 marks.	$55855

26.	27.	28.	29.	30.
34598	446789	512417	47565	98763
71623	615323	387694	62317	89689
24534	558394	757575	89497	78959

31.	32.	33.	34.	35.	36.
123	612	4567	2349	53427	77777
597	759	8912	3716	43251	86546
638	387	3456	5438	70698	71234
245	621	7890	3251	37056	56789
761	348	1357	7791	83799	66666

37.	38.	39.	40.
87653	95673	459543127	45325987654
12761	80170	745241436	91357048653
30508	19787	538579375	67324537167
39175	95068	477889901	45757863448
42078	49632	664543176	37666539399
66779	47819	836842566	85765684784

41.	42.	43.	44.	45.
27561	328005	35742	935684	138
3425	2796	35	71063	76
9	30	1276	29	924
342	420	328	576	92576
2700	7651	14514	3217	4231
6021	38	38	550	47
756	2071	1207	341256	561

46.	47.	48.	49.	50.
3278	9123	1179	6846	8784
8673	7168	7856	7845	7383
1075	3127	3412	3374	6486
3218	6503	3876	4063	3179
4716	9076	6719	5732	1954
7509	3795	5043	6118	2832
4123	4038	6132	9475	4085
8340	9987	7168	5684	5644
9999	7698	1876	7893	6756
6327	4389	1999	4952	7278
8909	6070	9871	3107	8393
5632	4395	7040	1736	9428
3067	7778	9328	8321	4595
1678	8666	9937	9614	6976
3915	4791	7064	7578	7833

37. Addition as a Science and as an Art.—The simplifications which arise from treating quantity as made up of *units;* from the grouping aggregates of units according to a simple systematic plan, so that they can be denoted by a *few number words* and a *few number symbols;* from the use of the system of *positional notation;* from the resulting possibility of performing the additions of all numbers, however large, by the use of the *addition table*—these simplifications together result in making addition all that can be desired as a practical *science.*

For by it, for instance, the general of an army in his tent can determine (and have before him in a form easy to comprehend and use) a representation of the number of men in each part and the whole of his army. A government can by it readily determine the number of its school-children or population, or state its wealth in numbers, etc.

But to be mastered as an *art,* the process of addition requires long practice. To add long columns of figures with absolute accuracy and great rapidity is a power which is obtained only after long and varied practice. The next exercise gives examples adapted to develop this power.

With practice the student will form habits (often instinctive and peculiar to the individual) of adding the figures in a column in certain special ways.

Thus, he may add them in groups of two, three, or four figures; or he may pick out in a column each group of figures that make 10 or 20, add these by themselves, add the other digits by themselves, and take their sum; or he may add two columns at a time. Practice and attention are the main factors which go to form a skillful calculator.

EXERCISE 5.

Let the teacher dictate numbers of one figure to the class, to be added mentally during the dictation and the sum reported immediately. Thus:

1. Add . . . 7, 1, 6, 3, 5, 4, 5, 8, 7, 4, 3, 8; sum is 61.
2. Add . . 8, 4, 3, 1, 7, 9, 6, 5, 3, 1, 7, 9, 6, 5, 6; sum is 80.
3. Add . . . 4, 5, 2, 9, 7, 1, 6, 5, 3, 7, 8, 9, 1, 8, 4, 6, 5; sum is 90.

EXERCISE 6.

1. In six bins there are 45 bu., 82 bu., 96 bu., 124 bu., 43 bu., and 215 bu. How many bushels are there in all?

2. A man owns a farm of five fields which contain 23, 46. 51, 17, and 30 acres respectively. How many acres in the farm?

3. During the six days of one week a merchant received on sales the following amounts: $765, $350, $917, $479, $807, $987. What was the total for the week?

4. There are ten schools in a city, and they enroll 171, 230, 165, 187, 301, 287, 517, 176, 215, 351 pupils respectively. How many school-children in that city?

5. In a township there are six farms which contain 175, 400, 236, 355, 278, 196 acres. How many acres in the township?

6. Find the total expenses of running a bank, if the items for a week are as follows: Salaries and wages $875, postage $11, rent $46, stationery $23, printing $40, books $8, and legal fees $127.

7. A man at death left $4500 to the widow, $1635 to each of three sons, and $958 to a daughter. What was the value of the estate?

8. From A to B is 812 miles, from B to C is 406 miles, from C to D is 615 miles, and from D to A is 786 miles. What is the distance around the whole circuit?

9. Let the teacher give the number of days in each of the months, and the class find the number of days in a year.

10. Let each pupil tell the number of people in his family, and then the whole class find the number of people in all the homes.

11. In the same way let each report the number of examples he has solved, and then the class compute the aggregate.

12. Direct each pupil to count the letters in his full name. Then by telling the number of them, the class can find the number of letters it will take to write the full names of all the members of the class.

13. From the geographies or elsewhere, find the population of each of the New England States. Then find the total.

14. Find the same for the Middle States and for the South Atlantic States.

15. Find the population of the capital of your own State, and of the capitals of all the States which touch it, and then find the total.

16. Find the number of square miles in the six largest States, and then the aggregate.

17. Add seventy-six, three hundred nine, twelve thousand six hundred ten, and forty thousand sixteen.

18. The English army at Waterloo consisted of 26661 infantry, 8735 cavalry, 6877 artillery, and 33413 allies. What was the total?

19. A man owns bonds worth $43765, real estate worth $37050, merchandise valued at $17980, and other property worth $50379. What is the total value of his property?

20. New York contains 49170 sq. mi.; New Jersey, 7815; Pennsylvania, 45215; Delaware, 2050; Maryland, 12210; Virginia, 42450; West Virginia, 24780; and Texas contains 82090 sq. mi. more than all of these put together. How many square miles has Texas?

21. Add 753284 + 95603 + 887653 + 47328 + 867547 + 37895 + 90384 + 7056 + 19948 + 38756 + 938765.

22. Add 77563 + 987635 + 447 + 88956 + 327654 + 887654 + 963558 + 79658 + 9976 + 885432 + 796 + 147785.

CHAPTER III.

SUBTRACTION.

38. Illustration.—John has 7 marbles and gives James 4 of them. How many marbles has John left?

If we take a group of 7 marbles, and remove 4 marbles one at a time, counting off 6, 5, 4, 3, we obtain 3 marbles as the number of marbles left.

But if we are familiar with results of former countings-off, and can recall these, we can say that if 4 marbles be taken from 7 marbles, 3 marbles will be left.

This process is called *subtraction.*

Or we can recall from the addition table the number which, added to 4, makes 7, and say: since 4 + 3 makes 7, when 4 is taken from 7 the number 3 must be left. In either of these two latter processes we substitute the less labor of memory for the greater labor of counting off one number from another.

39. Definitions.—**Subtraction** is the process of finding with least labor what number is left when a number of units is taken away from a larger number of units of the same kind.

The larger number is called the **minuend.**

The smaller number, to be taken from the minuend, is called the **subtrahend.**

The number left is called the **difference** or **remainder.**

Thus, in the illustrative example of Art. 38, we have

7 marbles, *Minuend.*
4 marbles, *Subtrahend.*
3 marbles, *Difference.*

40. The sign of subtraction is the horizontal dash, —, which reads "minus." Placed between two numbers the number of letters it will take to write the full names of all the members of the class.

13. From the geographies or elsewhere, find the population of each of the New England States. Then find the total.

14. Find the same for the Middle States and for the South Atlantic States.

15. Find the population of the capital of your own State, and of the capitals of all the States which touch it, and then find the total.

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If we take a group of 7 marbles, and remove 4 marbles one at a time, counting off 6, 5, 4, 3, we obtain 3 marbles as the number of marbles left.

But if we are familiar with results of former countings-off, and can recall these, we can say that if 4 marbles be taken from 7 marbles, 3 marbles will be left.

This process is called *subtraction*.

Or we can recall from the addition table the number which, added to 4, makes 7, and say: since 4 + 3 makes 7, when 4 is taken from 7 the number 3 must be left. In either of these two latter processes we substitute the less labor of memory for the greater labor of counting off one number from another.

39. Definitions.—**Subtraction** is the process of finding with least labor what number is left when a number of units is taken away from a larger number of units of the same kind.

The larger number is called the **minuend**.

The smaller number, to be taken from the minuend, is called the **subtrahend**.

The number left is called the **difference** or **remainder**.

Thus, in the illustrative example of Art. 38, we have

7 marbles, *Minuend.*
4 marbles, *Subtrahend.*
3 marbles, *Difference.*

40. The sign of subtraction is the horizontal dash, —, which reads "minus." Placed between two numbers the minus sign means that the second number is to be subtracted from the first.

Thus, $9 - 5$ reads "nine minus five," and means that 5 is to be subtracted from 9.

41. Subtraction Table.—Just as addition is performed to the best advantage by committing to memory certain primary sums (viz., the sum of each pair of digits), and performing the addition of all larger numbers by their use, so subtraction is performed to the best advantage by committing to memory certain primary differences, and performing the subtraction of all larger numbers by their use.

Thus, from 7 units we count off 4 like units and get 3 as a remainder, and, to save the labor of again counting off, commit the result to memory, $7 - 4 = 3$.

Similarly we obtain and commit to memory every difference in which the subtrahend and remainder are both single digits. Arranging these differences in a table, placing the minuend over the subtrahend as they usually occur in actual subtraction, and leaving it to the pupil to supply the remainders, we have the table on the opposite page.

So convenient is the system of numeration adopted that numbers, however large, may readily be resolved into digits and pairs of digits, and all subtractions performed by means of this table.

42. I. Subtraction when each digit of the subtrahend is less than the corresponding digit of the minuend.

The process is illustrated by the following example:

Ex. Subtract 345 from 597.

OPERATION.	EXPLANATION.
597, *Minuend.* 345, *Subtrahend.* 252, *Difference.*	We place the subtrahend under the minuend so that units of the same order shall stand in the same column. Beginning at the right, 5 units from 7 units leaves 2 units, and we write 2 in the units place; 4 tens from 9 tens leaves 5 tens, and we write 5 in the tens place; 3 hundreds from 5 hundreds leaves 2 hundreds, and we write 2 in the hundreds place. Hence, we obtain the remainder 252.

SUBTRACTION TABLE.

1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11
2	2	2	2	2	2	2	2	2	2
3	4	5	6	7	8	9	10	11	12
3	3	3	3	3	3	3	3	3	3
4	5	6	7	8	9	10	11	12	13
4	4	4	4	4	4	4	4	4	4
5	6	7	8	9	10	11	12	13	14
5	5	5	5	5	5	5	5	5	5
6	7	8	9	10	11	12	13	14	15
6	6	6	6	6	6	6	6	6	6
7	8	9	10	11	12	13	14	15	16
7	7	7	7	7	7	7	7	7	7
8	9	10	11	12	13	14	15	16	17
8	8	8	8	8	8	8	8	8	8
9	10	11	12	13	14	15	16	17	18
9	9	9	9	9	9	9	9	9	9

43. II. Subtraction when any figure of the subtrahend is greater than the corresponding figure of the minuend.

In this case, before subtracting, increase the figure in the minuend, which is too small, by borrowing a unit from the digit of next higher order of the minuend.

Ex. 1. Subtract 129 from 653.

OPERATION.	EXPLANATION.
653 129 524, *Remainder.*	Since we cannot subtract 9 units from 3 units, we take or borrow 1 ten from 5 tens and add it to the three units; then 9 units subtracted from 13 units leaves 4 units, which we write in the units place of the remainder. We now subtract the 2 tens from the 4 tens which

remain after taking away 1 ten, and obtain 2 tens, which we write in the tens place; 1 hundred taken from 6 hundreds leaves 5 hundreds. Hence, the entire remainder is 524.

It may be necessary to borrow several times in succession.

Ex. 2. Subtract 2358 from 5346.

Operation.	Explanation.
5346	For 8 from 16 leaves 8.
2358	5 " 13 " 8.
2988, *Remainder.*	3 " 12 " 9.
	2 " 4 " 2.

Instead of borrowing a higher unit from the next figure of the minuend, the subtraction may be performed by adding 1 to the corresponding unit of the subtrahend. The result will be the same, since if two numbers be equally increased (as by the addition of 1 ten to the 4 tens and 2 tens in Ex. 1 above, making them 5 tens and 3 tens), the difference will remain unchanged. The process performed in this latter way is slightly easier, since addition is easier than subtraction.

Thus, in Ex. 1 the method of the subtraction would be

	In Ex. 2,
9 from 13 leaves 4.	8 from 16 leaves 8.
3 " 5 " 2.	6 " 14 " 8.
1 " 6 " 5.	4 " 13 " 9.
Difference is 524.	3 " 5 " 2.
	Difference is 2988.

44. Verification.—To test the accuracy of the work, add the difference and the subtrahend. Their sum should equal the minuend.

Thus, in Ex. 2 above, add 2358 and 2988; their sum is 5346. Hence, the difference obtained, 2988, is correct unless mistakes have been made in the two processes, of such a nature that they compensate. This is not likely to occur. There is another similar method of verification which the student should discover for himself.

45. General Rule for Subtraction.—*Write the subtrahend under the minuend, placing units of the same order in the same column; begin at the right and subtract each figure of the subtrahend from the corresponding figure of the minuend, and place the result beneath;*

If any figure of the subtrahend is less than the corresponding figure of the minuend, increase the latter by 10, and subtract; to compensate, diminish by 1 the figure of the next higher order in the minuend (or increase by 1 the figure of next higher order in the subtrahend), and continue the process.

EXERCISE 7.

	1.	*2.*	*3.*	*4.*	*5.*
From	67 men.	357 boys.	531 balls.	256 mi.	614
take	25 men.	135 boys.	311 balls.	245 mi.	110

	6.	*7.*	*8.*	*9.*	*10.*
From	265 men.	948 boys.	728 balls.	876 mi.	840
take	161 men.	446 boys.	315 balls.	463 mi.	637

	11.	*12.*	*13.*	*14.*	*15.*	*16.*	*17.*
From	53	41	63	106	358	502	752
take	19	26	45	37	176	387	173

	18.	*19.*	*20.*	*21.*	*22.*	*23.*
From	362	543	760	924	571	5071
take	194	367	368	367	478	1837

	24.	*25.*	*26.*	*27.*	*28.*
From	4705	13756	21504	34576	28765
take	3846	8607	8476	16268	9175

	29.	*30.*	*31.*
From	7265432	301705401	2706510547308
take	3514765	170643053	1607432813954

32. From 71532056176032 take 47063127159374.

33. From 6755307165322 take 2946834073163.

34. Subtract 17650321470063280 from 29560732165032761.

35. Subtract 6307532410769547 24 from 85032507650403208 0.

36. Take 1234567890987654321 from 5432101234567890123.

46. Computers' Method of Subtraction.—There is another method of subtraction much used by professional computers, which the pupil should at least understand. It is illustrated by the ordinary process of making change. Thus, if a storekeeper receives a dollar bill in payment of a bill of 78 cents, he makes change by paying out first 2 cents, which, with the 78 cents, makes 80 cents; and then paying out 2 dimes, which, with the 80 cents, makes \$1. By this method the required subtraction is converted into and performed as addition. In like manner any subtraction may be performed as an addition.

Thus, to subtract 723 from 968, we have—

968	Since 3 and 5 are 8.
723	2 " 4 " 6.
245, *Remainder.*	7 " 2 " 9.

We set down 245 as the remainder. Similarly

653	9 and 4 are 13.
129	2 " 2 " 4.
524, *Remainder.*	1 " 5 " 6.

47. Value of Subtraction.—The student should frequently call to mind the saving of labor effected by subtraction as compared with other processes of determining a remainder.

Thus, if a merchant knows that his original stock of potatoes was 1000 bushels, and his records show that he has sold 627 bushels, by a simple subtraction, and without the labor of counting off the number of bushels sold from the original number, or the labor of actual measurement of the number left, he can tell the number of bushels remaining, and whether he can supply a customer who wants 400 bushels.

Thus, also, if he has bought a hogshead of molasses containing 63 gallons, and has sold 27 gallons, by subtraction he can determine the number of gallons left, without the labor of actually measuring the remainder in gallons and counting them.

EXERCISE 8.

1. The following accounts were each paid with a dollar bill: how much change was due in each case?

40 cts.	60 cts.	14 cts.	85 cts.	61 cts.	57 cts.
55 cts.	70 cts.	27 cts.	39 cts.	78 cts.	83 cts.

2. If a bicycle cost \$87 and sold for \$98, how much was gained?

3. A house cost \$3205 and sold for \$3052. Find the loss.

4. A merchant having 2712 yards of cloth sold 1907. How many yards remained?

5. A farmer who raised 1600 bushels of corn retained 205 bushels. How many bushels did he sell?

6. I paid \$110 for a horse and \$78 for a wagon. I sold both for \$169. Did I gain or lose, and how many dollars?

7. An official receives \$2315 salary and \$1692 in fees. He spends \$2865. How many dollars does he save?

8. A grain dealer bought in one week 76321 bushels of grain, and in the next 33478 bushels. He then sold 67305 bushels. How many remained?

9. America was discovered in 1492. How many years was that before you were born? How many years was that before the year 2000?

10. A farmer bought a horse for \$231, and harness for \$87. He sold the horse for \$256, and the harness for \$54. How many dollars did he lose altogether?

11. Of 3728 men in an army, 276 were wounded, 193 were killed, and 705 deserted. How many remained at duty?

12. An estate of \$23675 was divided among a widow who received \$8525, a son who got \$756 less than the widow, and a daughter who received the remainder. What was the daughter's part?

13. Three men invest \$25600. The first invests \$7356; the second \$1728 more than the first. How much does the third invest?

14. Thomas Jefferson was born in 1743 and died in 1826.

How old was he? How old would he have been if he had lived till 1900?

15. From the sum of 6175 and 2857 take their difference.

16. I receive \$27, \$42, \$69, \$121, and pay out \$73, \$29, \$11, \$7, and \$130. How much remains?

17. There have been subscribed toward a million dollars by Mr. A. \$26310, by Mr. B. \$42225, by Mr. C. \$61700, by Mr. D. \$54655, by Mr. E. \$112950, and by Mr. F. \$87605. How much remains to be raised?

Find the number of dollars remaining in the bank in each of these three cases:

18. Deposits.	*18.* Withdrawals.	*19.* Deposits.	*19.* Withdrawals.	*20.* Deposits.	*20.* Withdrawals
\$137	\$38	\$75	\$195	\$9721	\$46
341	142	132	38	328	375
273	67	41	92	5263	8
564	9	67	140	56	417
	156	328	7	8	1376
	225	576	18	46	4251
	46		4	575	3765
	7		56	1250	48
	11		123		4
					976
					23

Compute the values of—

21. 18 + 15 − 26 + 17 − 30 + 16.

HINT.—Take the sum of those preceded by a + sign and of those preceded by a − sign; subtract the latter from the former. Thus, 18 + 15 + 17 + 16 = 66; 26 + 30 = 56; 66 − 56 = 10, *Ans.*

22. 35 + 19 − 26.
23. 75 − 23 + 14.
24. 96 − 48 − 27.
25. 67 + 84 − 125.
26. 128 − 104 + 71.
27. 376 − 291 + 167.
28. 895 − 397 − 299.
29. 39 − 65 + 42 + 31 − 28.
30. 501 − 373 − 192 + 215.
31. 983 + 185 − 467 − 324.
32. 5768 − 4297 + 3008.
33. 59 − 43 + 97 − 101 + 38.

34. 87 − 75 − 9 + 108 − 79 + 40.

35. 131 − 118 + 46 − 28 + 137 − 95.

36. 1767 + 487035 − 397516 + 42765.

37. 895632 − 765107 + 143200 − 97653 − 8765.

38. From nine hundred seven take seven hundred nine.

39. Subtract six thousand five hundred sixty-three from fourteen thousand one hundred eight.

40. To seven hundred sixteen add three hundred ninety and six thousand seventy-five. From this sum take three thousand two hundred ninety-nine.

41. Subtract the sum of five thousand forty-seven and seven hundred twenty, from the sum of four thousand six hundred and three thousand one hundred eight.

42. From the sum of twenty-six thousand eight hundred forty-two and ninety-three thousand four hundred eighty-two, take the difference between four hundred six thousand forty-five and two hundred ninety-six thousand three hundred nine.

Compute the values of:

43. 75 − (12 + 37).
44. 96 − (28 + 51).
45. (96 − 28) + 51.
46. 29 + (75 − 19).
47. 300 − (175 + 98).
48. 975 − (328 + 400 − 275).
49. 788 − 275 − (300 − 96).
50. 1887 − 438 + 756 − 432.
51. 1887 − (438 + 756 − 432).
52. 976 − (85 + 176) − (276 − 88).
53. 8865 − (775 + 896 − 483) − (99 + 387).
54. (99765 + 73876 − 47956) − (88763 − 47958 + 38176).

NOTE.—Let the pupil's parent furnish more examples like the first seventeen of this exercise. Compare altitudes of mountains; population of cities; of States. Pupils can often form examples for each other, and then correct the papers or slates of one another.

CHAPTER IV.

MULTIPLICATION.

48. Illustration.—A woman buys 7 yards of cloth at $4 a yard. How many dollars does she pay for the cloth?

$4
4
4
4
4
4
4
$28

The cost of the cloth may be obtained by addition; the sum of a column of seven 4's is 28.

If, however, the student is familiar with the results of former additions of columns composed of the same digit, he may remember that seven 4's added make 28. It is easier to recall the result of the former addition than to add the column again. We substitute the less labor of recollection for the greater labor of addition and for the still greater labor of counting together the different sets of 4 units each.

Similarly the sum of any set of equal numbers may be found by recalling the results of former additions. This process is called *Multiplication.*

49. Multiplication is the process of finding the sum of a set of numbers, all equal to each other, by the abbreviated method of recalling the results of former additions.

The **multiplicand** is one of the equal numbers which are to be added.

$4, *Multiplicand.*
7, *Multiplier.*
$28, *Product.*

In the example given, $4 is the multiplicand.

The **multiplier** is the number which indicates how many equal numbers are to be added.

In the above example, 7 is the multiplier.

The **product** is the result obtained by the multiplication.

In the above example, $28 is the product.

The multiplicand and multiplier are called the *factors* of the product.

Multiplication is usually viewed in its abbreviated form, and may then be defined as follows:

Multiplication is the process of finding a number (the product) which shall equal another number (the multiplicand) repeated as many times as there are units in a third number (the multiplier).

50. The **sign of multiplication** is the inclined cross, ×. Placed between two quantities it means that the one is to be multiplied by the other. Thus 4 × 7 means "4 multiplied by 7," or "7 multiplied by 4" (that is, "4 times 7"). When the numbers to be multiplied are placed one over the other, the lower one is regarded as the multiplier.

51. Multiplication Table.—If the product of each pair of digits be obtained and committed to memory, the product of all other numbers, however large, may be obtained by the use of these few primary products.

For, so convenient is the system of numeration and notation which we have adopted, that all numbers, however large, may be resolved into digits, and their products obtained by taking the products of different pairs of digits.

While it is sufficient to know the products of pairs of numbers up to 9, it is convenient to extend the table a little further, and to learn the product of each pair of numbers up to 12.

By the addition of columns of like digits the following results are obtained:

MULTIPLICATION TABLE.

Twice	Three times	Four times	Five times	Six times	Seven times
1 are 2	1 are 3	1 are 4	1 are 5	1 are 6	1 are 7
2 " 4	2 " 6	2 " 8	2 " 10	2 " 12	2 " 14
3 " 6	3 " 9	3 " 12	3 " 15	3 " 18	3 " 21
4 " 8	4 " 12	4 " 16	4 " 20	4 " 24	4 " 28
5 " 10	5 " 15	5 " 20	5 " 25	5 " 30	5 " 35
6 " 12	6 " 18	6 " 24	6 " 30	6 " 36	6 " 42
7 " 14	7 " 21	7 " 28	7 " 35	7 " 42	7 " 49
8 " 16	8 " 24	8 " 32	8 " 40	8 " 48	8 " 56
9 " 18	9 " 27	9 " 36	9 " 45	9 " 54	9 " 63
10 " 20	10 " 30	10 " 40	10 " 50	10 " 60	10 " 70
11 " 22	11 " 33	11 " 44	11 " 55	11 " 66	11 " 77
12 " 24	12 " 36	12 " 48	12 " 60	12 " 72	12 " 84

Eight times	Nine times	Ten times	Eleven times	Twelve times
1 are 8	1 are 9	1 are 10	1 are 11	1 are 12
2 " 16	2 " 18	2 " 20	2 " 22	2 " 24
3 " 24	3 " 27	3 " 30	3 " 33	3 " 36
4 " 32	4 " 36	4 " 40	4 " 44	4 " 48
5 " 40	5 " 45	5 " 50	5 " 55	5 " 60
6 " 48	6 " 54	6 " 60	6 " 66	6 " 72
7 " 56	7 " 63	7 " 70	7 " 77	7 " 84
8 " 64	8 " 72	8 " 80	8 " 88	8 " 96
9 " 72	9 " 81	9 " 90	9 " 99	9 " 108
10 " 80	10 " 90	10 " 100	10 " 110	10 " 120
11 " 88	11 " 99	11 " 110	11 " 121	11 " 132
12 " 96	12 " 108	12 " 120	12 " 132	12 " 144

Since multiplication occupies the leading place in almost every process in arithmetic, the multiplication table should be so thoroughly mastered that the pupil can give instantly, without a moment's reflection, the product of any pair of digits. In committing the table to memory he will be aided by various simple expedients, thus:

In the *table for 5*, each product ends in 5 or 0.

In the *table for 9*, the sum of the digits of each product is 9 (except in 99), the tens digit increasing 1 and the units digit decreasing 1 in each successive product.

In the *table for 11*, the two digits in each product are alike up to 99.

The *labor of committing the table to memory is also diminished one-half* by remembering that, for instance, the product 9×7 is the same as 7×9.

EXERCISE 9.

ORAL.

1. There are 4 quarts in a gallon. How many quarts in 7 gallons? In 9 gallons? In 12 gallons?

2. There are 7 days in a week. How many days in 5 weeks? In 7 weeks? In 11 weeks? In 12 weeks?

3. How many working days are there in 8 weeks? In 10 weeks? In 12 weeks?

4. If I study 9 hours each day, how many hours will I study in 4 days? In 6 days? In 9 days? In 11 days?

5. If wood is worth $5 a cord, what must be paid for 3 cords? For 5 cords? For 9 cords? For 12 cords?

6. A boy spends $8 a month. How much will he spend in 4 months? In 6 months? In 9 months?

7. If 1 bucket of water costs 0 cents, what will 5 buckets cost? 7 buckets? 12 buckets?

8. What is the product of 8×7? 9×4? 11×6? 7×6? 5×9? 8×5? 12×9? 5×12? 4×7? 11×12? 8×9?

9. Find the value of $7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7$ without the table. Then by multiplication. Which is easier?

Compute the values of each of the following:

10. $3 \times 2 \times 5$.
11. $4 \times 3 \times 8$.
12. $5 \times 2 \times 9$.
13. $6 \times 2 \times 7$.
14. $7 \times 1 \times 8$.
15. $2 \times 5 \times 8$.
16. $3 \times 4 \times 2$.
17. $3 \times 3 \times 8$.
18. $8 \times 1 \times 12$.
19. $6 \times 2 \times 11$.
*20.** $4 \times 3 + 7$.
21. $6 \times 7 + 1$.
22. $3 \times 8 - 5$.
23. $4 \times 9 - 8$.
24. $9 \times 5 + 7$.
25. $2 \times 9 - 6$.
26. $3 \times 8 - 7$.
27. $9 \times 7 - 5$.
28. $7 \times 7 - 7$.
29. $8 \times 8 + 10$.
30. $5 \times 6 - 5$.
31. $7 \times 8 + 6$.
32. $4 \times 12 - 8$.
33. $11 \times 12 + 1$.
34. $9 \times 9 - 9$.
35. $7 \times 9 + 5$.
36.† $9 - 2 \times 3$.
37. $27 - 4 \times 6$.
38. $4 + 7 \times 3$.
39. $8 + 5 \times 6$.
40. $9 + 12 \times 3$.
41. $30 - 7 \times 4$.
42. $60 - 5 \times 12$.
43. $70 - 6 \times 11$.
44. $4 + 8 \times 12$.
45. $8 + 9 \times 6$.
46. $3 + 10 \times 5$.
47.‡ $2 + 3 \times 5 - 7$.
48. $12 - 3 \times 2 - 2$.
49. $5 \times 5 - 5 + 8$.
50. $14 - 6 + 3 \times 4$.
51. $3 \times 0 + 7 \times 2$.
52. $8 \times 7 - 6 \times 4$.
53. $7 \times 6 + 9 \times 1$.
54. $9 \times 12 - 8 \times 6$.
55. $7 - 3 \times 2 + 6 \times 8$.
56. $3 \times 4 + 5 - 2 \times 7$.
57. $9 \times 0 + 5 \times 12 - 8$.
58. $2 \times 5 \times 7 - 3 \times 4 \times 4$.
59. $8 - 2 \times 3 - 7 \times 0 + 4 \times 8$.
60. $13 + 7 \times 2 - 3 \times 5 + 6 \times 3$.
61. $75 - 7 \times 5 - 6 \times 6 + 6$.
62. $30 + 7 \times 9 - 8 \times 7 + 4 \times 0$.
63. $100 - 7 \times 6 - 5 + 4 \times 12 - 20$.

52. Use of Abstract and Concrete Numbers in Multiplication.—The use of abstract numbers is of the first importance in multiplication.

For if we made no use of abstract number and dealt with concrete numbers only, as marbles, apples, dollars, feet, etc., it would be necessary, for instance, to verify the multiplication table for each particular kind of concrete number before using the table to multiply that particular kind of

* In each example, the multiplication must be performed first.
† This is $9 - 6 = 3$. ‡ This is $2 + 15 - 7 = 10$.

concrete number. But if we form the multiplication table for abstract number (represented, for instance, by strokes or dots), we may then apply the table to any particular kind of concrete number.

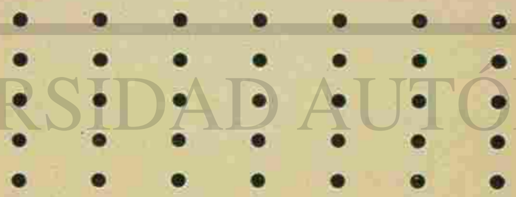
In the use of abstract and concrete number, certain limitations, however, are to be observed. Thus, the multiplicand may be either an abstract or concrete number, but the number of times the multiplicand is taken (that is, the multiplier) must be an abstract number. Thus, 7 apples can be multiplied by 8, but not by 8 marbles. Hence,

1 *The multiplicand may be either an abstract or concrete number.*

2. *The multiplier must be an abstract number.*

3. *The product is the same kind of number as the multiplicand;* i. e., *if the multiplicand is abstract, the product is abstract; if the multiplicand is concrete, the product is concrete and of the same kind.*

From these laws it follows that there are certain limitations also in interchanging the multiplicand and multiplier. If both are abstract numbers, it is evident that they are interchangeable. For if we have a series of 5 rows, each containing 7 dots, and each dot stand for an abstract unit, then the



total number of dots in the group is denoted either by 7×5 or 5×7; that is, *the factors of an abstract number are commutative.* Hence, we can perform the multiplication of two abstract numbers in either order, as may be most advantageous.

But if, of two factors, one is concrete and the other abstract, as $\$4 \times 7$, they are not commutative, and, taking the second number as the multiplier, we cannot write $\$4 \times 7 = 7 \times \$4$, since the latter form requires us to use a concrete number as a multiplier.

In such cases, however, we can obtain the advantage of the commutative principle in the actual process of the multiplication, by setting aside the concrete unit involved, and making the multiplication abstract till it is performed, and then restoring the concrete unit to the product; or we can, for purposes of computation, transfer the concrete unit from one factor to the other, thus—

$$\$4 \times 7 = \$1 \times 4 \times 7 = \$1 \times 7 \times 4 = \$7 \times 4.$$

If each dot in the rectangular array of dots given on opposite page represent one dollar, this is the same as saying that 7 columns, each containing 4 units of 1 dollar each, are the same as 4 rows, each containing 7 units of 1 dollar each; or 7 yds. of cloth, at \$4 each, cost as many dollars as 4 yds. at \$7 each, and the one computation may be exchanged for the other, if any advantage is gained thereby.

We now proceed to show that by the use of the multiplication table any two numbers, however large, may be multiplied together.

53. I. When the Multiplier is a Single Digit.—The process is illustrated by the following example:

Ex. Multiply \$264 by 7.

OPERATION.	EXPLANATION.
\$264, *Multiplicand.* 7, *Multiplier.* \$1848, *Product.*	We write the multiplier, 7, under the units figure, 4, of the multiplicand, and multiply the 4 units by 7, obtaining 28 units, or 2 tens and 8 units,

set down the 8 units in the units place, and reserve the 2 tens to be added to the next partial product. We then multiply the 6 tens by 7 and obtain 42 tens, to which we add the 2 tens reserved, and obtain 44 tens, or 4 hundreds and 4 tens. Setting down the 4 tens in the tens place, and reserving the 4 hundreds, we next multiply 2 hundreds by 7, and obtain 14 hundreds. To this we add the 4 hundreds reserved, and obtain 18 hundreds, which we set down in the proper place. Hence, the product of \$264 by 7 is \$1848.

The student should observe that the multiplication has been performed by taking the product of single pairs of digits separately and combining the partial products obtained. If the student will set down the number 264 seven times in a column, and obtain the sum by addition, he will realize the labor saved by the process of multiplication, even in a simple example like this.

EXERCISE 10.

1. There are 24 hours in one day. How many are there in 3 days? In 5 days? In 7 days? In 9 days?

2. One bushel of corn weighs 56 pounds. How many pounds will 4 bushels weigh? 9 bushels? 6 bushels?

3. There are 66 feet in one chain. How many feet in 4 chains? In 7 chains? In 8 chains?

4. A barrel of flour weighs 196 pounds. How many pounds do 4 barrels weigh? 5 barrels? 9 barrels?

5. There are 365 days in one year. How many days in 3 years? In 5 years? In 7 years?

6. If 725 men have each $4, how many have they all? If each has $6? $8?

Multiply:

7. 256 boys by 7.
8. 439 girls by 8.
9. 716 feet by 6.
10. 1729 men by 5.

11. $2803 by 6.
12. $6753 by 8.
13. $3926 by 9.
14. 42307 yds. by 7.

	15.	*16.*	*17.*	*18.*
Multiply	9013 feet	14706 inches	3278 min.	35544 rods
by	6	8	4	7

	19.	*20.*	*21.*	*22.*	*23.*
Multiply	2758	3759	46789	78697	53716
by	3	4	5	6	7

	24.	*25.*	*26.*	*27.*	*28.*
Multiply	7791	4567	3803	6175	40461
by	7	8	9	7	3

	29.	*30.*	*31.*	*32.*
Multiply	701508	517032	876531	90470680327
by	4	5	6	8

What must be paid for the following purchases:

33. 12 yards silk at $3 a yard, and 17 yards cloth at $2 a yard?

34. 16 spools thread at 5 cts. a spool, and 7 yards cord at 2 cts. a yard?

35. 8 cows at $28 each, and 6 oxen at $46 each?

36. 9 horses at $127 each, and 4 carriages at $108 each?

Find amounts of these two memoranda:

37. 3 mo. rent at $28.
6 tons coal at $6.
7 loads wood at $5.
12 barrels oil at $5.

38. 376 hats at $2.
428 coats at $8.
714 tickets at $6.
125 employees at $5.

Suppose here, too, the pupils make examples for each other. Let parents help at home. Teacher can give several examples whose answers can be told at a glance, such as: What is the difference between six times 5324, and 7 times the same number? Multiply 93076 first by 3, and then by 7, and add the results, etc., etc.

54. II. When the multiplier is a digit with one or more zeroes annexed.

Ex. Multiply $843 by 40.

OPERATION.
$843
40
$33720

EXPLANATION.

In this case we multiply by the digit, 4, and annex the zeroes to the product. For if 40 groups of 843 dollars each are to be added together, we can, if convenient, separate the 40 groups required into 10 groups of 4 each. $843 × 4, or $3372, will then give the number of units in each of the 10 groups, and the product of $3372 by 10 will give the entire number of units in the product of $843 × 40.

Similarly, if the multiplier is composed of any two factors, the multiplication, if it is desirable, may be separated into two steps. We first multiply the multiplicand by one factor of the multiplier, and then multiply the product so obtained by the other factor. Thus, to multiply 2608 by 63, since $63 = 9 \times 7$, we may first multiply 2608 by 9 and obtain 23472, and then multiply 23472 by 7 and obtain 164304 as the product of the two original numbers. We may proceed similarly if the multiplier is separable into three or more factors.

55. III. When the multiplier contains two or more digits.

Ex. 1. Multiply $384 by 237.

OPERATION.	ABBREVIATED FORM.
$384	$384
237	237
2688, *First Partial product.*	2688
11520, *Second* " "	1152
76800, *Third* " "	768
$91008, *Entire Product.*	$91008, *Product.*

EXPLANATION.—We regard the multiplier as composed of three parts, viz., 7 units, 3 tens or 30, and 2 hundreds or 200, multiply by each separately, and then add the partial products obtained. It is customary to omit the zeroes which indicate the order of the second, third, etc., partial products.

It is to be noted that if one or more of the figures of the multiplier is a zero, the corresponding partial product is a zero and need not be written.

Ex. 2. Multiply 56308 by 4007.

OPERATION.

```
    56308
     4007
   394156
 225232
 225626156, Product.
```

56. Verification.—One method of verifying the work of multiplication—that is, of testing it with a view to detecting any errors that may have been made—is to let the multiplier and multiplicand change places, and perform the multiplication again. We are at liberty to do this by Art. 52.

Ex. Multiply 437 by 26, and test the accuracy of the result.

OPERATION.	VERIFICATION.
437	26
26	437
2622	182
874	78
11362	104
	11362

As the two products obtained are identical, and it is not likely that mistakes have been made in such a way in the two different processes that they exactly compensate, we assume that the result obtained is correct.

If the student is familiar with the process of division, it is left as an exercise for him to devise another method of verifying multiplication by its use.

57. **A general rule for multiplication** may now be formally stated. When the multiplier is a single figure,

Write the multiplier under the units figure of the multiplicand;

Multiply each figure of the multiplicand by the multiplier;

If the product is less than 10, place it under the figure multiplied; if greater, set down the right-hand figure, adding the other figure to the next partial product.

When the multiplier is greater than 9,

Write the multiplier under the multiplicand, with units of the same order in the same column;

Begin with the units figure of the multiplier and multiply the multiplicand by each figure of the multiplier in succession, placing the right-hand figure of each partial product under the term by which it was obtained;

Add the partial products.

EXERCISE 11.

	1.	*2.*	*3.*	*4.*	*5.*
Multiply	$75	$83	96 boys	128 pens	365 days
by	16	25	33	46	58

	6.	*7.*	*8.*	*9.*	*10.*
Multiply	$387	$476	375 feet	406 days	365 days
by	67	89	123	314	236

	11.	*12.*	*13.*	*14.*	*15.*	*16.*
Multiply	$760	$809	550	828	512	891
by	325	426	433	567	471	635

	17.	*18.*	*19.*	*20.*
Multiply	32071	76203	93761	56714
by	50	420	305	270
	21.	*22.*	*23.*	*24.*
Multiply	56702	67094	76543	67038
by	508	1320	2045	3029
	25.	*26.*	*27.*	*28.*
Multiply	67563	76805	80984	66894
by	1300	7503	3370	3502
	29.	*30.*	*31.*	*32.*
Multiply	46780	33607	23716	55007
by	1432	5071	7632	3896
	33.	*34.*	*35.*	*36.*
Multiply	34256	68327	32765	33856
by	4017	4009	12706	10708

37. How would you do the first example in this exercise by addition? How the 24th? How the 35th? Can you now understand how useful multiplication is?

38. There are 24 hours in a day, and 365 days in a year. How many hours in a year? In 6 years? In 75 years?

39. What will 743 horses cost at $130 each?

40. One mile contains 1760 yards. How many yards in 425 miles? In 720 miles?

41. There are 5280 feet in a mile, and 12 inches in a foot. How many inches in a mile? In 25 miles?

42. Methuselah lived 969 years. How many days did he live?

43. A book of 674 pages contains 38 lines on each page. How many lines in the book?

44. If there are 14 words in each line, how many words in the book of Example 43?

45. A bicyclist rides 16 miles an hour for 36 days of 7 hours each. How many miles did he ride altogether?

46. Which is the greater, and how much,

$756 \times 243 + 965$, or $3328 \times 79 - 69300$?

47. Multiply $261 \times 325 - 83476$ by $66750 - 307 \times 209$.

58. Value of Multiplication.—The value of the process of multiplication as compared with addition should be frequently recalled in connection with examples like the following:

Ex. If it costs $23562 to build a single mile of railroad, how much will it cost to build 273 miles?

If this example were solved by addition, it would require the setting down of 23562 two hundred and seventy-three times, and the addition of five columns of figures, each containing 273 figures. The process of multiplication (by resolving the numbers into pairs of digits and recalling the results of a few simple additions) gives the same result with scarcely one hundredth of the labor.

The saving of labor effected by multiplication is still further realized when some adequate idea of the extent of its application is formed. The largeness of its field of application is due to the fact that wherever possible, groups of units are made uniform in the number of units which each group contains. Thus, each foot in linear measure is composed of 12 inches, all the milk in a can is sold for the same number of cents per quart, each week contains 7 days, etc. Also, where the number of units is not the same in different groups considered, it is frequently made uniform by taking the average of all the groups, so that multiplication may be applied. Thus, measurements to determine areas, volumes, etc., are usually so taken that the multiplicative principle may be applied. Thus, in measuring the area of a floor which is, say, 7 yards long and 5 yards wide, we do not mark off the floor into actual square yards and count or add the number of them, but we measure the length and breadth of the floor, and thus, in effect, arrange the square yards contained in the floor into 5 rows, each containing 7 square yards, and then obtain the number of square yards in the area by the multiplication $7 \times 5 = 35$.

EXERCISE 12.

1. Multiply 750 by 430 and the product by 48.

2. From $87 \times 43 \times 56$ subtract 235×260.

3. Take $370 \times 80 \times 420$ from $567 \times 203 \times 50$.

4. What is the value of $46 \times 70 - 28 \times 45 + 160 \times 23$?

5. What will 763 acres of land cost at $85 an acre?

6. A man 600 miles from New York walked toward that city 28 days, 17 miles each day. How far away was he then?

7. A railroad train travels 45 miles an hour for 38 days of 24 hours each. How many miles must it still run to have gone 45000 miles?

8. Supposing there are 85 apples in a bushel, how many will there be in a crop of the same kind, of 560 bushels?

9. There are 32 quarts in a bushel. How would you determine the number of grains in a bushel of wheat without counting them all? In 75 bushels?

10. There are 43 rows of trees in an orchard and 61 trees in each row. How many trees in the whole orchard? How many in another orchard having twice as many rows and twice as many trees in each row?

11. Thirteen books contain respectively 261, 295, 304, 247, 283, 311, 219, 276, 253, 309, 267, 238, 294 pages. If they each had 293 pages, how many more pages would there be?

12. Mr. Dash sold Mr. Blank 24 rolls of cloth, each containing 45 yards, at $2 a yard. Mr. Blank sold Mr. Dash 6 lots of land, each containing 7 acres, at $52 an acre. Which gentleman owes the other and how much?

13. In a train-load of flour there are 78 cars, each containing 184 barrels and each barrel weighing 196 pounds. Find total weight.

14. Two bicyclists are 1800 miles apart and ride toward each other. One rides 11 miles an hour for 6 hours of each day, and the other rides 9 miles an hour for 8 hours every day. After riding thus for 13 days, how far apart are they?

15. A speculator bought 585 acres of land at $74 an acre. He sold at one time 87 acres at $64 an acre; at another time, 137 acres at $93 an acre; and at a third sale, 178 acres at $77 an acre. At what price must he sell all the remaining acres to gain $4280 on the transaction?

16. What would he have gained by selling the remaining acres at $75 an acre?

17. Which is larger, 753×427 or 691×538?

18. Find the difference between $97 \times 86 - 347$ and $97 \times 347 - 86$.

19. Bought 370 barrels of apples at $3 a barrel, and sold them so as to gain $2 on each barrel. What was the cost? What was my profit? What was the selling price?

20. A book-keeper whose salary is $4000 a year spends at the rate of $7 a day for 365 days every year. How much can he save in 28 years?

21. A man buys 326 sheep at $4 each and sells them so as to gain $125. What was the selling price?

22. A farm of 381 acres sold for $76 an acre, the owner thereby gaining $486. What was its cost to him?

23. When is there gain? When loss?

24. A capitalist bought 376 acres of land at $83 an acre. He sold from it, 96 acres at $92; 139 acres at $85; 63 acres at $81; and the remainder at $107. What was his total profit?

25. There are 2 pints in a quart, 4 quarts in a gallon, and 63 gallons in a hogshead. Without counting them all, how could one determine the number of drops of water in a hogshead? In 525 hogsheads?

26. Which is the greatest and which the least:

$$17 \times 13 + 15 \times 11 - 18 \times 19,$$
$$17 + 13 \times 15 - 11 \times 18 \times 19, \text{ or}$$
$$(17 - 13) \times 15 \times 11 - 18 \times 19?$$

27. Explain the several operations and the *order* of operations in: $3 + (4 + 5) \times 6 - (7 + 8 \times 9) + 10 \times 11$. And also in: $3 + 4 + (5 \times 6) - 7 + (8 \times 9 + 10) \times 11$. Find the value of each.

CHAPTER V.

DIVISION.

59. Illustration.—A man having $18 is staying at a hotel at a cost of $3 a day. How many days can he stay?

Since he spends $3 a day, the number of days he can stay can be determined by subtracting $3 from $18, then $3 from the remainder, and so on in succession till the $18 are exhausted, and then counting the number of times $3 has been subtracted. The number of subtractions will be the number of days required.

If, however, the student is familiar with the multiplication table, it is much easier to recall that the number which multiplies 3 and makes 18 is 6. We thus substitute the less labor of memory for the greater labor of repeated subtraction and the still greater labor of counting off dollars in groups of 3 each from 18, and counting the number of groups counted off.

60. Definitions.—The process of determining how many times one number may be subtracted from another—that is, is contained in another—by a brief method (as by the aid of the multiplication table above), is termed *division*.

The **dividend** is the number from which the successive subtractions are made.

The **divisor** is the number successively subtracted.

The **quotient** is the number of times the divisor is subtracted. Thus, in the example in Art. 59, $18 is the dividend, $3 is the divisor, and 6 is the quotient.

Since the short way of dividing is to recall that multiplier which, applied to a given number, will produce another given number, we may say that—

Division is the process by which, one factor and the product being given, the other factor is determined. The given factor is the *divisor*, the given product is the *dividend*, and the required factor is the *quotient*.

61. Exact and Inexact Division.—If we divide $18 by 3, we find that, after subtracting six times, no dollars remain. When there are no units left over from a division, the division is said to be **exact**.

If we attempt to divide $18 by $5, we find that, after subtracting $5 three times, $3 are left.

When units are left over from a division, the division is said to be **inexact**. The units left over are called the **remainder**.

62. Relation of Quantities in Division.—Since the two factors of a number multiplied together equal the number, it follows that—

1. In exact division, *dividend = divisor × quotient.*
2. In inexact division, *dividend = divisor × quotient + remainder.* Or, using symbols, and denoting the dividend by D, divisor by d, quotient by Q, remainder by R,

$$D = d \times Q + R.$$

Thus, $$\$18 = \$5 \times 3 + \$3.$$

63. Symbols for Division.—The ordinary sign of division is ÷, which reads "divided by." Placed between two numbers, it means that the first number is to be divided by the second. Thus, 36 ÷ 9 means that 36 is to be divided by 9. Division may also be indicated by a horizontal line, with the dividend above and divisor below, as $\frac{36}{9}$; or by a curved line, with the divisor on the left, and the dividend on the right, as 9)36.

64. Table for Division.—Just as the multiplication of all large numbers is performed by means of a few simple primary products, which are learned once for all at the outset, so the division of all large numbers may be resolved into a few simple primary divisions, to be learned once for all, and used repeatedly afterward. These simple primary divisions are formed by taking each product in the multiplication table, and determining one of its factors when the other is given.

Thus, since $9 \times 6 = 54$, we may write $\frac{9)54}{6}$, $\frac{6)54}{9}$;

Or, $54 \div 9 = 6$; $54 \div 6 = 9$.

Let the pupil form a division table in this way and thoroughly master it.

EXERCISE 13.

ORAL OR DRILL EXERCISE.

1. Tell immediately the quotient in each case:

15 ÷ 3.	22 ÷ 11.	36 ÷ 6.	72 ÷ 9.
27 ÷ 9.	24 ÷ 6.	18 ÷ 6.	15 ÷ 5.
24 ÷ 4.	25 ÷ 5.	42 ÷ 7.	84 ÷ 12.
35 ÷ 5.	90 ÷ 10.	36 ÷ 4.	77 ÷ 11.
21 ÷ 7.	84 ÷ 7.	96 ÷ 8.	48 ÷ 6.
36 ÷ 12.	72 ÷ 12.	81 ÷ 9.	32 ÷ 4.
28 ÷ 4.	49 ÷ 7.	54 ÷ 6.	60 ÷ 5.
56 ÷ 7.	30 ÷ 5.	63 ÷ 7.	132 ÷ 12.
63 ÷ 9.	45 ÷ 9.	16 ÷ 8.	88 ÷ 11.
48 ÷ 12.	48 ÷ 8.	20 ÷ 5.	121 ÷ 11.
64 ÷ 8.	66 ÷ 11.	40 ÷ 8.	108 ÷ 12.
42 ÷ 6.	72 ÷ 8.	60 ÷ 10.	56 ÷ 8.

3)36	7)49	9)36	11)132
5)45	8)72	4)44	12)84
7)63	6)42	8)24	7)35
12)96	5)55	9)108	12)60
11)99	8)32	6)72	6)30

2. If a man earns $6 a day, how many days must he work to earn $36? $48? $72?

3. How many weeks in 56 days? In 49 days? In 77 days?

4. How many sponges at 8 cents apiece can be bought with 64 cents? With 72 cents? With 96 cents?

5. There are 4 quarts in a gallon; how many gallons in 20 quarts? In 40 quarts? In 48 quarts?

6. What is the remainder in each case following?

25 ÷ 4.	38 ÷ 8.	80 ÷ 9.	63 ÷ 6.
29 ÷ 9.	43 ÷ 5.	90 ÷ 11.	100 ÷ 12.
30 ÷ 7.	47 ÷ 6.	70 ÷ 8.	105 ÷ 11.
35 ÷ 4.	68 ÷ 9.	75 ÷ 7.	115 ÷ 12.

7. What is the price of each orange when 9 oranges cost 45 cents? When they cost 72 cents?

8. If there are 9 square feet in a square yard, how many square yards in 108 square feet? In 90 square feet?

9. If there are 48 roods in 12 acres, how many roods in 1 acre?

10. When 9 quarts of milk cost 63 cents, what is the price of 1 quart?

11. If I save $132 in 12 months, how much must I average each month?

12. In 8 days a digger opened 72 yards of ditch. How many yards could he open in 1 day?

13. If a wheelman rides 96 miles in 12 hours, how far does he ride in 1 hour? In 7 hours?

65. Abstract and Concrete Numbers in Division.—Abstract number is of the first importance in division, since it enables us to form a division table good for all kinds of particular concrete quantity. But, as in multiplication, so in division, certain limitations must be observed in the use of abstract and concrete numbers. Since we can divide a number of units by a number of like units or by a number of groups (but not by a number of unlike units), it follows that

1. If the dividend is an abstract number, the divisor must be an abstract number also, and hence the quotient too will be abstract.

2. If the dividend is a particular kind of concrete number and the divisor also is concrete, it must be a concrete number of the same kind with the dividend. Thus, we can divide $18 by $3, or 18 marbles by 3 marbles, but not $18 by 3 marbles.

In this case the *quotient is abstract number.*

3. If the dividend is concrete number, the divisor may be abstract number. Thus, $18 may be divided by 6; that is, divided into 6 equal parts.

In this case the *quotient is a concrete number of the same kind with the dividend.*

This kind of division is sometimes called *partition*, since it consists essentially of dividing a given number into a number of equal groups or parts.

In the preceding oral exercise, let the pupil point out which examples are cases of partition.

We now proceed to show how, by the use of the division table, any number, however large, may be divided by a smaller number.

66. I. Short Division.—When the divisor is a single digit, the process is called *short division*. The dividend may in this case be resolved into small partial dividends mentally, the divisions performed mentally, and the figures of the quotient set down at once.

Ex. 1. Divide \$9452 into 4 equal parts.

OPERATION.

$$4\overline{)\$9452}$$
\$2363, *Quotient.*

EXPLANATION.

We set down the divisor at the left of the dividend, and divide it into the different orders of units of which the dividend is composed, beginning with the highest and dividing each separately. 4 is contained in 9 (thousands) 2 (thousands) times, with a remainder of 1 (thousand). Setting down the 2 beneath the thousands and combining the 1 (thousand) with 4 (hundreds), we have 14 (hundreds) as the next partial dividend. 4 is contained in 14 (hundreds) 3 (hundreds) times, with a remainder of 2 (hundreds). Setting down the 3 (in the hundreds place) and combining the 2 (hundreds) remainder with the 5 (tens), we have 25 (tens) as the next partial dividend. 4 is contained in 25 (tens) 6 (tens) times, with a remainder of 1 (ten). Setting down the 6 (tens) in the quotient, the last partial dividend is 12 (units). Into this, 4 is contained 3 times. Hence, the division is exact, and the entire quotient is \$2363.

Ex. 2. Divide \$31559 by 7.

OPERATION.

Divisor $7\overline{)\$31559}$, *Dividend,*
\$4508, *Quotient,* with a *Remainder* of \$3.

EXPLANATION.—Since 7 is not contained in 3 (ten thousands), we treat 3 as a remainder, and divide 7 into 31 (thousands) as the first partial dividend, obtaining the quotient 4 (thousands), with 3 (thousands) as a remainder, and proceed as in Ex. 1. Similarly when 7 is not contained in 5 tens, we set down 0 in the tens place in the quotient, and take 59 as the next partial dividend. 7 is contained in 59 (units) 8 times, with a remainder of 3 (units). Hence the division is inexact, and the quotient is \$4508, with a remainder of \$3. The remainder is sometimes written over the divisor, with a line between, and set down as a (fractional) part of the quotient. The entire quotient in this case would then be \$$4508\frac{3}{7}$.

67. Other Cases of Short Division.—The method of short division can be employed when the divisor is 11 or 12, or a larger number if the student is familiar with the table of the products of such numbers by the nine digits.

Ex. 1. Divide 42084 by 12.

OPERATION.

$$12\overline{)42084}$$
3507, *Quotient.*

Again, if the divisor consist of a single digit followed by two or more zeroes, the division is best performed as a short division.

Ex. 2. Divide \$897563 by 300.

Since the divisor is an exact number of hundreds, that part of the dividend which is less than \$100 (viz., \$63) may be set aside as a part of the remainder, and the number of hundreds in the dividend divided by the number of hundreds in the divisor; hence, we have

$$3\widehat{00})\overline{\$8975\widehat{63}}$$
\$2991, *Quotient* + \$263, *Remainder.*

68. Verification.—To test the accuracy of the work done in division, we may multiply the divisor and quotient together, and add the remainder, if there be any, to the product. If the result equals the dividend and there have been no compensating errors in the two processes, the work is correct (see Art. 62).

Thus, to test the accuracy of the work in the last example, we have

$$\begin{array}{r} \$2991 \\ 300 \\ \hline \$897300 \\ \$263 \\ \hline \$897563. \end{array}$$

This result equals the original dividend; hence, the work in the division is (probably) correct.

EXERCISE 14.

Divide rapidly and orally:

1. 60 by 3.
2. 84 by 4.
3. 96 by 3.
4. 48 by 2.
5. 69 by 3.
6. 93 by 3.
7. 86 by 2.
8. 56 by 4.
9. 42 by 3.
10. 54 by 2.
11. 57 by 3.
12. 58 by 2.
13. 64 by 4.
14. 65 by 5.
15. 68 by 4.
16. 72 by 3.
17. 75 by 5.
18. 78 by 6.
19. 81 by 3.
20. 94 by 2.
21. 240 by 4; by 8.
22. 360 by 9; by 6.
23. 630 by 7; by 10.
24. 720 by 12; by 9.
25. 560 by 8; by 4.
26. 840 by 7; by 12.
27. 960 by 8; by 6.
28. 880 by 11; by 4.
29. 420 by 6; by 3.
30. 600 by 5; by 10.

Copy and divide the following:

31. 3)426 inches.
32. 2)590 feet.
33. 3)714 days.
34. 2)592 men.
35. 3)477 tons.
36. 2)738 men.
37. 4)344 days.
38. 3)825 pecks.
39. 4)628 yards.
40. 5)645 feet.
41. 4)30248
42. 5)16785
43. 6)43974
44. 7)66668
45. 7)19999
46. 7)456939
47. 8)301376
48. 8)123456
49. 8)579752
50. 8)703144
51. 9)555579
52. 9)745083
53. 9)430056
54. 9)388971
55. 9)8035083
56. 9)559431
57. 11)345631
58. 11)499037
59. 12)494820
60. 12)688608
61. 3)1213526
62. 2)16181590
63. 3)1524291
64. 4)3231616
65. 4)8352396
66. 5)9541035
67. 5)47521015
68. 5)3800275
69. 6)3046824
70. 7)5666612
71. 7)19633328
72. 8)96111264

73. Divide 7488, 56703, 1341117, and 627144 by 3.
74. Divide 5208, 617102, 9031758, and 12345678 by 6.
75. Divide 175345, 420275, 753105, and 5123045 by 5.
76. Divide 302208, 4251612, 25801278, and 10002018 by 6.
77. Divide 30028, 1756320, 75690376, and 5004108 by 4.
78. Divide 66965, 3201305, 3710420, and 9876540 by 5.
79. Divide 33334, 7080906, 1230463, and 20230721 by 7.
80. Divide 520352, 2705608, 3391576, and 70001208 by 8.
81. Divide 23742, 1012347, 50123458, and 70123456 by 9.
82. 567886 by 11.
83. 587180 by 11.
84. 76488 by 12.
85. 65472 by 12.
86. 930156 by 12.
87. 5750412 by 12.
88. Divide 35816, 44781, 4075973, and 10170688 by 11.
89. Divide 28284, 609756, 888384, and 46818072 by 12.
90. A man divided $7434 equally among 6 children. How many dollars did each receive?
91. There are 7 days in a week. How many weeks in 2002 days?
92. At $5 a barrel, how many barrels of flour can be bought for $3940?
93. A boy receives $9 a week. How many weeks must he work in order that he may receive $7020?
94. There are 12 inches in a foot. How many feet in 63360 inches?
95. If there are 4 pecks in a bushel, how many bushels in 16300 pecks?
96. Change 35091 square feet to square yards, if there are 9 square feet in one square yard.
97. Multiply 504 by 231, and divide the product by 9.
98. Divide 75320 by 8, and multiply the quotient by 76.
99. Multiply 204 by 917, and divide the product by 12. Then divide this quotient by 7.
100. Find the product of $225 \times 716 \times 135$, and divide it by 5. Divide the quotient by 12. Divide this quotient by 9.
101. If the product is 43744, and the multiplier is 8, find the multiplicand.

Find the divisor if the:

102. Dividend is 3078 and the quotient is 9.
103. Dividend is 176072 and the quotient is 8.
104. Dividend is 463815 and the quotient is 11.

Find the quotient and remainder in each case:

105. 8)31415	*112.* 20)98750	*119.* 300)976580
106. 9)21763	*113.* 30)67670	*120.* 400)95670
107. 7)1416838	*114.* 40)841370	*121.* 500)1796300
108. 6)360517	*115.* 80)35670	*122.* 600)92370
109. 5)312059	*116.* 90)123456	*123.* 700)516784
110. 10)16758	*117.* 100)26750	*124.* 800)56793
111. 11)176531	*118.* 200)976350	*125.* 900)387650

Find the dividend if the:

126. Divisor is 7, quotient is 43, and remainder is 5.
127. Divisor is 12, quotient is 327, and remainder 10.
128. Divisor is 23, quotient is 76, and remainder 18.
129. Quotient is 416, divisor 207, and remainder 194.
130. Quotient is 356, remainder 401, and divisor 510.

69. II. Long Division.—When the divisor is so large (larger than 12) that the products of it by the nine digits cannot be retained mentally and subtracted, it is necessary in the division to set down in succession the partial dividends and the successive subtrahends and remainders. When division is performed in this way, it is called *long division.*

Ex. Divide 8746 by 37.

OPERATION.	VERIFICATION.
Dividend.	
Divisor 37)8746(236, *Quotient.*	236
74	37
134	1652
111	708
236	8732
222	14
14, *Remainder.*	8746, (= *Dividend.*)

EXPLANATION.—We determine the first partial dividend by beginning at the left and taking the smallest number of digits that will contain the divisor; 37 is not contained in 8, but is contained in 87 (hundreds), 2 (hundreds) times, with a remainder of 13 (hundreds). Setting down the 2 (hundreds) to the right as part of the quotient, and combining the 13 (hundreds) remainder with the 4 (tens), we have 134 (tens) as the next partial dividend. 37 is contained in 134 (tens), 3 (tens) times, with a remainder of 23 (tens). Setting down the 3 (tens) as part of the quotient, and combining the 23 (tens) remainder with the 6 (units), we have 236 (units) as the last partial dividend. 37 is contained in 236 (units), 6 (units) times, with a remainder of 14 (units). Hence, the entire quotient is 236, with a remainder of 14.

It is now left as an exercise for the pupil to write out a *general rule for long division.*

In regard to long division, it is to be remarked that

1. If the work is properly done, the remainder is in all cases less than the divisor. If in the course of the work a remainder is obtained equal to or greater than the divisor, it shows that the last figure of the quotient is too small.

2. If at any time the divisor multiplied by the last figure gives a product greater than the partial dividend under which it is placed, it shows that too large a number has been taken for the last figure of the quotient.

3. If at any time the partial dividend is less than the divisor, and hence will not contain it, a zero is to be set down in the quotient, another figure of the dividend brought down, and the work continued as before.

Ex. Divide 216912714 by 71873.

```
71873)216912714(3018, Quotient.
      215619
        129371
         71873
         574984
         574984
```

EXERCISE 15.

Divide

1. 322 inches by 14.
2. 540 feet by 15.
3. 391 men by 23.
4. 728 days by 26.
5. 1457 years by 31.
6. 1204 feet by 43.

7. 2530 tons by 55.
8. 2989 min. by 61.
9. 6300 hours by 75.
10. 5312 pages by 83.
11. 7990 by 94.
12. 2668 by 23.
13. 3166 by 42.
14. 9968 by 56.
15. 17152 by 67.
16. 24198 by 74.
17. 29602 by 82.
18. 36210 by 85.
19. 51428 by 92.
20. 71808 by 96.

21. 7380 by 36.
22. 13815 by 45.
23. 21268 by 52.
24. 32448 by 64.
25. 47479 by 79.
26. 25800 by 25.
27. 77666 by 38.
28. 185288 by 46.
29. 302328 by 57.
30. 595765 by 85.
31. 1968 by 123.
32. 2430 by 135.
33. 6048 by 108.
34. 8695 by 235.
35. 9222 by 318.

36. 31832 by 184.
37. 400189 by 187.
38. 70305 by 215.
39. 106953 by 231.
40. 64250 by 125.
41. 120239 by 193.
42. 182688 by 346.
43. 113057 by 207.
44. 442458 by 523.
45. 164984 by 328.
46. 434995 by 719.
47. 719037 by 891.
48. 444024 by 881.
49. 377289 by 927.
50. 306327 by 503.
51. 802392 by 998.
52. 155484 by 126.
53. 721644 by 231.
54. 1722855 by 331.
55. 2135328 by 354.
56. 2752461 by 423.
57. 3019272 by 429.
58. 4125954 by 509.
59. 5488416 by 608.

60. 5582225 and 7430600 by 1325.
61. 8010555 and 17266376 by 2461.
62. 13841372 and 29745036 by 3812.
63. 21801270 and 25961535 by 4115.
64. 43673280 and 54034432 by 6208.

65. 27300 by 350.
66. 70680 by 760.
67. 37800 by 1400.
68. 170100 by 2700.
69. 1663200 by 5400.
70. 9010000 by 17000.

Find the quotient and remainder in each case:

71. 96731 ÷ 309.
72. 59178 ÷ 421.
73. 96733 ÷ 1209.
74. 188576 ÷ 2761.
75. 9980736 ÷ 2047.
76. 13896789 ÷ 8308.

77. If there are 24 hours in 1 day, how many days are there in 2136 hours? In 3168 hours?

78. How many beeves at $35 each can a dealer buy with $2380? With $6650?

79. There are 144 square inches in 1 square foot. How many square feet are there in 11952 square inches?

80. How many barrels of flour in a load containing 40180 pounds, a barrel of flour weighing 196 pounds?

81. The distance from the earth to the sun is 93000000 miles. How many days would it take a cyclist to ride that distance, traveling 124 miles a day? How many years, if there are 365 days in a year?

82. How many days would it take a locomotive to run that far, at the rate of 744 miles a day?

83. How long would light require, to come from the sun to earth, at the rate of 186000 miles a second? What is the name of your quotient?

84. At $117 a share, how many shares can be purchased with $843336?

85. What is the price of 1 acre of land, when 371 acres cost $87185?

86. There are 792 inches in a chain, and 63360 inches in a mile. How many chains in a mile?

87. The product is 350102, and the multiplier is 386. Find the multiplicand.

88. The quotient is 567, and the dividend is 2442069. Find the divisor.

89. How many yards of cloth at $3 a yard must be returned for 7 barrels of flour at $6 a barrel?

90. How many tons of hay worth $18 a ton must be given in exchange for 24 loads of coal at $9 a load?

91. At $36 an ox, how many oxen will be required, to pay for a farm of 104 acres, worth $81 an acre?

92. The 425 families of a village agree to bear equally the expense of paving 612 rods of street, costing $25 a rod. How many dollars will each family contribute?

Divide:

93. 5168254 by 1898.
94. 26638950 by 4314.
95. 69908524 by 7543.
96. 9111878 by 3954.
97. 32393290 by 4978.
98. 68115312 by 7165.
99. 37318800 by 3405.
100. 17829888 by 3072.
101. 83555703 by 8701.
102. 25360303 by 6329.
103. 32658915 by 5435.
104. 73728456 by 9208.
105. 280967545 by 6435.
106. 422582680 by 5785.
107. 60088326832 by 76048.
108. 3380002995264 by 67008.
109. 803000835205 by 200705.

Find the value of each of the following:

110. $7 \times 6 + 63 \div 9.$
111. $9 + 40 \times 2 - 72 \div 3.$
112. $80 + 121 \div 11 - 90.$
113. $196 \div 14 + 324 \div 18.$
114. $8 \times 7 - 18 \div 3 + 45 \div 15 - 105 \div 5.$
115. $256 \div 16 - 7 \times 2 + 11 \times 14 - 90 \div 9.$
116. $1728 \div 12 - 169 \div 13 - 289 \div 17 - 361 \div 19.$

70. Factorial Division.—If a divisor can be separated into two or more small factors, the work of division can often be diminished by dividing the dividend by the factors of the divisor in succession, instead of dividing by the entire divisor at once.

Ex. 1. Divide 11060 by 35.

The factors of 35 are 7 and 5.

Hence, to divide 11060 by 35, we divide first by 7, and then divide the quotient obtained by 5, thus:

```
7)11060
 5)1580
    316, Quotient.
```


By dividing by 7 we separate 11060 into 1580 groups of 7 each, and by dividing by 5 we separate the 1580 groups of 7 each into 316 groups of 5×7, or 35 each.

The method of determining the remainder in factorial division is best shown by an example.

Ex. 2. Divide 6083 by 84 by the factorial method.

OPERATION.

```
3)6083
4)2027 groups of 3 units, with a remainder of 2 units.
 7)506 groups of 3 × 4 units, with a remainder of 3 groups of 3 units.
    72 groups of 3 × 4 × 7 units, with a remainder of 2 groups of 3 × 4 units.
```

Hence, the quotient is 72, and the entire remainder consists of three parts, viz., 2 units, and 3×3 or 9 units, and $2 \times 3 \times 4$ or 24 units; or $2 + 9 + 24$, 35.

Hence, to find the entire remainder in factorial division,

Multiply each remainder by the divisor which produced the given remainder and by all preceding divisors, and add the results.

EXERCISE 16.

By the factorial method, divide:

1. 888 by 24.
2. 1736 by 28.
3. 3570 by 42.
4. 4230 by 45.
5. 7074 by 54.
6. 15744 by 64.
7. 13794 by 66.
8. 25632 by 72.
9. 31350 by 75.
10. 41426 by 77.
11. 50787 by 81.
12. 68712 by 84.
13. 38304 by 96.
14. 41846 by 98.
15. 49665 by 105.
16. 79596 by 108.
17. 91056 by 112.
18. 124272 by 144.

Find the quotient and the remainder in each, by the factorial method:

19. 355 by 36.
20. 927 by 48.
21. 791 by 64.
22. 999 by 56.
23. 975 by 121.
24. 3446 by 135.
25. 4289 by 144.
26. 4873 by 150.
27. 5327 by 216.
28. 582 by 243.
29. 6899 by 324.
30. 7163 by 385.
31. 8563 by 420.
32. 8873 by 540.
33. 9997 by 756.

71. The Value of Division, even in a single problem, is realized on comparing the labor employed in division with what the labor would be if the same problem were solved by repeated subtraction. Take, for example, the following problem:

A railroad contractor has $400000, and is paying out $3276 a day; he desires to know how many days his money will last. To determine the number of days by successive subtractions of $3276 would require the setting down and subtracting of this number more than 120 times. To obtain the required number of days in this way would require at least 50 times as much labor as by the method of long division.

The value of division is still further realized when some adequate notion is formed of the extent of its possible application. Division, like multiplication, has this wide application owing to the extended use of groups uniform in the number of units which they contain. Thus it is an advantage to have the barrels employed for a given purpose, as to measure oil, contain the same number of gallons; to sell all the yards in a given piece of cloth at the same price per yard, etc.

Where uniform groups of units are not explicitly given, they may often be formed by the proper analysis, and the solution of the given problem effected by division.

Ex. 1. In a given election where 10896 votes were cast, the successful candidate had a majority of 324. How many votes did each candidate receive?

We may regard the number of votes received by the defeated candidate as the primary or unit group of the problem. The successful candidate received this unit group and 324 votes besides. Hence, the total number of votes cast, 10896, equals twice the unit group and 324 votes besides. Hence, subtracting 324 votes from 10896 votes, we have 10572 votes as equal to twice the unit group, and the unit group may be obtained by dividing 10572 by 2. Hence, 10572 ÷ 2, or 5286, is the number of votes received by defeated candidate; and 5286 + 324, or 5610, is the number of votes received by successful candidate.

Ex. 2. A man dying left an estate of $84000, of which his wife was to receive a certain part, his daughter half as much as his wife, and his niece half as much as his daughter. How many dollars did each inherit?

If we regard the number of dollars received by his niece as the unit, then his daughter received twice this unit, and his wife four times it. Hence, the number of dollars bequeathed must equal 7 times the unit taken. If 7 times the unit equals $84000, the unit group itself may be obtained by dividing $84000 by 7, giving $12000 as a quotient. From this the value of each share is readily determined.

EXERCISE 17.

1. In an election in which 22795 votes were cast for 2 candidates, the successful candidate received a majority of 461 votes. How many votes were cast for each?

2. John has twice as many marbles as William, and William has three times as many as George; all have 130 marbles. How many has each?

3. A boy has four times as many examples as his sister, and both together have 100. How many has each?

4. Divide 90 into two parts so that one is twice as large as the other. One is 5 times as large as the other.

5. How could the examples of Exercise 15 have been done other than by division? How else could you do Example 21 there?

6. How could you do Example 60 but by division? Which is more simple? Could you do Example 81 the other way?

7. Without multiplication or division, how could the 92d example be done?

8. There are 10 times as many boys in a certain school as there are teachers, and 3 times as many girls as boys; altogether there are 492. How many teachers, boys, and girls separately?

9. Separate the number 132 into two parts such that one is 11 times the other. Into two other parts such that one is 21 times the other.

10. A has twice as many acres as B, B has twice as many as C, C has three times as many as D, and all have 1100 acres. How many has each?

11. In a certain election 56329 votes were cast for 2 candidates, the successful candidate receiving a majority of 1071 votes. How many did each receive?

CHAPTER VI.*

ABBREVIATED PROCESSES.

72. Fundamental Arithmetical Operations and their Abbreviations.—The four operations, addition, subtraction, multiplication, division, are called the fundamental operations of arithmetic, since all subsequent arithmetical work consists of these in various combinations. Hence, it is important that every possible method of abbreviating these fundamental processes be investigated and mastered. They are themselves, it is to be remembered, abbreviations of more tedious work, as of the counting and grouping of units; yet, in many cases, the four operations can be still further abbreviated. The abbreviations used diminish not only the amount of work, but also the likelihood of error.

ABBREVIATED MULTIPLICATION.

73. Multiplication by 25.—Since $25 = 100 \div 4$, to multiply a number by 25,

Annex two zeroes to the multiplicand (i. e., multiply by 100) and divide by 4.

Ex. Multiply \$8769 by 25.

$$\begin{array}{r} 4)\$876900 \\ \hline \$219225, \textit{ Product.} \end{array}$$

The multiplication, when performed in this way, calls for less mental effort and the making of fewer figures. It can be readily mastered and made a purely mental process.

74. Multiplication by an Aliquot Part of 100.—Of like nature is the multiplication by any other aliquot or exact

*In teaching young or backward pupils, the teacher should omit all of Chapter VI, except Arts. 73, 74, 79, 80, 84.

part of 100. This calls for a slight knowledge of fractions, but should be mentioned in this connection.

Thus,	$12\frac{1}{2} = \frac{1}{8}$ of 100.	$50 = \frac{1}{2}$ of 100.
	$16\frac{2}{3} = \frac{1}{6}$ of 100.	$62\frac{1}{2} = \frac{5}{8}$ of 100.
	$25 = \frac{1}{4}$ of 100.	$66\frac{2}{3} = \frac{2}{3}$ of 100.
	$33\frac{1}{3} = \frac{1}{3}$ of 100.	$75 = \frac{3}{4}$ of 100.
	$37\frac{1}{2} = \frac{3}{8}$ of 100.	$87\frac{1}{2} = \frac{7}{8}$ of 100.

Hence, for example, to multiply a number by $33\frac{1}{3}$, *annex two zeroes to the multiplicand and divide by* 3. Let the student state and illustrate a rule for multiplying by each of the other aliquot parts of 100. Examples illustrating this kind of multiplication will be found in Exercise 18.

75. When the Multiplier is a Series of 9's except the last Digit.—In this case the multiplication may be abbreviated into the process of *first multiplying by* 1 *followed by as many zeroes as there are figures in the multiplier* (*i. e., annexing this many zeroes to the multiplicand*), *and then deducting the product of the multiplicand by the excess of the multiplier used over the given multiplier.*

Ex. Multiply 13721685 by 99998.

Since $99998 = 100000 - 2$, we first multiply 13721685 by 100000, and then deduct 13721685×2 from the product so obtained. Thus we have

$$\begin{array}{r} 1372168500000 \\ 27443370 \\ \hline 1372141056630, \textit{ Product.} \end{array}$$

If the student will now perform the multiplication directly, he can estimate the labor saved by the abbreviated process.

76. Multiplication in which the Partial Products are not set Down, but are Added Mentally.—When multiplying by a number of but two or three digits, it is often of advantage to form in immediate succession all the partial products of the same order and add them mentally, setting down only the last figure of each result in its proper place in the product, and carrying the other figure mentally.

Ex. 1. Multiply 47 by 63.

OPERATION.

```
  47
  63
2961, Product.
```

EXPLANATION.

We have the following partial products: 7 (units) × 3 (units) = 21 (units). Set down 1 (unit) and carry 2 (tens); 3 (units) × 4 (tens) + 6 (tens) × 7 (units) + 2 (tens) carried = 56 (tens). Set down 6 (tens) and carry 5 (hundreds); 6 (tens) × 4 (tens) + 5 (hundreds) carried = 29 (hundreds), which we set down.

Ex. 2. Multiply 587 by 346.

OPERATION.

```
   587
   346
203102, Product.
```

The partial products, with figures carried, may be grouped as follows:

(Units) 7 × 6 = 42 units.
(Tens) 4 + 8 × 6 + 4 × 7 = 4 + 48 + 28 = 80 tens.
(Hundreds) 8 + 6 × 5 + 4 × 8 + 3 × 7 = 91 hundreds.
(Thousands) 9 + 4 × 5 + 3 × 8 = 53 thousands.
(Tens of thousands) 5 + 3 × 5 = 20 tens of thousands.

77. Abbreviations due to Limitations of Accuracy in Measuring Quantities to be Multiplied.—As has been explained in Art. 19, each digit of a number has a value depending on the place which it occupies in the number (as well as the absolute value of the digit). Thus the figure 6, when in the thousands place, has a hundred times the value it has when in the tens place. It is important that the student use every means to keep in mind this difference. For convenience in printing and writing, the digits of a number in the units, tens, etc., places are made of uniform size; but it is useful, in thinking of them, to regard them as of different sizes proportional to the positional values. Thus, while looking at the number 586, let the pupil think of the 5 as a digit ten times as large as the 8, and the 6 as a digit only one-tenth as large as the 8, thus ☞ 5_{8_6}

Similarly in the number 7652138, if 7 be given a size in proportion to its positional value as compared with the number, it would be a figure 8 feet long, while the 8 in comparison with the 2 would vanish into invisibility.

If numbers be thought of in this way, we easily realize that the digits composing a number are of less and less importance as we go to the right, and in certain cases their value vanishes into insignificance, and computations with them may be limited accordingly.

78. Computations Based on Measurements.—No measurement is accurate beyond the sixth or seventh figure; this is owing to the limitations of our eyesight and sense of touch-perception, and to the ultimate imperfections in all our instruments of measurement.

Thus, a mile (63360 inches) can be measured only to within $\frac{1}{16}$ inch of its true length; an inch can be measured only to within a millionth part of itself, etc. So great a degree of accuracy, however, can be obtained only by applying every possible refinement of accuracy. Ordinary measuring, such, for instance, as that done by a carpenter, is accurate only to the second or third figure, that is, to within $\frac{1}{100}$ or $\frac{1}{1000}$ part. Hence,

Computations based on measurements cannot be accurate beyond the fifth or sixth place of figures, and in ordinary work not beyond the third figure.

All numerical work, therefore, which does not affect the accuracy of the result within the required limits, may be omitted.

Ex. Multiply 3274 by 4125 so that the result shall be accurate to three places.

We carry out the work to five places, so that on adding the partial products the figure in the third place may be accurate, thus,

OPERATION.

```
 3274
 4125
13096
  327
   64
   15
13502, thousands.
```

EXPLANATION.

We first multiply by 4, the digit of highest denomination in the multiplier. Since this gives a partial product containing five figures, as required, before multiplying by 1, the next digit in the multiplier, we strike out the units figure 4 in the multiplicand and then multiply. Similarly we strike out one figure in the multiplicand before multiplying by each successive figure in the multiplier. The number of figures struck out fixes the denomination of the product.

13500000 is the product correct to the third place.

EXERCISE 18.

Perform the following multiplications by the shortest method:

1. 356 × 25.	*10.* 584 × 125.	*19.* 77 × 97.
2. 477 × 33⅓.	*11.* 648 × 375.	*20.* 296 × 98.
3. 968 × 12½.	*12.* 743 × 750.	*21.* 447 × 96.
4. 777 × 25.	*13.* 26 × 29.	*22.* 738 × 98.
5. 324 × 37½.	*14.* 32 × 36.	*23.* 315 × 998.
6. 586 × 62½.	*15.* 42 × 37.	*24.* 716 × 997.
7. 420 × 66⅔.	*16.* 53 × 61.	*25.* 1763 × 999.
8. 676 × 87½.	*17.* 78 × 43.	*26.* 2385 × 996.
9. 736 × 75.	*18.* 85 × 76.	*27.* 3485 × 997.

Find approximately the following products:

28. 231 × 463.	*32.* 4671 × 3085.
29. 378 × 512.	*33.* 7681 × 4321.
30. 918 × 761.	*34.* 8506 × 7533.
31. 4231 × 3248.	*35.* 4761 × 3289.

ABBREVIATED DIVISION.

79. Division by 25.—The result obtained by dividing a group of units by 25 is the same as that obtained by dividing four times as large a group by 100. The latter process usually involves less labor. Hence, to divide a number by 25,

Multiply the given number by 4 and divide by 100.

Ex. Divide 6237 by 25.

OPERATION.

$$\begin{array}{r} 6237 \\ \underline{4} \\ 100)\underline{24948} \\ 249\tfrac{48}{100}, \text{ or } 249\tfrac{12}{25}. \end{array}$$

Hence, the quotient is 249, with a remainder 12.

80. Division by an Aliquot Part of 100 may be abbre-

viated in a manner similar to that used in abbreviating the multiplication by such numbers.

Ex. Divide 89676 by 33⅓.

$$\begin{array}{r} 89676 \\ \underline{3} \\ 100)\underline{269028} \\ 2690\tfrac{28}{100} \text{ or } 2690\frac{9\frac{1}{3}}{33\frac{1}{3}}; \end{array}$$

that is, the quotient is 2690, while the remainder is 9⅓. Let the student formulate rules for dividing a number by aliquot parts of 100.

For examples, see Exercise 19.

81. When the divisor is a series of 9's except the last digit.

Ex. Divide 7865923 by 98.

The abbreviated process consists essentially in dividing by 100, and then forming remainders which are divided in succession.

OPERATION.

Quotients.	*Remainders.*
78659	23
1573	18
31	46
	62
1	49
	2
80264	51

EXPLANATION.

Dividing 7865923 by 100, we obtain 78659 as a quotient and 23 as a remainder. 98 is contained the same number of times as 100, with a further remainder of 2 for every time 100 is contained; that is, a further remainder of 157318. Dividing 157318 by 100, and continuing the process as above, we obtain a series of quotients, 78659, 1573, 31 (which are to be added together), and a series of remainders, 23, 18, 46, 62. Adding the remainders, we obtain 149; dividing this by 100, and then by 98, we obtain a further quotient, 1, and a final remainder, 51. Adding the quotients, the final quotient for the divisor 98 is 80264, with remainder, 51.

82. Abbreviation by Setting Down only the Successive Partial Dividends.—Students with especial aptitude for numerical work may learn to abbreviate long division by performing mentally the subtractions required, and setting down only the partial dividends.

Ex. Divide 460686 by 71.

OPERATION.

```
71)460686(6488, Quotient.
    346
     628
      606
       38, Remainder.
```

EXPLANATION.

In ordinary long division we would multiply 71 by 6 (the first figure of the quotient), and set down the product, 426, under 460, and subtract. In the abbreviated process we multiply each figure of 71 by 6, and subtract figure by figure, setting down only the remainder. Thus, $6 \times 1 = 6$, and 6 from 10 leaves 4; set down 4. 6×7 is 42, and 42 from 45 leaves 3, which we set down. Annexing 6, we have 346 as the next partial dividend. We proceed in like manner till the division is completed.

83. Division of Numbers Determined by Measurements.—If one or both of the numbers employed in a division be determined by measurement, the result of the division cannot be accurate beyond the fifth or sixth figure. Hence, all numerical work which does not affect the accuracy of the result beyond the fifth or sixth place may be omitted. This is done by beginning at a certain stage of the work to strike off the final digit of the divisor, instead of bringing down a new figure from the dividend and annexing it to the partial dividend.

Ex. Divide 872372500 by 15273, so that the quotient shall be accurate to 4 figures.

To be accurate to 4 places, we must also determine the fifth figure. If we obtain two figures of the quotient in the ordinary way, and then begin to strike off the final digits of the divisor, it will insure that two digits of the divisor will remain at the close of the process.

OPERATION.

```
15273)872372500(57118
      76365
      108722
      106911
        1811
        1527
         284
         153
         131
         122
```

Hence, the required quotient is 57120.

It is to be observed that striking out the last figure on the right of the divisor, the last figure struck out is in each case multiplied and the product carried, if it is more than 5; thus, in forming the last subtrahend, $8 \times 15 = 120$, but since 2 has just been struck out, and $8 \times 2 = 16$, the last subtrahend is $120 + 2$, or 122.

EXERCISE 19.

Divide most briefly:

1. 76125 by 25.
2. 89375 by 25.
3. 88900 by 33$\frac{1}{3}$.
4. 63850 by 33$\frac{1}{3}$.
5. 64350 by 75.
6. 47250 by 37$\frac{1}{2}$.
7. 679500 by 62$\frac{1}{2}$.
8. 460600 by 87$\frac{1}{2}$.
9. 385500 by 37$\frac{1}{2}$.
10. 832300 by 87$\frac{1}{2}$.
11. 56780 by 125.
12. 35716 by 375.
13. 87632 by 625.
14. 57650 by 875.
15. 48765 by 125.
16. 94670 by 875.
17. 1598 by 47.
18. 3216 by 67.
19. 4088 by 73.
20. 5747 by 79.
21. 3888 by 43.
22. 7163 by 59.
23. 58951 by 61.
24. 38765 by 83.
25. 17973 by 85.
26. 47381 by 87.
27. 76327 by 92.
28. 56345 by 95.
29. 87632 by 132.
30. 179630 by 241.
31. 31086 by 99.
32. 42751 by 98.
33. 87630 by 98.
34. 88561 by 97.
35. 92176 by 97.
36. 98753 by 98.
37. 147632 by 998.
38. 187632 by 999.
39. 256319 by 998.

Perform the following divisions accurately to 4 figures:

40. 1112223334 by 3567.
41. 3216789567 by 7238.
42. 5634217689 by 16328.
43. 87632765176 by 29045.

COMBINATIONS OF OPERATIONS.

84. Order of Operations.—When several operations are combined in a single process, it is often possible to perform these operations in different orders; by performing them in one order rather than another, much labor is often saved.

Ex. 1. Find the difference between the products

$$3217 \times 85 \text{ and } 3217 \times 83.$$

FIRST COMPUTATION.

```
  3217      3217
    85        83
 16085      9651
25736      25736
273445    267011
267011
  6434, Difference.
```

SECOND COMPUTATION.

```
85 - 83 = 2
3217
   2
6434, Difference.
```

The pupil will notice that the second computation requires less than one fourth the labor of the first computation, and that in the second computation as compared with the first the likelihood of error is diminished even more.

Ex. 2. Compute $\dfrac{156 \times 892}{78}$.

FIRST COMPUTATION.	SECOND COMPUTATION.
156	78)156(2
892	156
312	
1404	$892 \times 2 = 1784$, *Result.*
1248	
78)139152(1784, *Quotient.*	

In general, it is evident that if a, b, c, n be symbols denoting any numbers,

1. $a \times b \times c = a \times (b \times c) = (a \times b) \times c.$

2. $(a + b)\,c = ac + bc.$

3. $(a - b)\,c = ac - bc.$

4. $\dfrac{a \times b}{c} = \dfrac{a}{c} \times b.$ *6.* $\dfrac{a}{b} = \dfrac{a \times n}{b \times n}.$

5. $\dfrac{a + b}{c} = \dfrac{a}{c} + \dfrac{b}{c}.$ *7.* $\dfrac{a}{b} = \dfrac{a \div n}{b \div n}.$

The symbol, ( ), is called the **parenthesis.** When two or more numbers are inclosed in a parenthesis, it means that they are to be treated as a whole and subjected to the same operation.

Let the student state the above principles in general language. For instance, 7 may be expressed thus,

If the divisor and dividend be divided by the same number, the quotient is not changed.

EXERCISE 20.

1. I bought 460 acres at \$26 an acre, and sold them at \$37 an acre. Required my total gain.

2. During June a butcher took in \$37 on each of 25 days.

During July he took in \$43 on each of 25 days. What were his total receipts? What was the July excess?

3. Monday, 3126 people attended the theatre; and Tuesday, 4719 people attended. If the tickets were \$2 each, what were the total receipts? The Tuesday excess?

4. A train leaves New York for Chicago, 900 miles away, at the rate of 30 miles an hour, and at the same time another leaves Chicago at 45 miles an hour. How long before they meet? How far from Chicago do they meet? How long between the arrivals of the trains at destinations?

5. The excursion fare between two places is \$17. What does the R. R. Co. receive from the passengers in a train of 8 cars, each carrying 63 passengers?

6. From the sum of 1436, 2785, 43697, 5638, take the product of 341 and 75.

7. Divide the difference between 3467519 and 5321963 by the quotient of 38836 divided by 73.

8. Multiply the sum of 2076, 35941, 763, 9876, and 21638, by the difference between the greatest two of them.

9. When the divisor is 409, the quotient 1703, and the remainder 245, what is the dividend?

10. When the minuend is 57632051, and the remainder 14678932, what is the subtrahend?

11. When the dividend is 100, the quotient 8, and the remainder 4, what is the divisor?

12. When the dividend is 2606526, the quotient 1478, and the remainder 812, what is the divisor?

13. An officer in distributing an appropriation of \$697000 among the counties of a State, gave each county \$33190, and had \$10 remaining. How many counties were there?

85. Logarithms.—By the use of a series of auxiliary numbers called logarithms, numerical work may be further abbreviated. By the use of logarithms, multiplication is converted into addition, division into subtraction, etc. The consideration of these methods, however, comes later.

CHAPTER VII.

FACTORS AND ANALYSIS.

86. The **factors** of a number (see Art. 49) have already been defined as the numbers which, multiplied together, produce the given number.

Thus, the factors of 187 are 11 and 17; of 60 are 3, 4, and 5.

87. Illustration of the Value of a Knowledge of the Factors of Numbers.—If it is required to determine the value of

$$\frac{252 \times 240}{54 \times 35}$$

a knowledge of the factors of the given numbers enables us greatly to abbreviate the work. For, since dividing the dividend and divisor by the same number leaves the value of the quotient unchanged, we may divide 252 and 54 by the number 9, which is a factor of both, and proceed in like manner till all the factors common to both divisor and dividend are removed. Thus,

$$\frac{\overset{\overset{4}{\cancel{28}}}{\cancel{252}} \times \overset{\overset{8}{\cancel{48}}}{\cancel{240}}}{\underset{6}{\cancel{54}} \times \underset{7}{\cancel{35}}} = 4 \times 8 = 32, \textit{ Result.}$$

Similarly, an indicated quotient of two large numbers may often be reduced to a simple form by means of a knowledge of the factors of the numbers.

Thus, if we have $\frac{235}{376}$, and know that $235 = 47 \times 5$, and $376 = 47 \times 8$, we have

$$\frac{235}{376} = \frac{47 \times 5}{47 \times 8} = \frac{5}{8}.$$

These illustrations show the importance of as thorough a knowledge as possible of the factors of numbers and of the processes of determining them.

88. Prime and Composite Numbers.—A **prime number,** or **prime**, is a number which is not divisible by any number except itself and unity.

Thus, 2, 3, 17, 47, etc., are prime numbers.

A **composite number** is a number which can be divided by one or more numbers besides itself and unity.

Thus, 4, 6, 18, etc., are composite numbers.

89. Even and Odd Numbers.—As a rule, the most important possible factor of a number is 2.

An **even number** is a number exactly divisible by 2; as 2, 4, 6, etc.

An **odd number** is one not exactly divisible by 2; as 1, 3, 5, etc.

90. Powers and Exponents.—A power is the product of two or more identical factors. Thus, since

$$125 = 5 \times 5 \times 5$$

125 is said to be the *third* power of 5.

A **second power** is called a **square,** thus 25 is the square of 5; a **third power** is called a **cube,** thus 125 is the cube of 5.

An **exponent** is a small figure written above and to the right of a number to indicate how many times the number is taken as a factor (and to save the labor of writing out all the identical factors).

Thus, since $729 = 3 \times 3 \times 3 \times 3 \times 3 \times 3$, we may write $729 = 3^6$, the 6 being the exponent of 3.

Similarly, since $400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5$, we may write $400 = 2^4 \times 5^2$.

91. Short Methods of Determining Whether 2, 3, 4, 5, 8, 9, are Factors of a Given Number.—To determine whether a given number contains another number as a factor, the direct method is to divide the given number by the supposed factor. This work, however, may be greatly abbreviated in the case of 2, 3, 4, 5, 8, 9, and some other numbers. Thus, in order to determine whether 2 is a factor of a number, it is not necessary to divide the entire number by 2, but only the last or right-hand digit. Hence, we substitute the less labor of dividing the last figure of a number for the greater labor of dividing the entire number.

The following abbreviated methods of determining the factors are of especial importance:

A number is divisible,

1. By 2, if its last or right-hand figure is divisible by 2.

Thus, 765918 is divisible by 2 since 8 is divisible by 2.

2. By 4, if the number expressed by its last two digits is divisible by 4.

Thus, 742368 is divisible by 4 since 68 is.

3. By 8, if the number expressed by the last three digits is divisible by 8.

Thus, 659256 is divisible by 8 since 256 is.

4. By 3, if the sum of the digits composing the number is divisible by 3.

Thus, 8721561 is divisible by 3 since the sum of the digits of the number is 30, which is a number divisible by 3.

5. By 9, if the sum of the digits composing the number is divisible by 9.

Thus, 87219 is divisible by 9 since the sum of the digits is 27, which is a number divisible by 9.

6. By 5, when the last digit in the number is either 5 or 0.

7. By 6, when the number is divisible by both 2 and 3.

The reasons for the first three of the above tests of divisibility are similar. For, in the first case, any number may be regarded as made up of a number of tens and an additional group of units. Since 10 is divisible by 2, any number of tens is also divisible by 2; hence, if the last figure is also divisible by 2, the entire number is divisible by 2.

Similarly, in the second case, any number may be regarded as made up of a number of hundreds and an additional number composed of two digits. Since 100 is divisible by 4, any number of hundreds is divisible by 4; hence, if the additional number expressed by the last two digits is divisible by 4, the entire number is divisible by 4.

Similarly, the test for divisibility by 8 depends on the fact that 8 is an exact divisor of 1000. Let the student make a formal statement of the reasoning involved.

The reason for the test of the divisibility of a number by 3 is as follows: Any number larger than 10 may be separated into two parts, viz.: a part which is a multiple of 9 (and hence divisible by 3); and a second part which is equal to the sum of the digits of the number. Hence, if the latter part is divisible by 3, the entire number is. Thus, to test 852 as to its divisibility by 3,

$$\begin{aligned}
&\text{Since } 800 = 8\,(99 + 1) = 8 \times 99 + 8\\
&\quad\text{and } 50 = 5\,(9 + 1) = 5 \times 9 + 5\\
&\quad\text{and } 2 = 2\\
&852 = (8 \times 99 + 5 \times 9) + (8 + 5 + 2).
\end{aligned}$$

Since the multiples of 9 are divisible by 3, the divisibility of the entire number by 3 depends on whether the sum of the digits $8 + 5 + 2$ is divisible by 3.

Let the student state in like manner the reason for the abbreviated method of determining whether a number is divisible by 9.

Let the student also state the reason for the test of the divisibility of a number by 5.

The student should observe that the positional system of notation adopted in representing numbers makes possible these abbreviated tests of divisibility. Let the pupil determine which of them could be applied to numbers expressed in the Roman notation.

92. Prime Factors of a Number.—To determine the prime factors of a number, it is sufficient to *divide the given number by a prime factor (it is generally best to divide first by the smallest prime factor), then divide the quotient obtained by another prime factor, and so on till a quotient is obtained which is itself prime.*

Ex. Separate 2040 into its prime factors.

OPERATION.

$$\begin{array}{r|l} 2 & 2040 \\ 2 & 1020 \\ 2 & 510 \\ 3 & 255 \\ 5 & 85 \\ & 17 \end{array} \qquad \therefore 2040 = 2^3 \times 3 \times 5 \times 17, \textit{ Factors.}$$

EXERCISE 21.

Find the prime factors of:

1. 15, 18, 20, 25, 27, 28, 32, 36, 40, 42, 48.
2. 50, 56, 60, 64, 72, 80, 84, 88, 92, 98, 105.
3. 108, 112, 124, 128, 136, 148, 150, 165.
4. 224, 396, 480, 600, 842, 873, 919, 960.
5. 1315, 1599, 3003, 2145, 3696, 4081, 12121.

Tell by inspection whether or not each of the following numbers is divisible by 2, 3, 4, 5, 6, 8, 9, 10, or 20.

6. 120, 130, 140, 156, 171, 217, 240, 498.
7. 3428, 7653, 9345, 76532, 97605, 123456.
8. 98010, 152460, 216216, 445038, 876543210.

Determine whether the following numbers are prime or composite:

9. 81, 83, 87, 93, 111, 201, 271, 343, 427.
10. 319, 507, 533, 851, 917, 1189, 1927.
11. Which are the more numerous, odd numbers or even numbers? Prime numbers or odd numbers? Are all prime numbers odd?
12. Obtain tests for the divisibility of a large number by 12. By 15. By 18. By 36. By 40. By 32.
13. Ascertain what the sieve of Eratosthenes is, and by its use form a list of prime numbers from 1 to 500.
14. By the aid of this table determine whether 839 is a prime number. Is it necessary to divide by *all* of the prime numbers less than 1333 to determine whether it is prime? Which may be omitted?

93. Cancellation.—It has been shown that dividing both divisor and dividend by the same number does not change the quotient. So much labor is saved by this means that the process is frequently used, and it is convenient to give it a special name.

Cancellation is the operation of striking out a factor common to both divisor and dividend.

Ex. 1. Compute $\dfrac{50 \times 18 \times 84}{12 \times 3 \times 75}$ by cancellation.

OPERATION.

$$\frac{\overset{2}{\cancel{50}} \times \overset{\overset{2}{\cancel{6}}}{\cancel{18}} \times \overset{7}{\cancel{84}}}{\cancel{12} \times \cancel{3} \times \underset{\cancel{3}}{\cancel{75}}} = 28, \textit{ Quotient.}$$

EXPLANATION.

50 and 75 have the common factor, 25, which may be canceled, giving 2 in the place of 50, and 3 in the place of 75. 12 will divide 12 and 84, giving 1 and 7. 3 will divide 3 and 18, giving 1 and 6. 3 again will divide 3 and 6, giving 1 and 2. Hence, the quotient is $2 \times 2 \times 7$, or 28.

The quotient is not changed in its denomination by cancellation, but this is not the case with the *remainder*, if there be one. To obtain the true remainder, it is necessary to multiply the remainder after the cancellation, by all the factors cancelled out.

Ex. By cancellation $\dfrac{\overset{\cancel{4}}{\cancel{36}} \times \overset{14}{\cancel{42}} \times \overset{2}{\cancel{6}}}{\underset{\cancel{3}}{\cancel{12}} \times \cancel{9} \times \underset{5}{\cancel{15}}} = \dfrac{28}{5} = 5,$

with apparent remainder of 3. But the true remainder is obtained by multiplying 3 by $12 \times 9 \times 3$ (the factors cancelled out), giving 972, the remainder which should have been obtained if the division had been performed without any cancellation.

It is left as an exercise for the pupil to discover the reason of this process.

EXERCISE 22.

Reduce:

1. $\frac{6 \times 10}{20}$.

2. $\frac{24}{3 \times 2}$.

3. $\frac{8 \times 15}{20 \times 3}$.

4. $\frac{24 \times 25}{75}$.

5. $\frac{56 \times 27}{9 \times 42}$.

6. $\frac{65 \times 84}{26 \times 25}$.

7. $\frac{144 \times 125}{100 \times 12}$.

8. $\frac{88 \times 105}{20 \times 77}$.

9. $\frac{216 \times 343}{98 \times 42}$.

10. $\frac{5 \times 6 \times 20}{4 \times 15 \times 2}$.

11. $\frac{7 \times 9 \times 16}{8 \times 14 \times 3}$.

12. $\frac{17 \times 20 \times 27}{9 \times 34 \times 5}$.

13. $\frac{38 \times 40 \times 55}{44 \times 19 \times 50}$.

14. $\frac{48 \times 68 \times 77}{84 \times 187}$.

15. $\frac{60 \times 75 \times 81}{45 \times 50 \times 54}$.

16. $\frac{90 \times 96 \times 98}{168 \times 630}$.

17. $\frac{205 \times 666 \times 18}{81 \times 185 \times 84}$.

18. $\frac{360 \times 405 \times 7}{75 \times 504}$.

19. $\frac{384 \times 162 \times 275}{216 \times 200 \times 99}$.

20. Divide $16 \times 18 \times 24 \times 30$ by $45 \times 32 \times 72$.

21. Divide $60 \times 70 \times 85 \times 96$ by $42 \times 125 \times 64$.

22. Divide $128 \times 132 \times 150$ by $275 \times 48 \times 84$.

23. Divide $345 \times 396 \times 425$ by $187 \times 276 \times 375$.

Ascertain the value and the true remainder in each:

24. $\frac{15 \times 28 \times 96}{77 \times 40}$.

25. $\frac{18 \times 25 \times 126}{45 \times 28 \times 35}$.

26. $\frac{48 \times 50 \times 51}{60 \times 34 \times 16}$.

27. $\frac{58 \times 57 \times 56}{21 \times 24 \times 87}$.

28. $\frac{63 \times 64 \times 198}{42 \times 99 \times 72}$.

29. $\frac{95 \times 96 \times 98}{343 \times 38 \times 36}$.

ANALYSIS.

94. Units Used for Computation Purposes.—Besides units in general use, such as \$1, 1 yard, etc., certain special units are often employed in solving particular examples, simply as an aid in computation.

95. Analysis is the solution of problems by the aid of special units devised to aid in the computation.

Ordinarily we have given in the problem the value of the unit when taken a given number of times. The process of analysis consists (1) in determining the value of the unit taken once, and (2) the value of the unit when taken a required number of times. These two steps are called reasoning *to the unit* and *from the unit*.

Ex. 1. If 6 horses cost \$420, what will 15 horses cost?

Analysis.—The unit considered in this problem is the cost of 1 horse. Thus,

$$\$420 = \text{cost of 6 horses (6 units).}$$

$$\frac{\$420}{6} = \text{cost of 1 horse (1 unit).}$$

$$\frac{\$420 \times 15}{6} = \text{cost of 15 horses (15 units).}$$

$$\frac{\overset{70}{\cancel{\$420}} \times 15}{\cancel{6}} = \$1050, \text{ cost of 15 horses.}$$

It should be noticed that ordinarily it is of advantage merely to indicate the division which gives the value of the single unit, and not to obtain the quotient itself by actual division (thus we write $\frac{\$420}{6}$ and not \$70 as the cost of 1 horse), in order to take advantage of possible cancellations in the final computation.

Though the above statement of the analysis is all that the student need write down as the solution of the problem, he should be able to give clearly and exactly the reasoning used. Thus, in the above example, if \$420 is the cost of 6 horses, 1 horse will cost as many dollars as 6 is contained times in \$420, or $\frac{\$420}{6}$; and if 1 horse costs $\frac{\$420}{6}$, 15 horses will cost 15 times $\frac{\$420}{6}$ or \$1050.

Ex. 2. If 9 books cost \$20, what will 54 books cost?

The unit is the cost of 1 book.

$\$20 =$ cost of 9 books.

$\frac{\$20}{9} =$ cost of 1 book.

$\frac{\$20 \times 54}{9} =$ cost of 54 books.

$\frac{\$20 \times \overset{6}{\cancel{54}}}{\cancel{9}} = \$120$, *Result.*

Two or more steps are often necessary in obtaining the value of the computation unit (*i. e.*, in reasoning to the unit) and also at times in reasoning from the unit. Thus:

Ex. 3. A workman received \$21 for 15 days' work of 7 hours each. How many dollars will he receive for 17 days' work of 10 hours each?

The unit which controls the computation is the number of dollars received for 1 hour's work.

Hence,

$7 =$ No. hours in 1 day's work.

$15 \times 7 =$ No. hours in 15 days' work.

$\$21 =$ wages for 15×7 hours' work.

$\frac{\$21}{15 \times 7} =$ wages for 1 hour's work (the unit).

$\frac{\$21 \times 10}{15 \times 7} =$ wages for 10 hours' work.

$\frac{\$21 \times 10 \times 17}{15 \times 7} =$ wages for 17 days' work of 10 hours each.

$$\frac{\overset{3}{\cancel{\$21}} \times \overset{2}{\cancel{10}} \times 17}{\underset{3}{\cancel{15}} \times \cancel{7}} = \$34, \textit{ Result.}$$

After some practice in working similar examples, the above statement may be conveniently abbreviated as follows:

$\frac{\$21}{15 \times 7} =$ wages for 1 hour's work.

$\frac{\$21 \times 10 \times 17}{15 \times 7} =$ wages for 10×17 hours' work.

The unit of computation in a problem is often given explicitly, it being required to determine the number of times the unit is used.

Ex. 4. How many pounds of sugar at 6 cents a pound can be obtained in exchange for 10 dozen eggs at 21 cents a dozen?

The cost of 1 pound of sugar is the unit which controls the computation.

21 cents $=$ value 1 dozen eggs.

21 cents $\times$ 10 $=$ value 10 dozen eggs.

6 cents $=$ value 1 pound sugar.

$\frac{21 \text{ cents} \times 10}{6 \text{ cents}} =$ No. pounds sugar at 6 cents a pound which can be obtained for 21×10 cents.

$$\frac{\overset{7}{\cancel{21}} \times \overset{5}{\cancel{10}}}{\underset{2}{\cancel{6}}} = 35, \textit{ No. of pounds.}$$

Ex. 5. A milkman has 20 cows, each of which gives 8 quarts of milk daily. He sells the milk for 6 cents a quart. How many pieces of cloth, each containing 40 yards and costing 15 cents a yard, can he obtain for the milk of 10 days?

6 cents $\times 8 \times 20 \times 10 =$ value of milk for 10 days.

15 cents $\times 40 =$ cost of 1 piece of cloth.

$\frac{6 \text{ cents} \times 8 \times 20 \times 10}{15 \text{ cents} \times 40} =$ No. of pieces of cloth received for 6 cents $\times 8 \times 20 \times 10$.

$$\frac{\overset{2}{\cancel{6}} \times 8 \times \overset{5}{\cancel{20}} \times \cancel{10}}{\underset{3}{\cancel{15}} \times \underset{4}{\cancel{40}}} = 16, \textit{ No. of pieces.}$$

Ex. 6. 12 men working 8 hours a day do a piece of work in 15 days. How many days will it take 8 men working 10 hours a day?

The computation unit is the work done by one man in 1 hour; then

$12 \times 8 \times 15 =$ No. units work done by 12 men in 15 days of 8 hours each.

$8 \times 10 =$ No. units work done by 8 men in 1 day of 10 hours.

$\frac{12 \times 8 \times 15}{8 \times 10} =$ No. days it will take 8 men working 10 hours a day to do $12 \times 8 \times 15$ hours' work for 1 man.

$$\frac{\overset{6}{\cancel{12}} \times \cancel{8} \times \overset{3}{\cancel{15}}}{\cancel{8} \times \underset{2}{\cancel{10}}} = 18, \textit{ No. of days.}$$

EXERCISE 23.

1. If 6 stamps cost 30 cents, what will 14 stamps cost? 45 stamps?

2. If 9 pads cost 72 cents, what will 7 pads cost?

3. If 12 pounds of candy cost 216 cents, how much will 21 pounds cost?

4. When $415 will buy 5 acres of land, how many dollars are 17 acres worth?

5. What will 25 cattle cost when 7 cattle are worth $161?

6. If a bar of iron 12 feet long weighs 192 pounds, how much will a similar bar 19 feet long weigh?

7. If a stock of 45 chairs is worth $279, what is another stock of 55 similar chairs worth?

8. If a class of 74 men weigh together 11766 pounds, about what will a similar class of 111 men weigh?

9. A family of 7 drink 17 quarts of water each day. How many quarts will a town of 14000 people drink?

10. If 16 people use $20 worth of meat each week, how much will 50 people use in a year of 52 weeks?

11. A workman receives $6 for working 5 days of 9 hours each. How many dollars should he receive for the labor of 15 days of 10 hours each?

12. If a laborer receives $25 for the work of 15 days of 7 hours, how many dollars will be paid him after 28 days' work of 9 hours each?

13. If $264 are paid 11 men for the labor of 16 days, each 10 hours, how much should be paid 40 men for 25 days' labor, each day of 6 hours?

14. It cost $290 to print and bind 75 books, of 812 pages each. What will be the cost of printing and binding 81 books, of 560 pages each, at the same rate?

15. If a force of 63 men can do a certain task in 7 days, of 8 hours each, how many men, working 6 hours a day, will be needed to do a similar task in 12 days?

16. For the construction of a certain wall, 8 rods long, 36 men were required, working 10 hours each, of 24 days. How many days, of 8 hours each, will it take 55 men to construct a like wall, 22 rods long?

17. On the erection of a wall, 75 feet long, 6 feet wide, and 8 feet high, 30 men worked 17 days, of 8 hours each. How long a wall, 4 feet wide and 7 feet high, can a force of 34 men build in 40 days, of 7 hours each?

18. How many pounds of rice, at 8 cents a pound, can be bought for 12 pounds of butter, at 20 cents a pound?

19. A merchant exchanges 45 yards of cloth, worth $2 a yard, for silk, worth $5 a yard. How many yards of silk does he receive?

20. 12 casks of vinegar, each containing 16 gallons, and worth 10 cents a gallon, are given in exchange for potatoes, worth 60 cents a bushel. How many bushels of potatoes are received?

21. How many firkins of butter, each containing 50 pounds, worth 23 cents a pound, will be returned for 115 bales of hay, at 90 cents a bale?

22. A farmer sells the wool from 60 sheep, at 13 cents a pound, each fleece weighing 4 pounds. How many rolls of matting, at 52 cents a yard, can he buy with the money, if each roll contains 15 yards?

23. Each line of a book, of 150 pages, contains 12 words, and there are 30 lines on a page. If the printing costs 3 cents a word, how many bales of paper, each containing 10 bundles, of 20 quires each, and worth 18 cents a quire, can be bought with the proceeds of printing the book?

CHAPTER VIII.

GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE.

GREATEST COMMON DIVISOR.

96. Common Factors.—In order to simplify work as much as possible (as for instance in cancellation), we need to know not only *a* number that will divide each of two given numbers, but also the *largest* number that will divide both of them, and to have methods of determining this number which will cover all cases.

Thus, in order to simplify the indicated quotient $\frac{1128}{705}$, we need to know that the largest number that will divide both dividend and divisor is 141, or, if we do not know it, to have some way of determining it.

97. The **greatest common divisor** (or G. C. D.) of two or more numbers is the largest number that will divide them all.

It is also sometimes called the highest common factor, or H. C. F.

Thus, \$12 is the G. C. D. of \$24, \$36, \$60.

The G. C. D. may also be described as the largest unit group which can be used to measure all of a set of numbers.

When two numbers have no common factor except unity, the numbers are said to be *prime to each other;* thus, 8 and 15 are prime to each other, though neither of them is itself a prime number.

98. Aids in Finding the G. C. D.—It is helpful in finding the G. C. D. of two or more numbers to understand that

First, if a number be a factor of each of two or more numbers, it must be a factor of their G. C. D.

Thus, since 5 is a factor of 30, 75, and 90, it must be a factor of their G. C. D.

This principle enables us to separate the process of finding the G. C. D. of two or more numbers into several often comparatively simple steps of finding the prime factors common to all of the numbers, and a last step of taking the product of these common prime factors.

Second, if a number be a factor of two numbers, it must be a factor of the sum or difference of any multiples of these numbers.

This principle is illustrated if, for instance, we tie toothpicks into bundles, of 12 each, and have 8 bundles in one heap and 5 bundles in another heap. 12 toothpicks will evidently be a divisor of each entire heap, or of the sum of the two heaps, or of their differences, or of the sum or difference of any multiples of such heaps (since such a sum or difference will be composed entirely of bundles containing 12 each).

Thus, again, if 9 is a factor of both 333 and 855, it is a factor of their sum or difference, or of $855 - 2 \times 333$; that is, of the number 189, smaller than either of the original numbers.

This principle enables us to simplify the work of finding the G. C. D. of two large numbers, by using smaller and smaller numbers, obtained by successive subtractions of multiples of a smaller number from a larger. See Art. 101.

99. I. Short Division Method of Finding the G. C. D.—If the numbers whose G. C. D. is sought, be small, the most convenient method of proceeding is to *arrange the given numbers in a row, and divide by any number that will divide all the given numbers; similarly divide the quotients obtained till there is no number which will divide all the quotients; the product of all the divisors will be the G. C. D.*

Ex. Find the G. C. D. of 84, 126, 210.

OPERATION.

2)84, 126, 210
3)42, 63, 105
7)14, 21, 35
2, 3, 5

Hence, $2 \times 3 \times 7 = 42 =$ G. C. D.

If the pupil be already thoroughly acquainted with the prime factors of the numbers whose G. C. D. is sought, it is sometimes convenient to separate each of the given numbers into its prime factors, and multiply together the prime factors that are common to all the numbers.

Ex. Find the G. C. D. of 24, 72, 120.

OPERATION.

$$24 = 2^3 \times 3$$
$$72 = 2^3 \times 3^2$$
$$120 = 2^3 \times 3 \times 5$$
$$\therefore 2^3 \times 3 = 24 = \text{G. C. D.}$$

EXERCISE 24.

Find the G. C. D. of the following groups of numbers:

1. 18, 42.
2. 24, 60.
3. 60, 105.
4. 90, 198.
5. 112, 256.
6. 168, 273.
7. 216, 384.
8. 630, 924.
9. 30, 42, 72.
10. 32, 56, 88.
11. 36, 84, 180.
12. 180, 144, 198.
13. 168, 288, 216.
14. 135, 378, 324.
15. 210, 300, 450.
16. 192, 288, 416.
17. 144, 252, 400.
18. 432, 540, 918.
19. 24, 40, 56, 104.
20. 42, 98, 112, 140.
21. 40, 70, 90, 150.
22. 72, 120, 144, 264.
23. 108, 162, 198, 270.
24. 288, 480, 720, 336.

100. II. Long Division Method of Finding G. C. D.— When the numbers whose G. C. D. is sought are large, it is best to proceed by the method indicated in a general way in Art. 98. For, by the aid of this principle, it can be shown that the G. C. D. of any two numbers, taken as *divisor* and *dividend*, is the same as the G. C. D. of the *divisor* and *remainder*.

For since, denoting the quotient by m,

we have,
$$\textit{Divisor})\,\textit{Dividend}(m$$
$$\underline{m \times \textit{Divisor},}$$
$$\textit{Remainder},$$

and hence
$$\textit{Remainder} = \textit{Dividend} - m \times \textit{Divisor},$$

and also,
$$\textit{Dividend} = m \times \textit{Divisor} + \textit{Remainder}.$$

Now, every number that will divide both divisor and dividend exactly must also divide the remainder exactly (by Art. 98, Second Principle, since the

remainder is the difference between the dividend and a multiple of the divisor). Hence, all the common factors of both divisor and dividend are also common factors of the divisor and remainder.

Conversely, whatever number will divide the divisor and remainder exactly is also a factor of the dividend (by Art. 98, since the dividend is the sum of the remainder and a multiple of the divisor). Hence, all the common factors of the divisor and remainder are also common factors of the divisor and dividend.

Hence, every common factor of the one pair of numbers is a common factor of the other pair also; hence, the two pairs have the same G. C. D. Hence, we may substitute the smaller pair, the divisor and remainder, for the larger pair, the divisor and dividend, and by successive uses of this principle, finally determine the G. C. D.

Ex. Find the G. C. D. of 841, 1740.

OPERATION.

```
841)1740(2
    1682
      58)841(14
         58
         261
         232
          29)58(2
             58
G. C. D. = 29
```

EXPLANATION.

Dividing 841 into 1740, we obtain 2 for a quotient and 58 for a remainder. But by the principle proved above, the G. C. D. of 841 and 58 is the same as the G. C. D. of 841, 1740. Proceeding in like manner, the G. C. D. of 29 and 58 is the same as the G. C. D. of the original pair of numbers, 841 and 1740.

By the use of symbols, the proof given above that the G. C. D. of the divisor and remainder is the same as the G. C. D. of the divisor and dividend, may be put in an abbreviated form thus:

Denote the smaller of two numbers (the divisor) by A, the larger (the dividend) by B, the quotient by m, the remainder by R. We have

$$\begin{array}{lll} \text{A})\,\text{B}(\text{m} & & \\ \underline{\text{m A}} & \text{or} & \text{R} = \text{B} - \text{m} \times \text{A} \\ \text{R} & \text{also} & \text{B} = \text{R} + \text{m} \times \text{A} \end{array}$$

Then every factor of the numbers A and B is also a factor of R (by Art. 98, since R = B − m × A); and, hence, is a factor of the pair of numbers A and R.

Conversely, every factor of the pair of numbers A and R is also a factor of B (Art. 98, since B = R + m × A), and, hence, is a factor of the pair of numbers A and B.

Hence, every factor of the one pair of numbers is a factor of the other pair ∴ the G. C. D. of A and B = G. C. D. of A and R.

Hence, to find the G. C. D. of two numbers, *divide the less number into the greater, the remainder into the divisor, and thus continue until there is no remainder; the last divisor will be the G. C. D. of the two original numbers.*

To find the G. C. D. of three or more large numbers, first find the G. C. D. of two of the numbers by the above method, then obtain the G. C. D. of this result and a third number, and so on till all the numbers have been used. The last G. C. D. obtained is the G. C. D. of all the original numbers.

EXERCISE 25.

Find the G. C. D. of:

1. 55, 75.	*6.* 189, 261.	*11.* 390, 675.
2. 68, 92.	*7.* 176, 275.	*12.* 882, 903.
3. 90, 138.	*8.* 252, 480.	*13.* 918, 675.
4. 96, 152.	*9.* 187, 510.	*14.* 1457, 899.
5. 126, 153.	*10.* 182, 533.	*15.* 2736, 4389.

16. 182, 196, 357.	*21.* 270, 315, 735.
17. 209, 198, 473.	*22.* 546, 455, 702.
18. 272, 400, 816.	*23.* 1584, 2772, 3276.
19. 782, 969, 1156.	*24.* 2088, 2349, 3016.
20. 216, 360, 280.	*25.* 3330, 2035, 3663.

LEAST COMMON MULTIPLE.

101. A **common multiple** of two or more numbers is a number which is exactly divisible by all of them.

Thus, \$600 is a common multiple of \$15, \$20, and \$30.

The **least common multiple** (or L. C. M.) of two or more numbers is the least number which is divisible by them all.

Thus, \$60 is the L. C. M. of \$15, \$20, \$30.

The most useful application of the L. C. M. is in determining the least common multiple of the denominators of a set of fractions. This enables us to determine the largest unit which will measure each of a set of fractional quantities, just as the G. C. D. enables us to determine the largest unit which will measure a set of integral quantities.

102. I. Short Division Method of Determining the L. C. M. of Several Numbers.—If the numbers whose L. C. M. is desired, are small, the most convenient method of proceeding in order to determine their L. C. M. is to *arrange the given numbers in a row; divide by any prime factor that will divide at least two of them, bringing down each undivided number along with the quotients; continue the process till the quotients are all prime to each other; the L. C. M. will be the product of all the divisors and final quotients.*

If any one of the numbers is contained exactly in ($\therefore$ is a factor of) any other of the given numbers, it may be struck out. For, in finding the L. C. M. of the larger number, we find that of the smaller number also.

Ex. 1. Find the L. C. M. of 12, 21, 30, 36, 63, 70.

OPERATION.

$$\begin{array}{r|l} 2 & \cancel{12},\ \cancel{21},\ 30,\ 36,\ 63,\ 70 \\ \hline 3 & 15,\ 18,\ 63,\ 35 \\ \hline 3 & \cancel{5},\ 6,\ 21,\ 35 \\ \hline & 2,\ \cancel{7},\ 35 \end{array}$$

$$2 \times 3 \times 3 \times 2 \times 35 = 1260, \text{ L. C. M.}$$

EXPLANATION.—12 is contained in 36, and 21 in 63; hence, they are struck out, and the L. C. M. of the remaining numbers is found. (Similarly 5 and 7 are struck out in the course of the process.) By dividing by the prime factors 2, 3, 3, and multiplying them and the final quotients together, each prime factor will occur in the final product the highest number of times it occurs in any one number; hence, the product thus obtained will be the L. C. M.

If the prime factors of the given numbers are well known, it is sometimes more convenient to separate each of the given numbers into its prime factors, and take the product of all the different factors, using each factor the greatest number of times it occurs in any single number.

Ex. 2. Find the L. C. M. of 48, 72, 120.

$$48 = 2^4 \times 3.$$
$$72 = 2^3 \times 3^2.$$
$$120 = 2^3 \times 3 \times 5.$$
$$\therefore \text{L. C. M.} = 2^4 \times 3^2 \times 5 = 720.$$

EXERCISE 26.

Find the L. C. M. of:

1. 8, 12.
2. 9, 15.
3. 10, 25.
4. 18, 30.
5. 24, 42.
6. 28, 70.
7. 6, 10, 15.
8. 12, 16, 36.
9. 18, 45, 70.
10. 24, 45, 40.
11. 32, 81, 72.
12. 48, 84, 210.
13. 105, 168, 120.
14. 280, 144, 216.
15. 180, 189, 315.
16. 210, 231, 330.
17. 182, 286, 308.
18. 385, 420, 660.
19. 12, 20, 36, 54.
20. 22, 44, 88, 108.
21. 12, 24, 63, 84.
22. 8, 15, 18, 120.
23. 10, 25, 40, 75.
24. 7, 16, 77, 132.
25. 20, 24, 25, 36, 45.
26. 30, 36, 35, 56, 80.
27. 33, 42, 66, 70, 84.
28. 45, 54, 65, 91, 63.
29. 63, 75, 81, 98, 105.
30. 85, 102, 105, 110, 165.

103. II. Long Division Method of Determining the L. C. M. of two or more Numbers.—When it is required to find the L. C. M. of two large numbers, which cannot be readily factored, it is best to proceed by first finding their G. C. D. by the long division method.

Ex. Find the L. C. M. of 841 and 1740.

We first find the G. C. D. of the numbers, thus:

```
841)1740(2
    1682
      58)841(14
         58
         261
         232
          29)58(2        ∴ G. C. D. = 29
             58
```

Dividing each of the given numbers by their G. C. D., we have

```
29)841(29                    29)1740(60
   58                            174
   261                             0
   261
∴ 841 = 29 × 29              ∴ 1740 = 29 × 60
```


Hence, to find the L. C. M., we may proceed as in Art. 102:

$$29\underline{)29\times 29,\ 29\times 60}$$
$$29,\quad 60$$

$$\therefore \text{L. C. M.} = 29\times 29\times 60 = 29\times 1740 = 50460.$$

Hence, in general, to find the L. C. M. of two large numbers, *find first the G. C. D. of the given numbers; divide one of the given numbers by the G. C. D. and multiply the quotient by the other number; the product will be the L. C. M. of the two numbers.*

To find the L. C. M. of three or more large numbers, first find the L. C. M. of two of the given numbers, then the L. C. M. of this result and another of the given numbers, and so on till all of the given numbers have been used. The last L. C. M. obtained is the L. C. M. of all the given numbers.

EXERCISE 27.

Find the L. C. M. of:

1. 264, 319.
2. 320, 408.
3. 506, 308.
4. 390, 525.
5. 450, 648.
6. 832, 650.
7. 252, 329, 357.
8. 288, 405, 477.
9. 456, 684, 720.
10. 280, 448, 640.
11. 396, 495, 660.
12. 945, 810, 1260.

EXERCISE 28.

1. Find the G. C. D. of 72, 96, 132.
2. Find the L. C. M. of 60, 75, 90.
3. Find the G. C. D. of 672 and 526.
4. Find the L. C. M. of 12, 15, 25, 28, 35.
5. Find the G. C. D. of 782, 867, 969.
6. Find the L. C. M. of 1066 and 962.

Find the G. C. D. and the L. C. M. of:

7. 18, 42, 54, 96.
8. 24, 40, 120, 160.
9. 84, 210, 378.
10. 1008, 1365.
11. 195, 510, 468.
12. 406, 945, 980.

13. What is the greatest width of carpet that will exactly fit three rooms of widths 15 feet, 24 feet, and 33 feet respectively?

14. A merchant having 54 yards of one kind of cloth, 84 yards of another, and 132 yards of a third, wishes to cut them into patterns of equal length. What is the greatest possible length of each pattern?

15. With a 4-quart, a 5-quart, and a 6-quart vessel, what is the size of the smallest can which may be filled exactly by each?

16. Find the length of the shortest line that can be measured exactly by rods of lengths 6 feet, 8 feet, 10 feet, and 12 feet.

17. What is the length of the longest rod which will exactly measure 209 feet, 242 feet, and 341 feet?

18. A farm produces 442 bushels of oats, 728 bushels of corn, and 585 bushels of wheat. The grain is removed in equal cases and all are full. What is the greatest capacity of each case, provided there is no mixing of the grains?

19. How can the L. C. M. of two numbers, which are prime to each other, be found? Of two prime numbers?

20. How many common multiples may 2 or more numbers have?

21. Find the difference between the G. C. D. of 480 and 520, and the L. C. M. of 5, 6, 15, 20.

CHAPTER IX.

COMMON FRACTIONS.

104. Derived Units.—A certain unit, as one pound, having been chosen for the purpose of weighing objects in general, it is often convenient to obtain from this primary unit (one pound) other derived units to be used for weighing special classes of objects. Thus, one ton (or 2000 pounds) is used in weighing objects of small value in proportion to their bulk, such as hay, coal, etc., and one ounce is used in weighing objects of great value in proportion to their bulk, as spices, gold, drugs, etc. Similarly, from any primary unit, derived units may be obtained adapted to special uses.

When the derived unit is an exact part of the primary unit, it is termed a *fractional unit* or *fraction*.

105. Fractional Units.—Thus, for measuring long distances, the *mile* is the convenient unit; but for many purposes, as, for instance, in running races in athletic games, it is convenient to divide the unit into 4 equal parts, and call one of them one-fourth of a mile; similarly we form other fractional units from the miles, as one-eighth, one-half, one-sixteenth of a mile, etc. These are all fractional units or fractions, and are expressed by writing the number of parts into which the given unit is divided under the figure 1. Thus, $\frac{1}{8}$ of a mile means one-eighth of a mile.

If a given fractional unit be taken any number of times, the result is still a fraction, and is denoted by writing the number of times the unit is taken above the line instead of 1. Thus, "$\frac{5}{8}$ mile" is an abbreviation for "5 units" of the value $\frac{1}{8}$ mile.

Sometimes a fractional unit receives a special name, as when one-twelfth of a foot is called an "inch"; or it may be made into a physical object, as

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Sometimes a fractional unit receives a special name, as when one-twelfth of a foot is called an "inch"; or it may be made into a physical object, as when a quarter of a dollar is coined and is known as a "quarter." But the great majority of fractions have no name beyond their numerical one, and many of them are used merely as aids in computations, or in mental estimates and comparisons, and have no physical existence.

Hence, the advantages in the use of fractional units lie in the ease with which such units can be devised for any purpose, temporary or permanent; the unlimited number of such units that can be formed; and the fact that when conceptions of their value as compared with the primary unit have been once formed, these conceptions can be used in connection with a set of similar fractions constructed from any other unit. Thus, having formed ideas of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{16}$, etc., of 1 inch, and of properties of these fractional units, this knowledge can be used at once in connection with similar fractions of any other primary unit, as 1 apple. This could not be done readily if each derived unit were denoted by a special name rather than in the above numerical, fractional way.

106. Fractions as Indicated Divisions.—Fractions may also be regarded as indicated divisions. It was found in Art. 84 that when a process consists of a number of multiplications and divisions, it is usually best not to perform any of the operations till all of them can be considered together and all possible cancellations made.

When the quotient of one number divided by another is indicated by writing the dividend above a line, and the divisor below, the indicated quotient is termed a *fraction* or ratio.

Hence, a **fraction** may be defined as

(1) *One or more of the exact parts of a unit*, or

(2) *the indicated quotient of one number divided by another.*

These two ways of regarding fractions are aspects of the same idea, the one or the other aspect to be used as advantage may dictate.

In investigating the properties of fractions, we will adopt one or the other point of view, as is most advantageous. When a property of fractions is obtained from one point of view, it will be left as an exercise to the student to show that the same property is true for fractions from the other point of view.

Let the student draw a line 3 inches long and divide it into 8 equal parts. Each part will be $\frac{3 \text{ inches}}{8}$ long. Let him also divide each of the three inches into eighths and take one-eighth from each inch. He will have 3 times $\frac{1}{8}$ inch, or $\frac{3}{8}$ inch. It will then be easy for him to see that $\frac{3 \text{ inches}}{8} = \frac{3}{8}$ inch.

107. Denominator and Numerator.—In a fraction, the *denominator* is the number below the line, the *numerator* is the number above the line.

The denominator denotes the number of equal parts into which a unit is divided; the numerator denotes the number of equal parts which are taken. Thus, $\frac{5}{8}$ inch denotes that an inch is divided into 8 equal parts, and that 5 of these parts are taken.

Hence, the denominator determines the size of the fractional units; the numerator determines the number of them.

The denominator and numerator taken together are called the **terms** of a fraction.

108. Proper and Improper Fractions.—A **proper fraction** is one whose numerator is less than its denominator, as $\frac{1}{3}$ or $\frac{12}{13}$.

An **improper fraction** is one whose numerator is equal to or greater than its denominator, as $\frac{8}{8}$ or $\frac{37}{25}$.

109. Integers and Mixed Numbers.—An **integer** is a number of entire units, as 5 dollars, 18.

An integer may be expressed in the form of a fraction by writing 1 under the integer as a denominator.

A **mixed number** is a number which is partly integral, partly fractional, as $\$4\frac{3}{4}$.

Thus, a mixed number consists of two different kinds of units, one integral or entire, the other fractional.

110. Simple, Compound, and Complex Fractions.—A **simple fraction** is a fraction, both of whose terms are integers, as $\frac{7}{11}$.

A **compound fraction** is a fraction of a fraction, as $\frac{2}{3}$ of \$$\frac{3}{4}$.

A **complex fraction** is one having a fraction in its numerator or in its denominator or in both. Ex. $\dfrac{2\frac{1}{3}}{\frac{5}{6}}$, $\dfrac{7}{\frac{3}{4}}$

When fractions are classified as proper or improper, they are classified as to their *value* (as greater or less than unity). When they are classified as simple, compound, or complex, they are classified as to their *form* (that is, as to the combination of operations in them).

111. Notation and Numeration of Fractions.—The preceding statements explain sufficiently the method of reading a given fraction expressed in figures, and also the method of expressing in figures, a fraction given in words. Let the student write out a formal rule for each of these processes.

112. Fundamental Properties of Fractions.—In order to use fractions with facility for various purposes, it is often desirable to transform them in different ways. Thus, for instance, it may be desirable to change the size of the fractional unit, without changing the value of the fraction.

Hence, we have the following first properties of fractions.

A. *If the numerator and denominator of a fraction be both multiplied, or both divided, by the same number, the value of the fraction is not changed.*

Thus, $\frac{6}{8}$ inch $= \frac{3}{4}$ inch $= \frac{12}{16}$ inch.

This is a mere restatement of the principle used in canceling out a factor common to both divisor and dividend.

It will aid the pupil in the present application of this principle to draw a line 6 inches in length, and to mark it off in fourths, eighths, and sixteenths of an inch, and then observe that six eighths, three fourths, and twelve sixteenths are exactly equivalent in length.

B. *Multiplying the denominator of a fraction by a given number divides the value of the fraction by that number.*

Thus, if we have \$$\frac{5}{4}$ and multiply the denominator by 3, we have \$$\frac{5}{12}$, the value of which is one-third the value of the original fraction.

For, multiplying the denominator of a fraction by a number increases the number of parts into which the original unit is divided, and hence diminishes the size of each fractional unit correspondingly.

Let the pupil show by drawing a line and subdividing it, that $\frac{5}{2}$ inch is four times as long as $\frac{5}{8}$ inch.

C. *Dividing the denominator of a fraction by a given number multiplies the value of the fraction by the same number.*

Thus, if we have the fraction \$$\frac{5}{4}$ and divide the denominator by 2, the fraction becomes \$$\frac{5}{2}$, the value of which is twice as large as the value of the original fraction.

For, dividing the denominator of a fraction by a number diminishes the number of parts into which the unit is divided, and hence increases the size of each fractional unit correspondingly.

Let the pupil show by drawing a line, and subdividing it, that $\frac{7}{16}$ of an inch is one-fourth of $\frac{7}{4}$ of an inch.

The following questions suggest two other first properties of a fraction which the student may state and prove.

If the numerator of a fraction be multiplied by a given number, what change is made in the value of the fraction?

If the numerator of a fraction be divided by a given number, what change is made in the value of the fraction?

These first properties of a fraction may all be combined as a single general principle, thus:

Multiplying or dividing the **numerator** *of a fraction by a number makes the* **same** *change in the value of the fraction that it makes in the value of the numerator; but multiplying or dividing the* **denominator** *of a fraction makes an* **opposite** *change in the value of the fraction from that which it makes in the value of the denominator.*

EXERCISE 29.

Name the kind of fraction in each case and read the following fractions:

1. $\frac{3}{4}$, $\frac{7}{10}$, $\frac{5}{8}$, $\frac{14}{9}$, $\frac{27}{19}$, $5\frac{1}{2}$, $\frac{2}{3}$ of $\frac{1}{5}$, $\frac{4}{5}$ of $2\frac{1}{6}$.
2. $\frac{5}{5}$, $\frac{17}{18}$, $\frac{18}{17}$, $1\frac{11}{23}$, $\frac{112}{345}$, $\frac{227}{891}$, $\frac{13}{5}$ of $7\frac{1}{3}$, $\frac{111}{382}$.
3. $\frac{5\frac{1}{3}}{7}$, $\frac{19}{14\frac{2}{3}}$, $\frac{2\frac{1}{4}}{7\frac{1}{5}}$, $\frac{\frac{1}{3}}{\frac{7}{8}}$, $\frac{3\frac{1}{3}}{8\frac{1}{2}}$, $\frac{1\frac{2}{3}}{\frac{7}{11}}$, $\frac{4\frac{2}{3}}{2\frac{1}{2}}$.

Write the following fractions:

4. Three-fifths; six-tenths; ten-thirds.
5. Nine-seventeenths; eleven-fortieths.
6. Eight and one-third; six and a half.
7. Ten and five-sixths; seventeen-hundredths.
8. One and a fourth over six-sevenths.
9. Two-thirds of four and three-elevenths.
10. Thirty-one and one-ninth over fourteen-fiftieths.
11. What fractions of an inch are commonly used by a carpenter?
12. What fractions of a yard are commonly used by a storekeeper?
13. What fractions of a pound are used by a grocer?

EXERCISE 30.

ORAL.

1. How many thirds in a yard? In 4 yards?
2. How many fifths in 2 yards? In 10 yards?
3. How many eighths in 6 miles? In half a mile?
4. How many twelfths in 7 years? In half a year?
5. How many sixteenths in 5 inches? In a quarter-inch?
6. Express $\frac{3}{4}$ of a dollar as eighths of a dollar.
7. Express $\frac{5}{6}$ of a yard as eighteenths of a yard.
8. Express $\frac{2}{3}$ of a year as twelfths of a year.
9. Multiply $\$\frac{4}{5}$ by 3; $\$\frac{7}{8}$ by 5; $\frac{1}{3}$ yard by 7.
10. Divide $\$\frac{14}{16}$ by 3; $\$\frac{8}{9}$ by 4; $\frac{6}{13}$ by 3.
11. Multiply $\frac{5}{8}$ by 4 in two ways; also $\frac{11}{15}$ by 5.
12. Divide $\frac{2}{3}$ by 2; $\frac{5}{8}$ by 3; $\frac{4}{5}$ by 7.
13. Multiply $\frac{3}{4}$ by 5; $\frac{23}{36}$ by 18; $\frac{14}{15}$ by 45.

TRANSFORMATIONS OF FRACTIONS.

113. I. To Reduce a Mixed Number to an Improper Fraction.—A mixed number is a number expressed by

means of two units, one integral, the other fractional. Thus, $\$7\frac{3}{4}$ expresses 7 units of \$1 each, and 3 units of $\$\frac{1}{4}$ each.

It is often convenient to express such a number in terms of the fractional unit alone. In the above example, this would be done by expressing \$7 as fourths of a dollar, or $\$\frac{28}{4}$, and adding the 3 fourths to the 28 fourths, giving $\$\frac{31}{4}$ as equivalent to $\$7\frac{3}{4}$.

In general, to reduce a mixed number to an improper fraction, *multiply the whole number by the denominator of the fraction, add the numerator to the product, and place the sum over the denominator.*

Ex. Reduce $23\frac{5}{7}$ to an improper fraction.

SOLUTION.

$$23\frac{5}{7} = \frac{161}{7} + \frac{5}{7} = \frac{166}{7}, \textit{ Result.}$$

EXERCISE 31.

Reduce the following mixed numbers to equivalent improper fractions:

1. $4\frac{1}{4}$ in.	9. $24\frac{4}{9}$.	17. $\$58\frac{3}{4}$.	25. $101\frac{23}{26}$.
2. $7\frac{2}{3}$ ft.	10. $25\frac{3}{13}$.	18. $\$65\frac{7}{8}$.	26. $400\frac{5}{48}$.
3. $5\frac{4}{7}$ mi.	11. $45\frac{1}{8}$.	19. $\$73\frac{1}{16}$.	27. $307\frac{1}{17}$.
4. $7\frac{5}{9}$ rds.	12. $17\frac{14}{15}$.	20. $270\frac{3}{5}$.	28. $11\frac{175}{347}$.
5. $8\frac{3}{16}$ yds.	13. $121\frac{1}{2}$.	21. $350\frac{10}{11}$.	29. $18\frac{42}{75}$.
6. $10\frac{7}{11}$ gal.	14. $235\frac{1}{4}$.	22. $700\frac{13}{15}$.	30. $19\frac{99}{100}$.
7. $9\frac{11}{12}$ qts.	15. $310\frac{4}{5}$.	23. $309\frac{17}{21}$.	31. $28\frac{417}{563}$.
8. $16\frac{4}{5}$ yr.	16. $108\frac{5}{12}$.	24. $111\frac{19}{20}$.	32. $43\frac{101}{222}$.

33. What are the two units of measurement in each example? What is the unit of the result?

114. II. To Reduce an Improper Fraction to a Mixed Number.—It is often desirable to reverse the process of the preceding article and convert a number expressed in terms of a fractional unit into a number expressed as far as possible in terms of the primary integral unit. Thus, to express $\$\frac{32}{4}$ in terms of the unit \$1 as far as possible, since in \$1 there are $\$\frac{4}{4}$, in $\$\frac{39}{4}$ there are as many unit dollars as 4 is contained times in 39, or \$9 with $\$\frac{3}{4}$ remaining,

$$\therefore \$\tfrac{39}{4} = \$9\tfrac{3}{4}.$$

Hence, in general, to reduce an improper fraction to a mixed number, *divide the numerator by the denominator, and to the quotient annex the remainder placed over the denominator.*

Ex. Reduce $\frac{221}{12}$ to a mixed number.

Since 12 is contained in 221, 18 times with a remainder 5,

$$\tfrac{221}{12} = 18\tfrac{5}{12}, \textit{ Result.}$$

EXERCISE 32.

Reduce the following improper fractions to equivalent mixed numbers:

1. $\frac{48}{7}$ qt.	8. $\$\frac{293}{10}$.	15. $\frac{728}{27}$.	22. $\frac{1000}{39}$.
2. $\frac{59}{8}$ mi.	9. $\$\frac{459}{20}$.	16. $\frac{598}{33}$.	23. $\frac{2001}{55}$.
3. $\frac{75}{4}$ in.	10. $\$\frac{877}{15}$.	17. $\frac{508}{35}$.	24. $\frac{3076}{63}$.
4. $\frac{124}{5}$ day.	11. $\$\frac{889}{25}$.	18. $\frac{875}{41}$.	25. $\frac{4809}{80}$.
5. $\frac{163}{7}$ wk.	12. $\$\frac{756}{11}$.	19. $\frac{487}{45}$.	26. $\frac{7598}{73}$.
6. $\frac{176}{9}$ gal.	13. $\frac{694}{13}$.	20. $\frac{920}{51}$.	27. $\frac{9569}{101}$.
7. $\frac{249}{12}$ ft.	14. $\frac{545}{17}$.	21. $\frac{970}{61}$.	28. $\frac{9639}{127}$.

115. III. To Reduce a Fraction to its Lowest Terms.—A fraction is reduced to an equivalent fraction in its lowest terms when its numerator and denominator have no common factor, that is, are prime to each other.

Reduction of a fraction to its lowest terms often saves labor in the further use of the fraction.

When a fraction is in its lowest terms, it is also easier to form a definite mental picture or conception of its value. Thus, $\frac{411}{548}$ cannot be realized definitely; but if the fraction be reduced to its lowest terms, $\frac{3}{4}$, an exact idea of its magnitude can be formed at once.

A fraction is reduced to its lowest terms by the use of Property A (Art. 112) of fractions. In general, *divide the numerator and denominator of the fraction by their G. C. D.*

Ex. 1. Reduce $\frac{8}{12}$ to its lowest terms.

Dividing 8 and 12 by their G. C. D., 4, we obtain

$$\tfrac{8}{12} = \tfrac{2}{3}, \textit{ Result.}$$

Ex. 2. Reduce $\frac{411}{548}$ to its lowest terms.

In this case the G. C. D. of the numerator and denominator is not evident on inspection, and must be obtained by the long division method (Art. 100). Thus,

```
411)548(1
    411
    137)411(3      ∴ G. C. D. = 137.
        411
```

$$\therefore \frac{411}{548} = \frac{137 \times 3}{137 \times 4} = \frac{3}{4}.$$

116. Ratios, or Expressing one Number as a Part or Fraction of Another.—To express one number as the part of another number it is necessary to *take the number which is the part as the numerator and the other number as the denominator of a fraction* (that is, divide the number expressing a part by the number expressing the whole).

Ex. What part of a mile is 440 yards?

Since a mile contains 1760 yards,

$$440 \text{ yards equal } \tfrac{440}{1760} \text{ of a mile} = \tfrac{1}{4} \text{ of a mile.}$$

EXERCISE 33.

Reduce each fraction to the equivalent fraction in its lowest terms:

1. $\frac{12}{21}$.	8. $\frac{72}{88}$.	15. $\frac{96}{176}$.	22. $\frac{968}{1320}$.
2. $\frac{20}{36}$.	9. $\frac{54}{99}$.	16. $\frac{104}{143}$.	23. $\frac{960}{1024}$.
3. $\frac{25}{45}$.	10. $\frac{60}{70}$.	17. $\frac{228}{950}$.	24. $\frac{1177}{2675}$.
4. $\frac{6}{48}$.	11. $\frac{84}{96}$.	18. $\frac{336}{528}$.	25. $\frac{1728}{2448}$.
5. $\frac{16}{56}$.	12. $\frac{48}{108}$.	19. $\frac{234}{315}$.	26. $\frac{1848}{2352}$.
6. $\frac{30}{54}$.	13. $\frac{70}{98}$.	20. $\frac{1152}{1792}$.	27. $\frac{2520}{2835}$.
7. $\frac{35}{63}$.	14. $\frac{90}{105}$.	21. $\frac{2310}{3570}$.	28. $\frac{2914}{3243}$.

29. A man invested \$360 and gained \$144. What part of the cost did he gain?

30. What part of 336 is 144? Is 168?

31. What part of 245 is 196? Is 210?

32. What part of a year is 146 days?

33. What part of a ton is 1275 pounds?

34. What part of a mile is 1320 yards?

35. \$230 is what part of \$322?

36. \$781 is what part of \$923?

37. Property valued at \$187200 was taxed for \$7020. Find the ratio of the assessment to the valuation.

38. Receiving \$8568 annually, I spent \$1904. What part of my income did I spend? What part did I save?

What part of:

39. 540 is 378? *41.* 1775 is 1491?

40. 864 is 630? *42.* 3154 is 2158?

117. IV. **To reduce two or more fractions to equivalent fractions having a common denominator.**

Similar fractions are fractions which have the same denominator. Hence, similar fractions express numbers in terms of the same fractional unit. Thus, \$$\frac{3}{10}$, \$$\frac{5}{10}$, \$$\frac{7}{10}$, are similar fractions.

If a series of fractions have different denominators, it is often useful to reduce them to fractions having the same denominator, that is, to express them in terms of the same unit. By combining their numerators, they may then, in many cases, be converted into a single fraction, and much labor saved by treating them in this form.

Also, in case it is required to compare the values of two or more dissimilar fractions, a direct comparison is often difficult or impossible. If, however, the fractions be reduced to a common denominator, their values can be compared at once by comparing the numerators obtained.

In reducing fractions to a common denominator, it is

important to reduce them to their least common denominator, in order to save as much labor as possible. The least common denominator of a set of fractions is the L. C. M. of their denominators.

In general, to reduce fractions to equivalent fractions having the L. C. D., *find the L. C. M. of the denominators of the given fractions; divide this L. C. M. by the denominator of each fraction; multiply each numerator by the corresponding quotient; the results will be the new numerators; write the L. C. D. under each new numerator.*

On which of the principles, A, B, C, of Art. 112, is this process based?

Ex. 1. Reduce $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{12}$, to their L. C. D.

The L. C. M. of 4, 8, and 12, is 24.

Dividing 24 by each of the numbers 4, 8, 12, we obtain quotients 6, 3, 2.

Multiply the numerators 3, 5, 7, by the corresponding quotients and setting each result over 24, we obtain

$\frac{18}{24}$, $\frac{15}{24}$, $\frac{14}{24}$. *Result.*

Ex. 2. Which is greater, \$$\frac{7}{12}$ or \$$\frac{11}{18}$?

Reducing the fractions to their L. C. D.,

\$$\frac{7}{12}$ = \$$\frac{21}{36}$.
\$$\frac{11}{18}$ = \$$\frac{22}{36}$.

Since $\frac{22}{36}$ is greater than $\frac{21}{36}$, \$$\frac{11}{18}$ is greater than \$$\frac{7}{12}$.

118. When the denominators of two or more fractions are prime to each other, the L. C. D. is the product of all the denominators, and the shortest way of reducing the fractions to their L. C. D. is to *multiply each numerator by all the denominators except its own, and set the result over the common denominator.*

Ex. Reduce $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$ to their L. C. D.

The L. C. D. = $3 \times 5 \times 7$.

Hence, we have $\frac{2 \times 5 \times 7}{105}$, $\frac{4 \times 3 \times 7}{105}$, $\frac{6 \times 3 \times 5}{105}$,

or $\frac{70}{105}$, $\frac{84}{105}$, $\frac{90}{105}$, *Result.*

EXERCISE 34.

Reduce to equivalent fractions having a common denominator:

1. $\frac{3}{4}, \frac{5}{6}$.
2. $\frac{1}{6}, \frac{3}{8}$.
3. $\frac{5}{4}, \frac{7}{10}$.
4. $\frac{2}{3}, \frac{3}{4}$.
5. $\frac{4}{5}, \frac{8}{9}$.
6. $\frac{11}{12}, \frac{13}{15}$.
7. $\frac{5}{16}, \frac{11}{20}$.
8. $\frac{13}{36}, \frac{23}{24}$.
9. $\frac{7}{9}, \frac{7}{10}$.
10. $\frac{1}{2}, \frac{2}{3}, \frac{1}{4}$.
11. $\frac{1}{3}, \frac{3}{2}, \frac{5}{6}$.
12. $\frac{3}{4}, \frac{1}{6}, \frac{3}{8}$.
13. $\frac{5}{8}, \frac{2}{9}, \frac{11}{12}$.
14. $\frac{5}{12}, \frac{2}{15}, \frac{7}{30}$.
15. $\frac{7}{12}, \frac{5}{16}, \frac{1}{24}$.
16. $\frac{17}{18}, \frac{23}{24}, \frac{15}{32}$.
17. $\frac{3}{2}, \frac{2}{3}, \frac{1}{5}$.
18. $\frac{1}{4}, \frac{3}{7}, \frac{1}{6}$.
19. $\frac{2}{9}, \frac{5}{4}, \frac{11}{12}$.
20. $\frac{4}{7}, \frac{9}{11}, \frac{7}{2}$.
21. $\frac{3}{11}, \frac{8}{15}, \frac{4}{21}$.
22. $\frac{3}{13}, \frac{4}{15}, \frac{5}{16}$.
23. $\frac{4}{15}, \frac{3}{25}, \frac{7}{20}$.
24. $\frac{5}{11}, \frac{3}{8}, \frac{5}{6}$.
25. $2\frac{2}{3}, 1\frac{3}{4}, 4\frac{1}{6}$.
26. $2\frac{1}{4}, 5\frac{1}{10}, 7\frac{3}{8}$.
27. $\frac{7}{12}, \frac{5}{18}, \frac{11}{24}, \frac{13}{30}$.

Which is the larger,

28. $\frac{5}{6}$ or $\frac{7}{8}$?
29. $\frac{3}{5}$ or $\frac{2}{3}$?
30. $\frac{1}{4}$ or $\frac{2}{9}$?
31. $\frac{7}{13}$ or $\frac{8}{15}$?
32. $\frac{7}{12}$ or $\frac{4}{7}$?
33. $\frac{9}{16}$ or $\frac{15}{28}$?

34. Which is the largest and which is the least,
$\frac{13}{27}, \frac{17}{36}$, or $\frac{11}{24}$? Also $\frac{13}{28}, \frac{16}{35}$, or $\frac{19}{40}$?

35. At a certain convention a measure which required a favorable ballot of 5 to 3, to pass, received 96 votes for and 57 votes against it. Did it pass?

36. A certain bill required two-thirds majority to become a law, and received 390 out of a total of 584 votes. Did it pass?

37. A boy has read 144 pages of a book, containing 300 pages. What part of the book remains to be read?

38. The 1st day of September is the 244th day of an ordinary year. Is the part gone as much as $\frac{2}{3}$ of the year? Is the year as much as $\frac{7}{8}$ gone? Does $\frac{10}{31}$ of the year remain?

OPERATIONS WITH FRACTIONS.

I. Addition of Fractions.

119. General Case.—We have found (see Art. 31) that any numbers which refer to the same unit may be added. Thus,

17 apples + 28 apples = 45 apples.

Numbers which refer to the same fractional unit may be added in the same way. Thus, five eighths (of a unit) + two eighths (of same unit) = seven eighths (of this unit),

or $\frac{5}{8} + \frac{2}{8} = \frac{7}{8}$.

If fractions are dissimilar, in order to add them it is necessary first to make them similar, by reducing them to a common denominator. Hence, to add fractions, *reduce the given fractions to equivalent fractions having the least common denominator; add the numerators, and write the sum over the L. C. D.; in all cases simplify the result, and, if it is an improper fraction, reduce it to a mixed number.*

Ex. Add $\frac{2}{3}, \frac{5}{6}, \frac{3}{8}$.

The L. C. D. is 24.

$$\frac{2}{3} + \frac{5}{6} + \frac{3}{8} = \frac{16}{24} + \frac{20}{24} + \frac{9}{24}$$
$$= \frac{45}{24} = \frac{15}{8} = 1\frac{7}{8}, \textit{ Sum.}$$

EXERCISE 35.

Add:

1. $\frac{3}{4}$ and $\frac{1}{6}$.
2. $\frac{4}{9}$ and $\frac{5}{6}$.
3. $\frac{7}{12}$ and $\frac{8}{15}$.
4. $\frac{3}{5}, \frac{4}{15}, \frac{1}{3}$.
5. $\frac{7}{8}, \frac{5}{6}, \frac{1}{12}$.
6. $\frac{1}{4}, \frac{1}{6}, \frac{17}{18}$.
7. $\frac{9}{10}, \frac{7}{15}, \frac{1}{5}$.
8. $\frac{5}{12}, \frac{11}{15}, \frac{7}{20}$.
9. $\frac{8}{15}, \frac{7}{18}, \frac{13}{30}$.
10. $\frac{9}{8}, \frac{5}{24}, \frac{17}{18}$.
11. $\frac{1}{16}, \frac{17}{20}, \frac{29}{80}$.
12. $\frac{2}{3}, \frac{4}{5}, \frac{7}{30}, \frac{9}{10}$.
13. $\frac{8}{15}, \frac{9}{20}, \frac{10}{21}, \frac{25}{28}$.
14. $\frac{4}{9}, \frac{5}{12}, \frac{11}{27}, \frac{23}{36}$.
15. $\frac{7}{8}, \frac{9}{10}, \frac{2}{3}, \frac{5}{6}, \frac{11}{15}$.
16. $\frac{5}{18} + \frac{7}{9} + \frac{11}{16} + \frac{5}{8} + \frac{11}{6} + \frac{7}{12}$.
17. $\frac{31}{45} + \frac{17}{30} + \frac{13}{36} + \frac{41}{90} + \frac{58}{75} + \frac{99}{100}$.
18. $\frac{75}{144} + \frac{29}{72} + \frac{31}{48} + \frac{23}{36} + \frac{107}{108}$.
19. $\frac{47}{48} + \frac{97}{112} + \frac{53}{63} + \frac{9}{56} + \frac{35}{72} + \frac{107}{126}$.
20. $\frac{5}{8} + \frac{7}{12} + \frac{10}{33} + \frac{19}{44} + \frac{29}{60} + \frac{124}{165} + \frac{99}{110}$.

120. Cases of Abbreviated Addition of Fractions.—1. The addition of two fractions each of whose numerators is unity, and whose denominators are prime to each other, may be abbreviated.

Thus, $\frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}$.

Or, in general, *the sum of the two denominators gives the numerator of the sum of the fractions; and the product of the two denominators gives the denominator of the sum of the fractions.*

2. The addition of a series of small fractions may often be facilitated by first adding the fractions in groups of two or three, and then taking the sum of the results.

Ex. Add $\frac{2}{3} + \frac{3}{4} + \frac{1}{3} + \frac{5}{12}$.

$$\frac{2}{3} + \frac{1}{3} = 1$$
$$\frac{3}{4} + \frac{5}{12} = \frac{7}{6}$$
$$2\frac{1}{6}, \textit{ Sum.}$$

EXERCISE 36.

Add:

1. $\frac{1}{3} + \frac{1}{7}$.
2. $\frac{1}{5} + \frac{1}{8}$.
3. $\frac{1}{6} + \frac{1}{10}$.
4. $\frac{1}{3} + \frac{1}{13}$.
5. $\frac{1}{12} + \frac{1}{17}$.
6. $\frac{4}{5} + \frac{1}{3} + \frac{1}{5}$.
7. $\frac{3}{11} + \frac{6}{7} + \frac{5}{11}$.
8. $\frac{4}{15} + \frac{1}{3} + \frac{2}{5}$.
9. $\frac{1}{4} + \frac{1}{5} + \frac{1}{6}$.
10. $\frac{1}{7} + \frac{1}{8} + \frac{1}{14}$.
11. $\frac{1}{9} + \frac{1}{6} + \frac{1}{18}$.
12. $\frac{1}{12} + \frac{1}{11} + \frac{1}{13}$.
13. $\frac{3}{16} + \frac{2}{5} + \frac{1}{2} + \frac{4}{5}$.
14. $\frac{4}{9} + \frac{3}{8} + \frac{5}{9} + \frac{1}{16}$.
15. $\frac{3}{5} + \frac{2}{7} + \frac{1}{15} + \frac{3}{14}$.
16. $\frac{14}{15} + \frac{1}{6} + \frac{2}{3} + \frac{4}{5}$.

121. Addition of Mixed Numbers.—To add mixed numbers, *first add the whole numbers, then add the fractions, then take the sum of the two results obtained.*

Ex. Add $11\frac{5}{8}$ and $10\frac{7}{12}$.

$$11\frac{5}{8} = 11\frac{15}{24}$$
$$10\frac{7}{12} = 10\frac{14}{24}$$
$$22\frac{5}{24}, \textit{ Sum.}$$

EXERCISE 37.

Add:

1. $3\frac{2}{3} + 2\frac{1}{2}$.
2. $4\frac{5}{6} + 2\frac{3}{4}$.
3. $5\frac{1}{5} + 7\frac{3}{10}$.
4. $6\frac{3}{8} + 2\frac{5}{12}$.
5. $4\frac{5}{9} + 4\frac{2}{15}$.
6. $5\frac{11}{12} + 6\frac{17}{18}$.
7. $8\frac{5}{24} + 1\frac{7}{36}$.
8. $1\frac{2}{7} + 3\frac{5}{14} + 7\frac{1}{2}$.
9. $3\frac{4}{5} + 4\frac{5}{6} + 6\frac{7}{10}$.
10. $8\frac{7}{12} + 5\frac{4}{15} + 10\frac{9}{20}$.

11. $2\frac{3}{7} + 4\frac{1}{3} + 6\frac{2}{5} + \frac{5}{6} + 1\frac{31}{35}$.
12. $5\frac{4}{5} + 3\frac{1}{6} + 2\frac{3}{4} + 4\frac{7}{10} + 3\frac{14}{15} + \frac{11}{12}$.
13. $3\frac{1}{5} + 4\frac{1}{3} + 5\frac{3}{25} + 7\frac{4}{9} + 5\frac{7}{15} + \frac{3}{10} + \frac{1}{2}$.
14. $\frac{3}{8} + \frac{7}{6} + 1\frac{1}{4} + 2\frac{1}{9} + \frac{35}{36} + 4\frac{7}{12} + 8\frac{1}{3}$.
15. $9\frac{1}{2} + 7\frac{2}{3} + \frac{4}{5} + 1\frac{5}{6} + 7\frac{7}{8} + 4\frac{1}{9} + \frac{9}{10}$.
16. $3\frac{3}{8} + \frac{5}{6} + \frac{4}{9} + 2\frac{7}{12} + 3\frac{11}{18} + \frac{7}{16} + 14\frac{2}{3}$.

ORAL EXERCISE.

17. $\frac{1}{2} + \frac{2}{3}$.
18. $\frac{3}{4} + \frac{5}{6}$.
19. $\frac{1}{5} + \frac{1}{7}$.
20. $\frac{4}{7} + \frac{2}{5}$.
21. $\frac{3}{10} + \frac{4}{15}$.
22. $\frac{1}{3} + \frac{1}{15}$.
23. $\frac{1}{13} + \frac{1}{11}$.
24. $1\frac{1}{3} + 2\frac{1}{4}$.
25. $1\frac{2}{3} + 3\frac{3}{4}$.
26. $3\frac{1}{5} + 2\frac{3}{4}$.
27. $4\frac{3}{5} + 2\frac{2}{3}$.
28. $1\frac{1}{6} + 5\frac{3}{8}$.
29. $7\frac{3}{16} + 5\frac{7}{12}$.
30. $2\frac{11}{18} + 3\frac{4}{27}$.
31. $\frac{2}{3} + \frac{4}{5} + \frac{1}{10}$.
32. $\frac{3}{5} + \frac{7}{10} + \frac{7}{15}$.
33. $\frac{5}{9} + \frac{7}{12} + \frac{1}{4}$.
34. $\frac{1}{2} + \frac{5}{2} + \frac{3}{4} + \frac{5}{4}$.
35. $\frac{4}{5} + \frac{1}{6} + \frac{3}{4} + \frac{1}{2}$.
36. $\frac{7}{4} + 1 + \frac{1}{6} + \frac{1}{24}$.
37. $\frac{3}{14} + 3 + \frac{5}{7} + \frac{1}{2}$.

II. Subtraction of Fractions.

122. General Case.—As in addition of fractions, if the fractions are dissimilar, it is necessary to make them similar before subtracting. Hence, in general, to subtract one fraction from another, *reduce the fractions to their L. C. D.; subtract the numerator of the subtrahend from the numerator of the minuend and place the difference over the L. C. D.; simplify the result.*

Ex. Subtract $\frac{3}{8}$ from $\frac{7}{12}$.

$$\frac{7}{12} - \frac{3}{8} = \frac{14}{24} - \frac{9}{24} = \frac{5}{24}, \textit{ Difference.}$$

EXERCISE 38.

Subtract

1. $\frac{2}{3}$ from $\frac{5}{6}$.
2. $\frac{5}{9}$ from $\frac{7}{12}$.
3. $\frac{8}{15}$ from $\frac{4}{5}$.
4. $\frac{11}{18}$ from $\frac{17}{24}$.
5. $\frac{8}{21}$ from $\frac{9}{14}$.
6. $\frac{7}{25}$ from $\frac{29}{30}$.
7. $\frac{8}{35}$ from $\frac{19}{42}$.
8. $\frac{9}{10}$ from $\frac{54}{55}$.
9. $\frac{3}{35}$ from $\frac{11}{56}$.
10. $\frac{6}{25}$ from $\frac{14}{15}$.
11. $\frac{8}{9}$ from $\frac{23}{24}$.
12. $\frac{75}{64}$ from $\frac{205}{96}$.

Find the value of

13. $\frac{6}{7}-\frac{1}{3}$.
14. $\frac{4}{9}-\frac{5}{24}$.
15. $\frac{17}{11}-\frac{11}{17}$.
16. $\frac{9}{8}-\frac{8}{9}$.
17. $\frac{15}{14}-\frac{40}{49}$.
18. $\frac{13}{35}-\frac{8}{25}$.
19. $\frac{7}{12}-\frac{19}{66}$.
20. $\frac{5}{18}-\frac{4}{81}$.
21. $\frac{61}{64}-\frac{55}{72}$.
22. $\frac{43}{48}-\frac{1}{44}$.
23. $\frac{67}{81}-\frac{59}{99}$.
24. $\frac{47}{120}-\frac{23}{75}$.
25. $\frac{35}{76}-\frac{31}{78}$.
26. $\frac{125}{84}-\frac{29}{119}$.
27. $\frac{87}{88}-\frac{9}{143}$.
28. $\frac{39}{92}-\frac{17}{76}$.
29. $\frac{97}{105}-\frac{57}{98}$.
30. $\frac{119}{144}-\frac{101}{132}$.

123. Special Case.—The subtraction of certain fractions may be simplified in a manner similar to that given for the addition of fractions in Art. 120.

Ex. $\frac{1}{3}-\frac{1}{5}=\frac{5-3}{15}=\frac{2}{15}$, *Difference.*

Let the student make a formal statement of this case.

124. Subtraction of Mixed Numbers.—If the fraction in the subtrahend is less than the fraction in the minuend, subtract the fractions and whole numbers separately, and combine the results.

Ex. 1. Subtract $\$3\frac{1}{4}$ from $\$5\frac{1}{3}$.

$$\begin{aligned}\$5\tfrac{1}{3} &= \$5\tfrac{4}{12}\\ \$3\tfrac{1}{4} &= \$3\tfrac{3}{12}\\ &\ \$2\tfrac{1}{12},\ \textit{Difference.}\end{aligned}$$

But if the fraction in the subtrahend is larger than the fraction of the minuend, it is necessary to increase the fraction in the minuend by borrowing 1 from the integral part of the minuend.

Ex. 2. Obtain the difference, $17\frac{1}{4}-12\frac{2}{3}$.

$$\begin{aligned}17\tfrac{1}{4} &= 17\tfrac{3}{12} = 16\tfrac{15}{12}\\ 12\tfrac{2}{3} &= 12\tfrac{8}{12} = 12\tfrac{8}{12}\\ &\phantom{= 17\tfrac{3}{12} =}\ 4\tfrac{7}{12},\ \textit{Difference.}\end{aligned}$$

Similarly, Ex. 3. $3-1\frac{2}{3}=2\frac{3}{3}-1\frac{2}{3}=1\frac{1}{3}$, *Difference.*

EXERCISE 39.

Subtract:

1. $1\frac{4}{7}$ from $3\frac{2}{3}$.
2. $2\frac{3}{10}$ from $2\frac{3}{5}$.
3. $4\frac{1}{8}$ from $5\frac{11}{12}$.
4. $3\frac{3}{4}$ from $5\frac{5}{6}$.
5. $6\frac{1}{4}$ from $9\frac{2}{3}$.
6. $\frac{1}{3}-\frac{1}{4}$.
7. $\frac{1}{5}-\frac{1}{8}$.
8. $\frac{1}{7}-\frac{1}{11}$.
9. $\frac{1}{8}-\frac{1}{18}$.
10. $\frac{1}{14}-\frac{1}{15}$.
11. $1\frac{1}{4}-\frac{1}{8}$.
12. $1\frac{1}{3}-\frac{1}{4}$.
13. $2\frac{1}{5}-\frac{1}{11}$.
14. $3\frac{1}{6}-2\frac{1}{8}$.
15. $4\frac{1}{10}-3\frac{1}{15}$.
16. $3\frac{1}{5}-2\frac{1}{2}$.
17. $6\frac{1}{8}-2\frac{1}{3}$.
18. $8\frac{1}{4}-5\frac{4}{5}$.
19. $7\frac{3}{11}-6\frac{2}{3}$.
20. $2\frac{5}{12}-1\frac{11}{18}$.
21. $4\frac{2}{3}-3$.
22. $5\frac{3}{4}-4$.
23. $6\frac{1}{12}-2\frac{1}{2}$.
24. $2-\frac{1}{2}$.
25. $3-1\frac{1}{3}$.
26. $14-5\frac{5}{9}$.
27. $8-3\frac{2}{11}$.
28. $13\frac{6}{7}-9\frac{3}{5}$.
29. $25\frac{11}{24}-8\frac{13}{15}$.
30. $19\frac{13}{18}-16\frac{7}{8}$.
31. $12\frac{14}{27}-7\frac{31}{45}$.

EXERCISE 40.

ORAL.

1. $\frac{1}{3}-\frac{1}{7}=?$
2. $\frac{1}{2}-\frac{1}{5}=?$
3. $\frac{2}{3}-\frac{1}{2}=?$
4. $\frac{5}{6}-1=?$
5. $\frac{1}{2}-\frac{1}{3}=?$
6. $\frac{3}{4}-\frac{2}{3}=?$
7. $\frac{3}{8}-\frac{1}{4}=?$
8. $\frac{5}{8}-\frac{1}{3}=?$
9. $1\frac{1}{2}-\frac{2}{3}=?$
10. $1\frac{3}{4}-\frac{4}{5}=?$
11. $1\frac{1}{3}-\frac{1}{4}=?$
12. $3-1\frac{1}{2}=?$
13. $2-1\frac{2}{3}=?$
14. $5-3\frac{4}{5}=?$
15. $5\frac{4}{5}-2\frac{1}{2}=?$
16. $3\frac{2}{5}-1\frac{2}{3}=?$
17. How much larger is $\frac{5}{6}$ than $\frac{2}{3}$?
18. How much larger is $1\frac{1}{3}$ than $\frac{1}{2}$? $2\frac{1}{2}$ than $1\frac{5}{6}$?
19. How much less is $2\frac{3}{4}$ than 3? $2\frac{1}{2}$ than $3\frac{1}{5}$?

EXERCISE 41.

REVIEW.

Find the value of:

1. $\frac{3}{4}+\frac{1}{2}-\frac{1}{5}$.
2. $\frac{1}{5}+\frac{3}{8}-\frac{3}{10}$.
3. $\frac{2}{3}-\frac{1}{6}+\frac{4}{9}$.
4. $1\frac{1}{2}-\frac{2}{3}+\frac{5}{6}$.
5. $3\frac{3}{4}-1\frac{1}{3}-\frac{1}{4}$.
6. $2\frac{1}{6}+1\frac{1}{2}+2\frac{1}{3}$.
7. $\frac{3}{5}+7\frac{1}{4}-5\frac{3}{10}$.
8. $2\frac{1}{2}+3\frac{3}{5}-3\frac{1}{4}$.
9. $7\frac{1}{3}-2\frac{1}{2}-4\frac{1}{6}$.
10. $8\frac{2}{3}-3\frac{1}{3}+2\frac{5}{6}$.
11. $7\frac{3}{8}-4\frac{1}{2}+3\frac{7}{8}$.
12. $6\frac{5}{12}+3\frac{1}{8}-7\frac{5}{6}$.
13. $3\frac{11}{15}-1\frac{7}{12}+\frac{13}{18}$.
14. $4\frac{1}{10}-3\frac{2}{5}+5\frac{11}{24}$.
15. $3\frac{1}{2}+2-4\frac{1}{6}+7\frac{1}{3}-4$.
16. $7\frac{1}{8}-4\frac{1}{6}+2\frac{1}{2}-4+8\frac{5}{12}-7\frac{2}{3}$.
17. $3\frac{7}{9}-1-1\frac{5}{6}+9\frac{3}{4}-3\frac{1}{3}+3-2\frac{11}{12}$.

18. If from a piece of cloth containing 50 yards there have been sold at one time $17\frac{3}{4}$ yards, at another $12\frac{5}{8}$, and at another $7\frac{7}{12}$, how many yards remain?

19. If during the month I spend $\frac{1}{3}$, $\frac{1}{5}$, $\frac{3}{10}$, and $\frac{1}{15}$ of my salary for the month previous, what part have I left?

20. A carpenter finds some boards of following lengths, $17\frac{1}{3}$, $12\frac{1}{4}$, $15\frac{2}{3}$, $16\frac{5}{8}$, $11\frac{7}{12}$, and $18\frac{1}{2}$ feet respectively. What was the total length?

21. From an account amounting to $\$175\frac{3}{4}$, I have drawn at different times, $\$19\frac{1}{2}$, $\$24\frac{2}{3}$, $\$4\frac{4}{5}$, $\$11\frac{3}{4}$, $\$41\frac{3}{5}$, and $\$35\frac{1}{12}$. How many dollars remain?

III. MULTIPLICATION OF FRACTIONS.

125. To Multiply a Fraction by a Whole Number.—Fractional units are multiplied by a whole number in the same manner that other units are, viz.: by obtaining the product of the number of units and the multiplier. Thus,

7 dollars $\times 5 = 35$ dollars.

Similarly, 7 twelfths (of any unit) $\times 5 = 35$ twelfths (of this unit).

Or, $\frac{7}{12} \times 5 = \frac{35}{12}$, *Product.*

Hence, to find the product of a whole number and a fraction, *take the product of the numerator by the whole number; set the result over the denominator; simplify by cancellation.*

$$\text{Ex. } \frac{7}{24} \times 15 = \frac{7 \times \overset{5}{\cancel{15}}}{\underset{8}{\cancel{24}}} = \frac{35}{8} = 4\tfrac{3}{8}, \textit{ Product.}$$

126. To Multiply a Fraction by a Fraction.—If it be required to multiply $\$\frac{7}{4}$ by $\frac{3}{5}$, we know that the product of $\$\frac{7}{4}$ by 1 equals $\$\frac{7}{4}$; hence, since, if we make the multiplier $\frac{1}{5}$ as large, the product will be $\frac{1}{5}$ as great, $\$\frac{7}{4}$ multiplied by $\frac{1}{5}$ equals $\frac{1}{5}$ of $\$\frac{7}{4}$, or $\$\frac{7}{20}$ (see Art. 112, B).

Hence, $\$\frac{7}{4}$ multiplied by $\frac{3}{5}$ equals 3 times $\$\frac{7}{20}$ or $\$\frac{21}{20}$ (see Art. 112, B), or $\$1\frac{1}{20}$,

or, in brief, $\frac{7}{4} \times \frac{3}{5} = \frac{7 \times 3}{4 \times 5} = \frac{21}{20} = 1\frac{1}{20}$, *Product.*

Hence, in multiplying one fraction by another, we do two things, (1) we diminish the size of the fractional unit (from one-fourth to one-twentieth of a dollar in the above example), (2) we increase the number of the units.

It may help the student to realize the above process to take 7 quarter dollars, multiply by $\frac{1}{5}$ by substituting nickels for quarters, then multiply by 3 by increasing the number of nickels threefold.

From the above, we obtain the following convenient mechanical method for multiplying two fractions:

Multiply the numerators together for the numerator of the product, and the denominators for the denominator of the product, abbreviating the work as far as possible by cancellation.

127. Continued Multiplication of Fractions.—To multiply three or more fractions together, multiply all the numerators together for a new numerator, and all the denominators for a new denominator, canceling when possible. For the (indicated) product of two fractions may be multiplied by another fraction, and so on for any number of fractions.

Ex. 1. Multiply $\frac{9}{10} \times \frac{32}{81} \times \frac{35}{48}$.

$$\frac{\cancel{9}}{\underset{2}{\cancel{10}}} \times \frac{\overset{16}{\cancel{32}}}{\underset{9}{\cancel{81}}} \times \frac{\overset{7}{\cancel{35}}}{\underset{3}{\cancel{48}}} = \frac{7}{27}.$$

A **Compound Fraction** is a fraction of a fraction. Hence, a compound fraction is the product of two fractions, looked at from another point of view.

Ex. 2. Find value of $\frac{3}{8}$ of $\frac{4}{7}$.

$\frac{3}{8}$ of $\frac{4}{7} = \frac{3}{8} \times \frac{4}{7} = \frac{3}{14}$, *Result.*

EXERCISE 42.

Multiply:

1. $\frac{3}{5} \times 4$.	*5.* $12 \times \frac{5}{21}$.	*9.* $\frac{4}{9} \times \frac{1}{2}$.
2. $\frac{4}{7} \times 14$.	*6.* $14 \times \frac{16}{35}$.	*10.* $\frac{7}{11} \times \frac{3}{5}$.
3. $\frac{5}{6} \times 9$.	*7.* $\frac{15}{44} \times 33$.	*11.* $\frac{5}{6} \times \frac{12}{25}$.
4. $10 \times \frac{3}{16}$.	*8.* $\frac{3}{7} \times \frac{2}{3}$.	*12.* $\frac{6}{7} \times \frac{21}{22}$.

13. $\frac{25}{27} \times \frac{36}{35}$.
14. $\frac{48}{49} \times \frac{63}{64}$.
15. $\frac{3}{5} \times \frac{4}{9} \times \frac{15}{16}$.
16. $\frac{14}{15} \times \frac{20}{21} \times \frac{9}{16}$.
17. $\frac{29}{49} \times \frac{63}{70} \times \frac{7}{18}$.
18. $\frac{72}{121} \times \frac{55}{56} \times \frac{77}{90}$.
19. $\frac{15}{46} \times \frac{69}{70} \times \frac{28}{27}$.
20. $\frac{48}{185} \times \frac{147}{64} \times \frac{49}{27}$.
21. $\frac{360}{343} \times \frac{77}{495} \times \frac{21}{32}$.

22. $\frac{2}{3}$ of 17.
23. $\frac{3}{4}$ of 14.
24. $\frac{7}{8}$ of 20.
25. $\frac{9}{10}$ of 56.
26. $\frac{7}{12}$ of 90.
27. $\frac{9}{11}$ of $\frac{2}{3}$.
28. $\frac{4}{5}$ of $\frac{10}{11}$.
29. $\frac{11}{12}$ of $\frac{20}{33}$.
30. $\frac{5}{7}$ of $\frac{4}{15}$ of $\frac{28}{29}$.

31. $\frac{9}{11}$ of $\frac{22}{27}$ of $\frac{17}{18}$.
32. $\frac{2}{3}$ of $\frac{3}{4} + \frac{1}{8}$.
33. $\frac{5}{6}$ of $\frac{9}{10} - \frac{5}{8}$.
34. $5 + \frac{2}{3}$ of 25.
35. $8\frac{1}{7} - \frac{4}{7}$ of $\frac{35}{36}$.
36. $\frac{8}{9}$ of $\frac{27}{28} - \frac{1}{4}$.
37. $\frac{1}{2} + \frac{2}{11}$ of $63 + \frac{3}{5}$.
38. $\frac{2}{3}$ of $75 - 14\frac{2}{7}$.
39. $\frac{9}{16}$ of $\frac{56}{33}$ of $\frac{44}{63}$.

128. To multiply a mixed number by another mixed number, it is best to reduce both mixed numbers to improper fractions.

Ex. 1. What will $3\frac{3}{8}$ yards of cloth cost at $\$1\frac{1}{3}$ a yard?

$$\$\tfrac{4}{3} \times \tfrac{27}{8} = \$\tfrac{9}{2} = \$4\tfrac{1}{2}, \textit{ Cost.}$$

If, however, an integer (especially if it be a large one) is to be multiplied by a mixed number, the labor of multiplication is often diminished by proceeding as follows:

Ex. 2. What will 47 yards of cloth cost at $\$2\frac{3}{8}$ a yard?

OPERATION.

```
    47
    2 3/8
8)141
  17 5/8
  94
 111 5/8, Product.
```

EXPLANATION.

We obtain first $\frac{3}{8}$ of 47, or $17\frac{5}{8}$; then multiply 47 by 2, obtain 94, and add the results. The entire cost is therefore $\$111\frac{5}{8}$.

EXERCISE 43.

Multiply:

1. $21 \times 2\frac{5}{7}$.
2. $24 \times 3\frac{5}{8}$.
3. $27 \times 1\frac{2}{9}$.
4. $30 \times 4\frac{3}{10}$.
5. $6 \times 3\frac{3}{4}$.
6. $7 \times 2\frac{1}{5}$.
7. $10 \times 4\frac{3}{11}$.
8. $28 \times 7\frac{5}{6}$.
9. $7\frac{1}{3} \times 80$.
10. $9\frac{4}{9} \times 100$.
11. $13\frac{2}{11} \times 51$.
12. $2\frac{1}{2} \times 4\frac{2}{5}$.
13. $6\frac{1}{4} \times 1\frac{1}{15}$.
14. $8\frac{5}{9} \times 5\frac{8}{11}$.
15. $20\frac{4}{13} \times \frac{39}{44}$.
16. $47\frac{3}{16} \times 6\frac{2}{3}$.
17. $64\frac{8}{15} \times 7\frac{3}{11}$.
18. $101\frac{3}{5} \times 9\frac{14}{39}$.

19. $\frac{3}{7}$ of $\frac{5}{6}$ of $23\frac{1}{10}$.
20. $\frac{7}{8}$ of $\frac{16}{25}$ of $33\frac{4}{7}$.
21. $\frac{13}{15}$ of $7\frac{6}{7} \times 7\frac{7}{11}$.
22. $\frac{19}{27} \times 2\frac{4}{7}$ of $15\frac{1}{6}$.
23. $8\frac{2}{5} \times 5\frac{1}{7} \times 3\frac{1}{18}$.
24. $\frac{1}{24}$ of $9\frac{3}{7} \times 11\frac{5}{11}$.

Find the areas of the following floors (or ceilings) in square feet:

25. 16 feet long, $12\frac{2}{3}$ feet wide.
26. 15 feet long, $13\frac{1}{2}$ feet wide.
27. 21 feet by $8\frac{1}{3}$ feet.
28. 19 feet by $15\frac{3}{4}$ feet.
29. 30 feet by $20\frac{11}{12}$ feet.
30. 28 feet by $18\frac{5}{6}$ feet.

Find the areas of the walls of the following rooms:

31. Distance around (perimeter) is 80 feet, and height is $10\frac{1}{4}$ feet.
32. 36 feet in perimeter, and $7\frac{2}{3}$ feet high.
33. 41 feet in perimeter, and $8\frac{5}{12}$ feet high.
34. $46\frac{1}{2}$ feet in perimeter, and $7\frac{3}{4}$ feet high.

EXERCISE 44.

ORAL.

Find:

1. $\frac{2}{3}$ of 90.
2. $\frac{4}{5}$ of 55.
3. $\frac{5}{38}$ of $1\frac{1}{5}$.
4. $2\frac{1}{2} \times \frac{9}{10}$.
5. $1\frac{2}{5} \times 1\frac{3}{7}$
6. $\frac{3}{4}$ of 50.
7. $\frac{4}{5}$ of 27.
8. $\frac{11}{12}$ of 80.
9. $\frac{3}{7}$ of 19.
10. $1\frac{4}{5} \times 2\frac{3}{5}$.
11. $3\frac{3}{4} \times 2\frac{2}{5}$.
12. $8\frac{7}{11} \times 1\frac{3}{19}$.

13. What will $7\frac{3}{4}$ ponnds of sugar cost at 6 cents a pound?

14. If $3\frac{5}{12}$ yards of cloth are necessary for a coat, how many yards will be required for 10 coats?

15. A man paid to each of nine laborers $2\frac{3}{4}$ dollars. How much did he pay to all?

16. If tea is worth $\$\frac{3}{4}$ a pound, what will $\frac{2}{3}$ of a pound cost?

17. If ribbon is worth $\$\frac{4}{5}$ a yard, what will $3\frac{3}{4}$ yards cost? What will $\frac{1}{2}$ a yard cost?

EXERCISE 45.

Required the cost of:

1. 27 books @ $\$1\frac{1}{4}$ apiece.
2. 42 pairs of shoes @ $\$2\frac{3}{4}$ each.
3. 125 dozen pencils @ $\$\frac{1}{8}$ a dozen.

4. $7\frac{7}{8}$ tons of hay @ \$$11\frac{1}{9}$ a ton.
5. 64 lamps @ \$$5\frac{5}{8}$ each.
6. 27 lbs. sugar @ $5\frac{3}{4}$ cents per pound.
7. 145 tons of coal @ \$$5\frac{2}{3}$ a ton.
8. $5\frac{5}{6}$ cords of wood @ \$$5\frac{1}{7}$ a cord.
9. $14\frac{4}{5}$ months of board @ \$$18\frac{3}{4}$ a month.
10. $76\frac{3}{7}$ acres of land @ \$$21\frac{7}{10}$ an acre.
11. $103\frac{1}{2}$ acres of land @ \$$61\frac{1}{6}$ an acre.

12. How many square feet in the ceiling of a room $19\frac{5}{8}$ feet long and $8\frac{2}{7}$ feet wide?

13. How many square feet in the walls of a room whose perimeter is $141\frac{1}{3}$ feet and height $17\frac{3}{4}$ feet?

14. If I withdraw from a bank $\frac{3}{8}$ of my deposit, and then $\frac{2}{5}$ of the remainder, what part do I draw the second time? What part of the whole deposit is left?

15. I owned $\frac{3}{4}$ of $\frac{2}{3}$ of a business and sold $\frac{1}{5}$ of my share. What part of the entire enterprise do I still own?

16. If a wagon-wheel $16\frac{1}{4}$ feet in circumference revolves $43\frac{1}{5}$ times in going a certain distance, how many feet in that distance?

17. Of a pole $\frac{1}{6}$ is red, $\frac{2}{9}$ is white, and the rest is black. What part is black?

18. Of another pole $\frac{1}{6}$ is red, $\frac{2}{5}$ of the remainder is white, and the rest is black. What part is black?

19. From a roll of cloth containing $35\frac{3}{4}$ yards, $16\frac{5}{8}$ yards were sold at one time, and, at another, $\frac{4}{5}$ of the remainder. How many yards still remain?

IV. Division of Fractions.

129. To Divide a Fraction by a Whole Number.—We may divide a number of fractional units just as we divide a number of any other units. Thus, just as

$$12 \text{ dollars} \div 3 = 4 \text{ dollars},$$

so, 12 fifteenths (of any unit) $\div$ 3 = 4 fifteenths (of this unit)

$$\text{or, } \tfrac{12}{15} \div 3 = \tfrac{4}{15}, \textit{ Quotient.}$$

Instead of dividing the numerator of the fraction by the divisor, it may be necessary to perform the division by multiplying the denominator by the divisor (see B, Art. 112).

Ex. If 4 yards of calico cost $\frac{7}{8}$ of a dollar, what will 1 yard cost?

$$\$\tfrac{7}{8} \div 4 = \$\frac{7}{8 \times 4} = \$\frac{7}{32}.$$

In this division we diminish the size of the fractional units (from eighths to thirty-seconds), but leave the number of units unchanged.

130. To Divide a Fraction by a Fraction.—If it be required to determine how many times \$$\frac{7}{10}$ is contained in \$$\frac{3}{4}$, we may proceed as follows:

\$$\frac{1}{10}$ is contained in \$1 ten times, hence, \$$\frac{7}{10}$ is contained in \$1, $\frac{1}{7}$ of 10 times, or $\frac{10}{7}$ times; if \$$\frac{7}{10}$ is contained in 1 dollar $\frac{10}{7}$ times, it is contained in $\frac{3}{4}$ of a dollar, $\frac{3}{4} \times \frac{10}{7}$ times, or $\frac{15}{14}$ times, or in brief,

$$\$\tfrac{3}{4} \div \$\tfrac{7}{10} = \tfrac{3}{4} \times \tfrac{10}{7} = \tfrac{30}{28} = \tfrac{15}{14}, \textit{ Quotient.}$$

Similarly, any number of fractional units may be divided by another number of fractional units of the same kind of quantity, or by an abstract number of fractional units. Hence, to divide one fractional number by another, we have the following convenient mechanical rule: *invert the divisor and proceed as in multiplication.*

Ex. Divide $\frac{18}{75}$ by $\frac{6}{15}$.

$$\tfrac{18}{75} \div \tfrac{6}{15} = \tfrac{18}{75} \times \tfrac{15}{6} = \tfrac{3}{5}, \textit{ Quotient.}$$

131. To Divide one Mixed Number by Another.—First reduce the mixed numbers to improper fractions.

Ex. 1. If one yard of cloth costs \$$2\frac{1}{4}$, how many yards may be bought for \$$40\frac{1}{2}$?

$$40\tfrac{1}{2} \div 2\tfrac{1}{4} = \tfrac{81}{2} \div \tfrac{9}{4} = \tfrac{81}{2} \times \tfrac{4}{9} = 18, \textit{ No. of yards.}$$

However, in dividing a large mixed number by an integer, labor may often be saved by first dividing the integral part of the mixed number by the divisor.

Ex. 2. If one ton of coal costs \$6, how many tons may be bought for \$142½?

$$\$6)\underline{\$142\tfrac{1}{2}}$$
$$23\tfrac{3}{4},\ \textit{No. of tons.}$$

Dividing \$6 into \$142½, we obtain 23 for a quotient, with a remainder of 4½. 4½ divided by 6 gives ¾. Hence, the entire quotient is 23¾.

EXERCISE 46.

Divide:

1. $\frac{12}{13}$ by 3.
2. $\frac{25}{29}$ by 5.
3. $\frac{8}{11}$ by 6.
4. $\frac{104}{105}$ by 8.
5. $\frac{104}{105}$ by 24.
6. $\frac{168}{231}$ by 36.
7. $\frac{45}{121}$ by 27.
8. $\frac{5}{6}$ by 3.
9. $\frac{4}{5}$ by 5.
10. $\frac{8}{9}$ by 15.
11. $\frac{315}{401}$ by 420.
12. $1\frac{79}{101}$ by 288.
13. $\frac{4}{9}$ by $\frac{2}{27}$.
14. $\frac{7}{22}$ by $\frac{14}{33}$.
15. $\frac{15}{16}$ by $\frac{21}{32}$.
16. $\frac{75}{224}$ by $\frac{125}{192}$.
17. $1\frac{3}{5}$ by $1\frac{1}{3}$.
18. $5\frac{1}{4}$ by $4\frac{1}{5}$.
19. $12\frac{2}{3}$ by $10\frac{5}{9}$.
20. $13\frac{3}{4}$ by $15\frac{1}{8}$.
21. $20\frac{5}{6}$ by $52\frac{1}{12}$.
22. $65\frac{1}{10}$ by 3.
23. $92\frac{4}{7}$ by 6.
24. $247\frac{2}{5}$ by 7.
25. $342\frac{2}{3}$ by 8.
26. $153\frac{1}{2}$ by 2.
27. $203\frac{1}{2}$ by 3.
28. $496\frac{2}{3}$ by 12.
29. 35 by $8\frac{1}{6}$.
30. 63 by $5\frac{2}{5}$.
31. 48 by $10\frac{2}{7}$.
32. 63 by $7\frac{7}{11}$.
33. 96 by $14\frac{2}{9}$.
34. 17 by $23\frac{4}{5}$.
35. 38 by $52\frac{1}{4}$.
36. 55 by $66\frac{11}{12}$.
37. $26\frac{8}{11}$ by $2\frac{54}{55}$.
38. $13\frac{1}{11}$ by $4\frac{7}{12}$.
39. $36\frac{4}{7}$ by $9\frac{5}{8}$.

V. Simplification of Complex Fractions. Reciprocals.

132. **A complex fraction** is a fraction containing a fraction in its numerator or in its denominator, or in both.

Exs. $\frac{5}{\frac{3}{7}}$, $\frac{3\frac{3}{4}}{5\frac{2}{7}}$, $\frac{\frac{2}{3}-\frac{1}{2}}{3\frac{1}{5}}$.

Hence, the quotient of one fraction by another may be indicated as a complex fraction.

Thus, $\frac{3}{4} \div \frac{5}{6}$ may be written $\frac{\frac{3}{4}}{\frac{5}{6}}$.

133. The **reciprocal** of a number is the quotient obtained by dividing 1 by that number.

Thus, the reciprocal of 3 is $\frac{1}{3}$; of $\frac{2}{5}$ is $\frac{1}{\frac{2}{5}}$ or $\frac{5}{2}$; of $2\frac{3}{4}$ is $\frac{1}{2\frac{3}{4}}$ or $\frac{4}{11}$.

134. Simplification of Complex Fractions.—To simplify a complex fraction, *simplify the numerator and denominator; divide the numerator by the denominator.*

Ex. 1. Simplify $\frac{3\frac{2}{3}}{4\frac{5}{6}}$.

$$\frac{3\frac{2}{3}}{4\frac{5}{6}} = \frac{\frac{11}{3}}{\frac{29}{6}} = \frac{11}{3} \times \frac{6}{29} = \frac{22}{29},\ \textit{Result.}$$

Ex. 2. Simplify $\frac{5\frac{1}{2}-3\frac{1}{3}}{\frac{5}{6}\text{ of }\frac{2}{3}}$.

$$\frac{5\frac{1}{2}-3\frac{1}{3}}{\frac{5}{6}\text{ of }\frac{2}{3}} = \frac{\frac{11}{2}-\frac{10}{3}}{\frac{5}{6}\times\frac{2}{3}} = \frac{\frac{33}{6}-\frac{20}{6}}{\frac{5}{9}} = \frac{\frac{13}{6}}{\frac{5}{9}} = \frac{13}{6} \times \frac{9}{5} = \frac{39}{10} = 3\frac{9}{10},\ \textit{Result.}$$

Ex. 3. Simplify $\dfrac{7}{5+\dfrac{2}{1-\dfrac{1}{4-\frac{3}{7}}}}$

$$\frac{7}{5+\dfrac{2}{1-\dfrac{1}{4-\frac{3}{7}}}} = \frac{7}{5+\dfrac{2}{1-\dfrac{1}{\frac{25}{7}}}} = \frac{7}{5+\dfrac{2}{1-\frac{7}{25}}} = \frac{7}{5+\dfrac{2}{\frac{18}{25}}}$$
$$= \frac{7}{5+\frac{25}{9}} = \frac{7}{\frac{70}{9}} = \frac{9}{10},\ \textit{Result.}$$

EXERCISE 47.

Simplify:

1. $\frac{\frac{5}{6}}{\frac{10}{9}}$.
2. $\frac{1\frac{2}{3}}{3\frac{1}{3}}$.
3. $\frac{2\frac{4}{5}}{4\frac{9}{10}}$.
4. $\frac{7\frac{2}{3}}{8\frac{5}{8}}$.
5. $\frac{10\frac{2}{5}}{3\frac{9}{10}}$.
6. $\frac{11\frac{4}{7}}{4\frac{10}{11}}$.
7. $\frac{9\frac{3}{8}}{9\frac{9}{16}}$.
8. $\frac{101\frac{7}{13}}{13\frac{5}{7}}$.
9. $\frac{17}{10\frac{1}{5}}$.
10. $\frac{25\frac{5}{6}}{93}$.
11. $\frac{\frac{2}{3}\text{ of }5\frac{1}{4}}{\frac{5}{6}+\frac{3}{4}}$.
12. $\frac{\frac{3}{5}\text{ of }9\frac{1}{6}}{1\frac{1}{3}-\frac{2}{5}}$.
13. $\frac{11\frac{2}{3}}{4\frac{1}{6}-3\frac{3}{4}}$.
14. $\frac{6\frac{1}{2}+8\frac{1}{5}}{10\frac{3}{8}-4\frac{3}{5}}$.
15. $\frac{\frac{8\frac{1}{3}}{5}}{7 \div 4\frac{1}{2}}$.

16. $\dfrac{\dfrac{2\frac{5}{6}-1\frac{5}{8}}{2\frac{5}{12}}}{8\frac{1}{3}-7\frac{5}{12}}$.

17. $\dfrac{\dfrac{\frac{4}{7}\text{ of }9\frac{4}{5}}{9\frac{3}{10}-5\frac{1}{15}}}{1\div 6\frac{1}{21}}$.

18. $\dfrac{\dfrac{\frac{7}{13}\times 4\frac{9}{14}\times 4\frac{2}{3}}{7\frac{1}{9}}}{3\frac{7}{8}-2\frac{11}{12}+1\frac{23}{24}}$.

19. Divide the sum of $7\frac{3}{5}$ and $9\frac{5}{7}$ by their difference.

20. Divide the sum of $4\frac{1}{6}$, $3\frac{3}{5}$, $6\frac{1}{10}$ by their product.

21. Multiply the sum of $\frac{7}{8}$, $\frac{5}{6}$, $\frac{23}{24}$, $\frac{8}{9}$, $\frac{17}{18}$ by the least, and divide the product by the greatest of the five fractions.

What part of:

22. $6\frac{1}{4}$ is $2\frac{1}{2}$?

23. $7\frac{1}{3}$ is $5\frac{1}{2}$?

24. $4\frac{4}{9}$ is $3\frac{1}{3}$?

25. $12\frac{5}{6}$ is $1\frac{3}{4}$?

26. $14\frac{7}{10}$ is $4\frac{1}{5}$?

27. 140 is $134\frac{1}{6}$?

28. $3\frac{1}{11}$ is what part of $5\frac{9}{22}$?

29. 6 is what part of $12\frac{3}{4}$?

Find the value of:

30. $\dfrac{1\frac{1}{2}}{9}+\dfrac{3\frac{1}{4}}{26}+\dfrac{25}{4\frac{2}{7}}$.

31. $\frac{3}{7}$ of $9\frac{4}{5}-3\frac{4}{10}$.

32. $7\frac{1}{5}-\frac{4}{9}$ of $8\frac{1}{7}$.

33. $\dfrac{4\frac{2}{3}}{1\frac{1}{6}}-\dfrac{2\frac{1}{8}}{5\frac{2}{3}}-\dfrac{3\frac{8}{9}}{8\frac{1}{6}}$.

34. $\dfrac{\dfrac{\frac{4}{7}\text{ of }8\frac{5}{9}}{4\frac{2}{5}}-\dfrac{7\frac{3}{8}-2\frac{2}{3}}{11\frac{3}{10}}}{\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+1\frac{5}{6}}$.

Simplify:

35. $2+\dfrac{1}{3+\frac{1}{4}}$.

36. $3+\dfrac{2}{5-\frac{1}{3}}$.

37. $5-\dfrac{6}{2+\dfrac{4}{3-\frac{1}{3}}}$.

38. $1+\dfrac{10}{2-\dfrac{5}{3+\frac{1}{2}}}$.

39. $3-\dfrac{1}{10-\dfrac{5}{3-\dfrac{3}{1+\frac{1}{4}}}}$.

40. $9+\dfrac{8}{7-\dfrac{5}{3-\dfrac{6}{2+\frac{3}{4}}}}$.

41. $1+\dfrac{3}{2+\dfrac{5}{4-\dfrac{3}{4+\frac{1}{2}}}}$.

42. $6-\dfrac{8}{1+\dfrac{1}{2+\dfrac{1}{1+\frac{2}{3}}}}$.

43. $5+\dfrac{7}{4+\dfrac{3}{2+\dfrac{8}{3+\frac{1}{5}}}}$.

VI. G. C. D. and L. C. M. of Fractions.

135. G. C. D. of Fractions.—In order to find the G. C. D. of two or more fractions, the simplest method is to *reduce the fractions to their lowest terms (mixed numbers to improper fractions); reduce the fractions thus obtained to their least common denominator; find the G. C. D. of the numerators, and set the result over the common denominator.*

This is equivalent to expressing the fractional quantities in terms of the same fractional unit, and finding their G. C. D. in this form.

Ex. Find the G. C. D. of $1\frac{1}{3}$ and $1\frac{3}{5}$.

$1\frac{1}{3}$ and $1\frac{3}{5}$ reduce to $\frac{4}{3}$ and $\frac{8}{5}$, and when reduced to their common denominator are $\frac{20}{15}$ and $\frac{24}{15}$.

The G. C. D. of 20 and 24 is 4.

Hence, the G. C. D. of $1\frac{1}{3}$ and $1\frac{3}{5}$ is $\frac{4}{15}$, *Result.*

136. L. C. M. of Fractions.—To find the L. C. M. of two or more fractions, we proceed similarly, thus, *reduce the given fractions to their L. C. D.; find the L. C. M. of the numerators of the fractions thus obtained, and set the result over the common denominator.*

Ex. Find the L. C. M. of $1\frac{1}{3}$ and $1\frac{3}{5}$.

These fractions reduce to $\frac{20}{15}$ and $\frac{24}{15}$; the L. C. M. of 20 and 24 is 120; hence, the L. C. M. of $1\frac{1}{3}$ and $1\frac{3}{5}$ is $\frac{120}{15}$, or 8.

EXERCISE 48.

Find the G. C. D. of

1. $\frac{4}{5}$ and $\frac{6}{7}$.
2. $\frac{5}{6}$ and $\frac{3}{10}$.
3. $\frac{5}{11}$ and $\frac{7}{10}$.
4. $4\frac{1}{6}$ and $1\frac{7}{8}$.
5. $5\frac{5}{8}$ and $7\frac{1}{12}$.
6. $\frac{5}{12}$, $\frac{9}{16}$, $\frac{4}{9}$.
7. $\frac{14}{15}$, $\frac{21}{20}$, $\frac{35}{12}$.
8. $4\frac{4}{7}$, $6\frac{3}{5}$, $2\frac{1}{21}$.
9. $2\frac{2}{5}$, $3\frac{3}{11}$, $4\frac{4}{7}$.
10. $17\frac{1}{7}$, $16\frac{2}{5}$, $5\frac{5}{14}$.

Find the L. C. M. of

11. $\frac{5}{6}$ and $\frac{1}{8}$.
12. $\frac{4}{15}$ and $\frac{8}{9}$.
13. $\frac{5}{7}$ and $\frac{9}{10}$.
14. $2\frac{1}{4}$ and $3\frac{3}{5}$.
15. $5\frac{1}{4}$ and $3\frac{1}{2}$.
16. $\frac{4}{15}$, $\frac{4}{9}$, $\frac{2}{3}$.
17. $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{2}$.
18. $\frac{7}{10}$, $1\frac{3}{5}$, $\frac{14}{15}$.
19. $3\frac{3}{8}$, $1\frac{4}{7}$, $1\frac{19}{21}$.
20. $4\frac{1}{6}$, $6\frac{1}{4}$, $2\frac{1}{12}$.

VII. Analysis Involving Fractions.

137. I. Given the value of a number of integral units, to find the value of another number of units.—We may proceed in the same way as in Art. 95, where only integers are involved. In all cases it is important to save labor by the use of cancellation wherever possible.

Ex. 1. If 7 chickens cost $\$4\frac{3}{8}$, what will 16 chickens cost?

Cost of 7 chickens = $\$4\frac{3}{8}$ or $\$\frac{35}{8}$.
Cost of 1 chicken = $\frac{1}{7}$ of $\$\frac{35}{8} = \$\frac{5}{8}$.
Hence, cost of 16 chickens = 16 times $\$\frac{5}{8}$ or \$10.

It may be that the value of a number of fractional units is required.

Ex. 2. If 10 acres of land cost \$1124, what will $5\frac{3}{4}$ acres cost?

Cost of 10 acres = \$1124.
Cost of 1 acre = $\$\frac{1124}{10}$.
Cost of $5\frac{3}{4}$ acres = $\$\frac{1124}{10} \times \frac{23}{4} = \$646\frac{3}{10}$, *Result.*

EXERCISE 49.

1. If 3 pounds of candy cost $8\frac{1}{4}$ cents, what will 8 pounds cost at the same rate? $3\frac{2}{5}$ pounds?

2. If 7 pairs of boots cost $\$23\frac{5}{8}$, what will be the cost of 12 pairs?

3. How many yards of cloth will be required for 16 coats, if 11 coats can be cut from $34\frac{5}{8}$ yards?

4. How many tons of hay will a horse require in 365 days, if he eats $1\frac{9}{10}$ tons in 133 days?

5. When \$355 will buy 15 acres of land, what are $8\frac{1}{2}$ acres worth?

6. A bar of metal 5 feet long weighs $35\frac{5}{8}$ pounds. What will a similar bar $3\frac{3}{7}$ feet long weigh?

7. If a load of 40 bushels of lime weigh 3210 pounds, what will be the weight of a like load containing $72\frac{2}{3}$ bushels?

8. If there are $404\frac{1}{4}$ cubic inches in 7 quarts of milk, how many cubic inches in $42\frac{2}{9}$ quarts?

9. In 15 links there are $118\frac{4}{5}$ inches. How many inches in $41\frac{1}{9}$ links?

10. If \$57 buy 9 rolls of cloth, how many rolls will $\$53\frac{1}{5}$ buy?

11. If 7 loads of lumber cost $\$95\frac{1}{3}$, how many loads can be bought with $\$162\frac{1}{2}$?

138. II. Given the value of a number of fractional units, to find the value of a another number of other units.

In this case the process consists in brief in *finding the value of a single fractional unit, then finding the value of a single integral unit, then finding the value of any number of other units.*

Ex. 1. The value of $\frac{2}{5}$ of a steamboat is \$12000. What is the value of the entire steamboat?

Value of $\frac{2}{5}$ of the vessel = \$12000.
Value of $\frac{1}{5}$ of the vessel = \$6000.
Value of $\frac{5}{5}$, or the whole of the vessel = \$30000, *Result.*

Ex. 2. If $\frac{3}{4}$ of an acre of land is worth \$72, what is the value of $\frac{5}{12}$ of an acre?

Value of $\frac{3}{4}$ acre = \$72.
Value of $\frac{1}{4}$ acre = \$24.
Value of $\frac{4}{4}$ of an acre, or of 1 acre = \$96.
$\therefore$ Value of $\frac{5}{12}$ acre = $\$96 \times \frac{5}{12} = \$40$, *Result.*

Ex. 3. A farmer sold $\frac{1}{3}$ of his flock of chickens, then $\frac{3}{4}$ of the remainder, and found that he had 20 chickens left. How many chickens did he have originally?

$1 - \frac{1}{3} = \frac{2}{3}$, the part of the flock left after the first sale.
$\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$, the part disposed of in the second sale.
$\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$, the part of flock left unsold.
If $\frac{1}{6}$ of the flock = 20 chickens,
$\frac{6}{6}$, or the whole of the flock = 120 chickens, *Result.*

EXERCISE 50.

1. If $\frac{3}{7}$ of a book contains 234 pages, how many pages in the entire book?

2. If $\frac{4}{9}$ of a gentleman's salary is \$3800, what is the whole salary? What is $\frac{3}{4}$ of it?

3. When a man owning $\frac{3}{5}$ of a vessel sells his portion for \$67318, what is the value of the rest?

4. Nine-tenths of a certain journey is 4770 miles; how long is the entire journey?

5. If $2\frac{2}{3}$ yards of cloth cost \$$11\frac{1}{5}$, what will $7\frac{2}{9}$ yards of the same cloth cost?

6. If $5\frac{1}{3}$ boxes of soap cost \$52, what will be the cost of $4\frac{4}{15}$ boxes?

7. When $\frac{3}{4}$ of a mile of fence can be built for \$$17\frac{1}{3}$, what will $4\frac{1}{8}$ miles cost?

8. If $\frac{4}{11}$ of a gallon contains 84 cubic inches, what will $\frac{13}{14}$ of a gallon contain?

9. If \$$25\frac{1}{5}$ purchase $3\frac{1}{5}$ cords of wood, how many cords will \$$184\frac{1}{2}$ secure?

10. If 75 hairs of a certain length weigh $3\frac{1}{2}$ drams, how many hairs of the same size will be required to weigh $21\frac{3}{4}$ drams?

11. When \$$57\frac{3}{4}$ will buy $5\frac{1}{7}$ acres of land, what are $7\frac{3}{7}$ acres worth?

12. A bar of iron $4\frac{3}{8}$ feet long weighs $26\frac{1}{4}$ pounds. What will a similar bar of iron $11\frac{1}{9}$ feet long weigh?

13. If for $7\frac{1}{3}$ days of labor a man receive \$$6\frac{7}{8}$, what will be due him for $\frac{4}{5}$ of 1 day?

14. If \$$454\frac{3}{10}$ will buy $11\frac{4}{5}$ acres of land, how many dollars will be required to buy $4\frac{2}{7}$ acres?

15. A boy loses $\frac{1}{4}$ of his marbles and gives $\frac{1}{3}$ away. He still has 20. How many had he at first?

16. After selling $\frac{2}{9}$ of my farm and giving $\frac{1}{6}$ to my son, I have 220 acres left. How many acres did I sell?

17. One day I read $\frac{1}{5}$ of the pages of a book; the next day $\frac{1}{3}$, and the next day $\frac{3}{10}$. There still remained 60 pages. How many pages in the book?

18. A pole is $\frac{1}{4}$ white, $\frac{2}{7}$ red, $\frac{5}{14}$ blue, and the rest, which is 12 feet, is in the ground. How long is the pole and how many feet are above ground?

19. A lad loses $\frac{2}{5}$ of his marbles and then gives $\frac{1}{3}$ of the remainder away. He finds that 12 remain. How many had he at first? How many did he give away?

20. After selling $\frac{2}{7}$ of my farm, I gave $\frac{3}{5}$ of the remainder to my son and have 142 acres left. How many acres had I at first? How many did I give to my son?

21. One day I read $\frac{1}{6}$ of a book; the next, $\frac{1}{2}$ of the remainder and had 155 pages left. How many pages in that book?

22. A gentleman left $\frac{1}{3}$ of his property to his wife; $\frac{1}{4}$ of the remainder to his son; $\frac{1}{4}$ of what still remained to his daughter, who received \$1575. What was the value of the estate?

139. Synopsis of Principles Relating to Fractions.—It will be a useful exercise for the pupil to collect and tabulate the essential principles relating to fractions. Thus, in outline,

FIRST PRINCIPLES OF FRACTIONS.

A. *If the numerator and denominator of a fraction be both multiplied, or both divided, by the same number, the value of the fraction is not changed.*

B. *Multiplying the denominator of a fraction by a given number divides the value of the fraction by that number.*

C. *Dividing the denominator of a fraction by a given number multiplies the value of the fraction by the same number.*

TRANSFORMATIONS OF FRACTIONS.

I. *To reduce a mixed number to an improper fraction, multiply the whole number by the denominator of the fraction, add the numerator to the product, and place the sum over the denominator.*

II. *To reduce an improper fraction to a mixed number, divide the numerator by the denominator, and to the quotient annex the remainder placed over the denominator.*

III. and IV., Etc., Etc.

PROCESSES WITH FRACTIONS.

Etc., Etc.

EXERCISE 51.

REVIEW.

Which is the greater and how much?

1. $\frac{2}{3}$ of $5\frac{1}{4}$ or $\frac{3}{5}$ of $4\frac{3}{16}$?

2. $\dfrac{\frac{3}{7}\text{ of }11\frac{1}{5}}{5\frac{1}{3}}$ or $\dfrac{7\frac{1}{6}-3\frac{3}{4}}{1\frac{3}{7}\times 2\frac{5}{8}}$?

Find the sum of:

3. $\frac{5}{13}+\frac{11}{26}+\frac{20}{39}+\frac{31}{52}$.

4. $21\frac{7}{9}+4\frac{1}{3}+2\frac{2}{5}+10\frac{1}{15}$.

5. $6\frac{1}{3}+7\frac{3}{7}+5\frac{1}{2}+4\frac{3}{14}$.

6. $11\frac{4}{11}+3\frac{1}{3}+4\frac{1}{5}+\frac{14}{15}+42\frac{23}{33}$.

7. $7\frac{5}{8}+4\frac{1}{12}+1\frac{23}{24}+\frac{7}{16}+7\frac{2}{3}+4\frac{5}{6}$.

8. $5\frac{3}{5}+4\frac{5}{6}+2\frac{1}{8}+17\frac{4}{15}+23\frac{13}{40}+\frac{13}{24}$.

9. $1\frac{13}{21}+\frac{13}{49}+3\frac{19}{35}+7\frac{19}{63}+\frac{4}{7}+5\frac{14}{15}+\frac{19}{45}$.

Find the value of each:

10. $7\frac{3}{11}-4\frac{5}{6}$.

11. $5-3\frac{11}{12}$.

12. $6\frac{1}{3}-5\frac{11}{15}$.

13. $10-8\frac{73}{101}$.

14. $18\frac{7}{10}-9\frac{16}{25}$.

15. $\frac{12}{13}-\frac{13}{19}$.

16. $3\frac{7}{52}-1\frac{113}{117}$.

17. $1\frac{7}{138}-\frac{200}{253}$.

18. $7\frac{103}{124}-5\frac{131}{155}$.

19. $8\frac{1}{111}-1\frac{45}{259}$.

20. $\frac{231}{235}-\frac{173}{893}$.

21. $5\frac{97}{210}-1\frac{137}{270}$.

22. $1\frac{2}{3}-\frac{5}{6}+7\frac{5}{8}-3\frac{7}{12}$.

23. $9\frac{2}{5}+11\frac{7}{10}-8\frac{4}{15}-3\frac{1}{5}$.

24. $15\frac{1}{4}-6\frac{2}{7}-1\frac{5}{14}-4\frac{1}{2}$.

25. $19-4\frac{2}{5}-6\frac{1}{10}-2\frac{13}{15}-5\frac{1}{4}+\frac{11}{12}$.

26. $15\frac{1}{2}+14\frac{3}{11}-4\frac{1}{5}+1\frac{7}{10}-3\frac{14}{55}+\frac{7}{22}$.

27. $4\frac{1}{6}-1\frac{2}{3}+(1\frac{1}{5}-\frac{1}{2})$.

28. $6\frac{1}{4}-(3\frac{1}{3}-1\frac{3}{4})+1\frac{5}{6}$.

29. $11+(4\frac{1}{5}+1\frac{2}{3})-12\frac{7}{15}$.

30. $(8\frac{1}{7}-3\frac{1}{3})-(9\frac{3}{14}-7\frac{1}{6})$.

31. $15\frac{13}{24}-(1\frac{7}{8}+7\frac{1}{6}-5\frac{7}{9})$.

Multiply together:

32. $7\frac{4}{5}\times 1\frac{2}{13}$.

33. $5\frac{1}{4}\times 4\frac{2}{7}$.

34. $9\frac{1}{7}\times 1\frac{11}{24}$.

35. $\frac{7}{11}\times 4\frac{3}{5}\times 1\frac{3}{7}$.

36. $9\frac{3}{5}\times 5\frac{5}{12}\times 6\frac{1}{13}$.

37. $15\frac{3}{4}\times 13\frac{1}{3}\times 1\frac{7}{15}$.

Divide:

38. $9\frac{3}{4}$ by $5\frac{5}{8}$.

39. $10\frac{4}{5}$ by $11\frac{4}{7}$.

40. $16\frac{5}{7}$ by $6\frac{4}{21}$.

41. $\frac{3}{8}$ of $19\frac{1}{5}$ by $3\frac{23}{25}$.

42. $19\frac{5}{7}$ by $(5\frac{1}{2}\times 3\frac{2}{7})$.

43. $15\frac{2}{11}\times 5\frac{3}{5}$ by $5\frac{41}{44}$.

What part of:

44. $2\frac{3}{4}$ is $\frac{1}{5}$?

45. $7\frac{1}{5}$ is $1\frac{23}{25}$?

46. $10\frac{1}{2}$ is $\frac{7}{8}$?

47. $\frac{4}{9}$ of $6\frac{3}{7}$ is $\frac{2}{3}$ of $\frac{5}{7}$?

48. $\frac{7}{12}$ of $7\frac{1}{5}$ is $\frac{17}{15}$ of $4\frac{4}{15}$?

49. $\frac{9}{11}$ of $15\frac{2}{5}$ of $7\frac{1}{14}$ is $19\frac{1}{8}$?

Simplify:

50. $\dfrac{5\frac{1}{4}}{3\frac{1}{2}}$.

51. $\dfrac{3\frac{2}{11}}{9\frac{1}{3}}$.

52. $\dfrac{7\frac{7}{10}}{9\frac{1}{6}}$.

53. $\dfrac{17\frac{5}{7}}{44\frac{9}{14}}$.

54. $\dfrac{7-3\frac{11}{12}}{32\frac{5}{8}}$.

55. $\dfrac{71\frac{1}{4}}{17\frac{1}{6}+2\frac{5}{8}}$.

56. $\dfrac{5}{6\frac{1}{4}}\times\dfrac{3}{4\frac{1}{2}}$.

57. $\dfrac{7\frac{1}{3}}{1\frac{5}{6}}\div\dfrac{4\frac{2}{3}}{9\frac{1}{10}}$.

58. $\dfrac{4\frac{7}{8}}{5\frac{1}{5}}\div\dfrac{8\frac{1}{3}}{11\frac{1}{2}}$.

59. $\dfrac{27}{37\frac{4}{5}}\times\dfrac{87\frac{3}{5}}{98\frac{1}{8}}\times\dfrac{\frac{7}{8}}{2\frac{1}{2}}\times\dfrac{89\frac{5}{11}}{128}$.

60. $\frac{37}{65}\times\frac{17}{296}\times\dfrac{76}{93\frac{1}{2}}\times\dfrac{44}{1\frac{2}{17}}\times\frac{7}{11}$.

61. $\frac{3}{5}$ of $7\frac{1}{7}+\frac{5}{8}$ of $9\frac{5}{11}-\frac{4}{11}$ of $7\frac{1}{14}$.

62. $\frac{7}{10}$ of $6\frac{2}{3}-\frac{5}{9}$ of $9\frac{3}{4}+\frac{8}{15}$ of $4\frac{11}{16}+3$.

63. $1+\cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}}$.

64. $2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{6}}}$.

65. $3+\cfrac{1}{4+\cfrac{1}{1+\frac{1}{4}}}$.

66. $1+\cfrac{1}{6+\cfrac{1}{2+\cfrac{1}{6+\frac{1}{2}}}}$.

67. $3+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3+\frac{1}{3}}}}$.

68. $4+\cfrac{1}{2+\cfrac{1}{3+\cfrac{7}{6-\frac{2}{5}}}}$.

69. $2-\cfrac{4}{2+\cfrac{3}{3+\cfrac{1}{5-\cfrac{6}{1+\frac{1}{2}}}}}$.

70. If one man earn \$$2\frac{1}{4}$ in one day, what will 70 men earn in $10\frac{3}{7}$ days?

71. If the dividend is $41\frac{1}{3}$ and the quotient $21\frac{6}{13}$, what is the divisor?

72. If the product is $89\frac{3}{8}$ and the multiplicand is $9\frac{5}{8}$, find the multiplier.

73. If the dividend is $54\frac{1}{10}$, the quotient $11\frac{7}{8}$, and the remainder $2\frac{4}{5}$, find the divisor.

74. When the divisor is $8\frac{1}{6}$, the quotient $9\frac{1}{7}$, and the remainder $4\frac{1}{4}$, what is the dividend?

75. What must $\frac{7}{11}$ of $6\frac{1}{17}$ be multipled by to produce $\frac{25}{38}$ of $8\frac{1}{7}$ of $6\frac{1}{15}$?

76. Three men, A, B, C, agree to do a piece of work for \$100, sharing equally. But upon completion it is found that A has done $\frac{2}{5}$ of it, and B $\frac{11}{40}$ of it. What part did C do, and how much money ought he have?

77. If $\frac{3}{11}$ of a piece of work is done in $22\frac{1}{2}$ days, how much of it will be done in $26\frac{3}{5}$ days?

78. The sum of $\frac{2}{5}$ and $\frac{1}{4}$ of a certain number is $170\frac{3}{10}$. What is the number?

79. The difference between $\frac{3}{7}$ and $\frac{2}{5}$ of a number is $26\frac{4}{21}$. Find the number.

80. A school of 150 pupils has only $\frac{2}{3}$ as many boys as girls. How many boys are there in the school?

81. There are 126 green and white balls in a box, but the number of green balls is $\frac{2}{5}$ the number of white ones. How many are there of each?

82. A farmer put his 1000 sheep into two pastures, and in one pasture he put $\frac{2}{3}$ as many as in the other. How many sheep are there in each?

83. After a man has walked $4\frac{1}{2}$ hours on a journey of $31\frac{1}{2}$ miles, he finds that he has traveled $\frac{2}{5}$ of the number of miles remaining. How fast is he walking?

84. How many tons of ore must be taken from the mine, so that after a loss of $\frac{2}{5}$ in roasting, and $\frac{6}{13}$ of the remainder in smelting, there may be 210 tons of pure metal left?

85. A farmer sowed $\frac{1}{5}$ of a field in oats, $\frac{5}{9}$ of the remainder in buckwheat, and planted $\frac{3}{4}$ of what was left in potatoes; there still remained 8 acres for grass. How many acres in the whole field?

86. There are 5 farms marked A, B, C, D, and E, respectively. Farm A contains $253\frac{1}{3}$ acres; farm B contains half as much as A; farm C is half as large as B, and so on to E. How many in E, and how many in them all together?

87. Divide $\frac{2}{3}$ of $\frac{4}{5}$ of $1\frac{2}{3}$ of $\frac{3}{4}$ of $6\frac{1}{4}$ by $\frac{4}{7}$ of $\frac{9}{13}$ of $\frac{7}{12}$ of $7\frac{1}{2}$.

88. From $27\frac{9}{10}$ acres I sell to one man $5\frac{5}{6}$ acres at $\$70\frac{2}{7}$ an acre; to another man $7\frac{1}{2}$ acres at $\$85\frac{3}{5}$ an acre; and to a third man the remainder at $\$92\frac{2}{3}$ an acre. Find proceeds of entire tract.

89. In exchange for $7\frac{1}{4}$ dozen eggs at $23\frac{1}{3}$ cents a dozen, and $15\frac{1}{4}$ pounds of butter at $27\frac{1}{2}$ cents a pound, a man takes oats at $5\frac{5}{6}$ cents a quart. How many quarts will he receive?

90. A merchant bought 3 pieces of silk for $\$655\frac{1}{2}$. The first contained $30\frac{7}{20}$ yards, the second $42\frac{1}{2}$ yards, and the third $47\frac{3}{8}$ yards. He wishes to sell the silk so as to gain $\frac{3}{8}$ of the cost. At what price must he sell it per yard? Find selling price of each piece.

91. A man has $\frac{1}{3}$ of his property invested in real estate, $\frac{3}{7}$ of the remainder in stocks, $\frac{4}{5}$ of what is still remaining in machinery, and the residue, which is \$3500, in the bank. What is the value of his entire property?

92. Simplify $\frac{3}{7}$ of $\dfrac{4\frac{5}{8}}{12\frac{1}{3}}$ of $\dfrac{3\frac{4}{11}}{11\frac{5}{7}} \div 3\frac{3}{11}$.

93. Divide $1\frac{1}{3} \times \dfrac{3\frac{3}{5}}{\frac{5}{6} \times 1\frac{2}{9}}$ by $\dfrac{7\frac{3}{5} - 4\frac{1}{2}}{\frac{1}{14} \div \frac{6}{35}} \times \dfrac{7\frac{3}{5}}{31\frac{5}{9}}$.

94. The sum of $\dfrac{3\frac{2}{5} \times \frac{5}{6}}{4\frac{7}{8} \div 9\frac{3}{4}}$ and $\dfrac{\frac{4}{7} \text{ of } \frac{7}{12}}{\frac{1}{3} \text{ of } 2\frac{1}{4}}$ is how many times their difference?

95. The product of 3 numbers is $453\frac{1}{3}$; two of them are $5\frac{5}{8}$ and $11\frac{1}{3}$; find the third.

96. If $\frac{5}{8}$ of a ton of coal cost $\frac{2}{3}$ of $\$9\frac{3}{8}$, what will $\frac{7}{10}$ of a ton cost? What will $9\frac{1}{10}$ tons cost?

97. If 6 be added to both terms of the fraction $\frac{9}{16}$, is the fraction increased or diminished, and how much?

98. Same question, if 6 be subtracted from both terms of same fraction.

99. There are 3 numbers, the least of which is $7\frac{7}{11}$. The second is $3\frac{3}{4}$ times as large as the first, and the third $3\frac{2}{3}$ times as large as the second. Find their sum.

100. A man sold $\frac{3}{8}$ of his farm to one neighbor and $\frac{5}{11}$ of it to another. There remained 90 acres. How many acres in the farm at the beginning?

101. If $3\frac{3}{4}$ yards of silk cost $\$10\frac{1}{2}$, what will $\frac{2}{3}$ of $8\frac{3}{5}$ yards cost at the same rate?

102. If a man saw $\frac{3}{4}$ of $4\frac{2}{3}$ cords of wood in a day, how many cords will he saw in $5\frac{3}{5}$ days?

103. If a man walk $28\frac{4}{7}$ miles in one day, how many days will he require to walk $177\frac{3}{8}$ miles?

104. Find the cost of $8\frac{2}{3}$ yards of carpet when $3\frac{1}{4}$ yards cost $\$10\frac{3}{5}$.

105. If $\frac{5}{8}$ of a ton of coal cost \$6, how many tons can be bought for $\$67\frac{1}{2}$?

106. If 4 be subtracted from both terms of the fraction $\frac{15}{23}$, is its value increased or diminished, and how much?

107. A horse and cow were bought for \$160, and the cow cost $\frac{3}{5}$ as much as the horse. Find the cost of each.

108. The sum of $\frac{2}{3}$ and $\frac{4}{5}$ of a certain number is $388\frac{2}{3}$. Find the number. Find the difference between $\frac{2}{3}$ and $\frac{4}{5}$ of it.

109. If a man can do a piece of work in 12 days and a boy can do it in 18 days, what part can the man do in one day? In 5 days? What part can the boy do in one day? In 8 days? What part can they both do in one day? In 4 days? How many days will they require to do it all, working together?

110. If a man can mow a field in 15 days and his son can do it in 27 days, answer the same seven questions about them.

111. Mr. A. can dig a certain ditch in 6 days, Mr. B. in 10 days, and Mr. C. in 15 days. Find the number of days required by each pair of men working together. Also the number required by all three together.

112. One pipe, X, can empty a cistern in 8 hours; Y, in 9 hours; Z, in 12 hours. The cistern is full and all pipes are open, how long will be required for them to empty the cistern?

CHAPTER X.

DECIMAL FRACTIONS.

140. Definitions.—If any integral unit (as one apple) be divided into 10 equal parts, each of these parts is called one-tenth. If one-tenth be divided into 10 equal parts, each of these parts is called one-hundredth. Similarly, from one-hundredth we form one-thousandth, etc. A set of fractional units so obtained is called a set of decimal units.

A **decimal fraction**, a decimal, is a fraction whose denominator is 10, or 100, or 1000, or some other power of 10.

A **mixed decimal** is a number composed partly of integers and partly of decimals.

141. Advantage in the Use of Decimal Fractions. Notation.—Since each decimal unit is one-tenth as great as the decimal unit which precedes it, a set of decimal fractions can be expressed in a simplified way similar to that used in expressing integral numbers in the decimal scale. This is done by the use of what is called the *decimal point*, and letting the position of each figure to the right of the decimal point determine the size of the decimal unit which this figure represents.

Thus, instead of 29 yds. + $\frac{3}{10}$ yds. + $\frac{7}{100}$ yds. + $\frac{4}{1000}$ yds. + $\frac{5}{10000}$ yds., we write 29.3745 yds.

Thus the labor of writing the denominators of the various fractions is saved, since the denominator of each decimal figure is determined by the decimal point and the number of figures between the decimal point and the figure considered.

Thus, in the above illustration, the unit represented by the figure 4, or *thousandths*, is determined by the decimal point and the two figures 3 and 7 intervening between the decimal point and the 4.

It should be observed that the source of this advantage lies in the fact that each figure is put to several uses. Thus, 3 not only expresses the number of tenths, but it also helps to determine the decimal denomination or local value of 7, 4, and 5, and hence serves four purposes at once.

This economy in representing fractions leads to other advantages in operating with the fractions after they are expressed in the decimal notation.

142. Illustrations of Decimal Fractions.—The most familiar illustration of decimal fractions is found in the money used in the United States. The primary unit, one dollar, is divided into ten equal parts called *dimes*, each dime is divided into ten equal parts called *cents*, and each cent into ten equal parts called *mills*. Thus, 12 dollars, 8 dimes, 6 cents, and 5 mills can be briefly expressed by the aid of the decimal notation as $12.865.

The ease and rapidity with which calculations can be made when money is expressed on a decimal scale will be appreciated by the student when he comes to reckon with money expressed in some other way, as, for instance, by pounds, shillings, and pence, as in English money.

So great are the advantages of subdividing a unit by the decimal method that this method is being applied more and more widely wherever possible. Thus, engineers divide the unit of length, the foot, not into inches, but into tenths and hundredths. Astronomers frequently divide the year decimally, indicating, for instance, April 1, 1879, by 1879.25. They also sometimes divide a degree of longitude decimally, instead of into degrees and minutes, using, for instance, 324.5° for 324° 30′. The United States Treasury Department uses tenths of a foot, pound, etc., instead of the ordinary fractions.

143. Metric System.—An entire system of weights and measures, based on decimal divisions of the fundamental units, has been devised and is in use in all civilized countries except Great Britain and the United States.

A unit of length is taken, called the *meter*, which is divided into tenths called *decimeters*; each decimeter is subdivided into ten equal parts called *centimeters*, etc. Similarly the unit of weight, the *gram*, is divided by the decimal system, as also are the units of area and volume, the *are* and the *stere*. This system of decimal units will doubtless come, in time, to be used by the entire civilized world. See page 326.

144. Notation and Numeration of Decimals.—The positional system of expressing fractions by the aid of the decimal point has been explained in Art. 15. The following table will enable the pupil to give readily the decimal unit which each figure in a decimal represents.

ORDER OF UNIT.	Ten millions.	Millions.	Hundred-thousands.	Ten-thousands.	Thousands.	Hundreds.	Tens.	Units.	Decimal Point.	Tenths.	Hundredths.	Thousandths.	Ten-thousandths.	Hundred-thousandths.	Millionths.	Ten-millionths.	Hundred-millionths.	Billionths.
NUMBER.	7	3	4	5	1	2	7	8	.	4	6	5	7	8	3	1	0	5
PLACE.	8TH.	7TH.	6TH.	5TH.	4TH.	3RD.	2ND.	1ST.		1ST.	2ND.	3RD.	4TH.	5TH.	6TH.	7TH.	8TH.	9TH.
	Integers.									Decimals.								

145. Reading Decimals.—The most convenient way of reading decimals is to express each decimal number in terms of the smallest decimal unit and read the number of such units.

Thus, to read 0.37, instead of reading three tenths and seven hundredths, we express the tenths as thirty hundredths, and read the entire decimal fraction as 37 hundredths. Similarly, the decimal fraction expressed in the above table (Art. 144), viz.: 0.465,783,105, is read 465 millions 783 thousands 105 billionths. Hence, in general, *read the decimal as if it were a whole number, and give it the name of the last decimal place.*

In reading *whole numbers* never use "and," but in reading a *mixed decimal put "and" in place of the decimal point.*

Thus, 462 reads "four hundred sixty-two."
4.062 reads "four *and* sixty-two thousandths."

146. Writing Decimals.—Similarly, to express in figures a decimal which is given in general language, *express the numerator in figures, and then fix the decimal point so that the name of the last figure shall express the denomination of the given decimal.*

Ex. 1. Express in figures "four hundred sixty-two thousandths."

We write 462 and place the decimal point immediately to the left of the 4, since 2 must come in the third or thousandths place, and obtain .462, *Result.*

Observe that four hundred *and* sixty-two thousandths would be written 400.062.

EXERCISE 52.

Express correctly as decimal fractions.

1. Forty-six hundredths.
2. Ten and sixteen hundredths.
3. Seven and fifty-one thousandths.
4. Thirty-six millionths.
5. Two hundred twelve hundred-thousandths.
6. Five and five millionths.
7. Seventy-five and forty-two ten-thousandths.
8. Seven hundred six thousandths.
9. Seven hundred and six thousandths.
10. One thousand five hundred and one tenth.
11. Four hundred and four hundred one thousandths
12. Two hundred forty-one and four hundred twelve millionths.
13. Eighty-nine and ninety-eight hundred-millionths.
14. One thousand and one thousandth.
15. Three thousand and three millionths.
16. Sixteen ten-millionths.
17. $\frac{7}{100}$; $\frac{51}{10000}$; $1\frac{2}{1000}$; $31\frac{701}{1000000}$; $25\frac{405}{1000000}$.
18. $3\frac{81}{1000}$; $7\frac{6}{10}$; $71\frac{17}{100}$; $45\frac{48}{1000000}$; $9\frac{9}{10}$; $9\frac{9}{100}$.

Read the following decimal fractions:

19. 0.78; 1.071; 20.05; 275.572; 0.4758.
20. 0.705; 0.0102; 100.0301; 51.0007; 0.003001.
21. 300.001; .301; 6175.0214; 5001.005001.

147. Primary Processes with Decimals.—The simplicity of the decimal system of fractions is such that certain elementary methods of operating with them arise immediately from the notation.

1. Shifting the decimal point one place to the right increases the value of the decimal tenfold; shifting it two places to the right increases its value one hundredfold, etc.

Thus, $0.063 = 10 \times .0063$.

For by moving the decimal point one place to the right, each figure in the decimal is made to express a number ten times as great as it did at first.

2. Shifting the decimal point one place to the left decreases the value of the decimal to one-tenth of what it was; two places, to one-hundredth, etc.

Thus, $.0055 = 0.055 \div 10$.

3. Any number of zeroes may be annexed to the right of a decimal without changing its value.

Thus, $0.3 = 0.30 = 0.300$, etc.

OPERATIONS WITH DECIMALS.

148. I. Addition of Decimals.—If the numbers to be added be so arranged that their decimal points shall be in the same column, all the decimal units of the same order, as tenths, hundredths, etc., will be in the same column, and may be added by columns.

Ex. Add \$5.69, \$100.257, \$37.015.

OPERATION.

```
$  5.69
 100.257
  37.015
$142.962, Sum.
```

EXPLANATION.

Arranging the numbers so that the decimal points are in the same column, we begin at the right hand, or thousandths, column to add. 7 thousandths + 5 thousandths make 12 thousandths, or 1 hundredth and 2 thousandths. Setting down the 2 thousandths, we carry 1 to the hundredths column, and continue the work, "carrying" wherever necessary, just as in the case of the addition of integers in the decimal system.

Hence, in general, to add decimals, *write the numbers so that the decimal points shall be in the same column; begin with the right-hand column and add; place a decimal point between the units and tenths of the result.*

Hence, in the addition of decimal fractions we are saved the labor of reducing fractions to fractions having a common denominator, which is necessary in the addition of common fractions.

Learn to add, subtract, multiply and divide decimals; Learn how to change a decimal to a common fraction (par 150 p 148) Learn how to change a fraction to a decimal (par 153 p 150)

EXERCISE 53.

Add:

1.	*2.*	*3.*
\$27.05	5.571 inches.	1.0071 square yards.
123.74	93.428 inches.	.0382 square yard.
6.735	.96 inch.	5.917 square yards.
2.045	.407 inch.	41.0328 square yards.
38.7	8.14 inches.	17.51 square yards.

4. \$57.13 + \$7.15 + \$0.61 + \$70.09.
5. \$125.74 + \$307.06 + \$51.075 + \$6.305.
6. 8.08 + 1.001 + 101.0101 + 3040.1304 + 0.1345.
7. 270.01 + 31.0031 + 0.0073 + 25 + 43.0106 + 4.008.
8. 27.35 mi. + 4.701 mi. + 34.375 mi. + 8.0704 mi.
9. 7.9324 + 79.324 + .079324 + 7932.4 + 0.79324.

149. II. **The subtraction of decimals** is similar in method to the addition of decimals; that is, *write the subtrahend under the minuend, so that the decimal points shall be in the same column; begin at the right hand to subtract.*

Ex. At six o'clock the mercury in a certain barometer stood at 39.3 inches; at 10 o'clock the mercury in the same barometer stood at 39.215 inches. How many inches had it fallen?

```
39.300 inches.
39.215 inches.
  .085 inch Difference.
```

EXERCISE 54.

	1.	*2.*	*3.*	*4.*
From	\$7.35	\$18.196	71.44 inches	65.03
take	3.21	4.75	38.67 inches	47.903

	5.	*6.*	*7.*	*8.*
From	51.7	301.04	19.4003	3.41
take	4.52	79.5281	9.876	2.5807

9. 0.54 − 0.37.	*13.* 0.185 − 0.0917.	*17.* 1. − 0.1.
10. 1.28 − 1.1.	*14.* 0.042 − 0.0318.	*18.* 0.01 − 0.003.
11. 9.53 − 7.99.	*15.* 70.07 − 6.408.	*19.* 10 − .001.
12. 3.01 − 2.714.	*16.* 301.5 − 30.105.	*20.* 2 − 0.010203.

21. Find the difference between \$75.08 and \$87.85.

22. What is the difference between 3.141592 and 3.142857?

23. From an account of \$175.43, a man drew \$46.95. How much remained?

24. Upon three days a gentleman deposited in a bank \$27.54, \$35.97, and \$71.16, and on the fourth day withdrew \$49.73. How much remained?

25. Find the difference between six hundred twenty-eight thousandths, and four hundred *and* sixty-nine thousandths.

150. III. **Multiplication of Decimals.**—To obtain a method of multiplying one decimal number by another, we shall take an example and work it first by the method of common fractions.

Ex. Multiply 3.372 by 2.28.

Expressing the decimal fractions as common fractions, we have,

$$3.372 \times 2.28 = \frac{3372}{1000} \times \frac{228}{100} = \frac{3372 \times 228}{100000} = \frac{768816}{100000}$$

$$= 7.68816, \textit{ Product.}$$

Hence, *the number of decimal places in the product is equal to the number of zeroes in the two denominators, that is, to the number of decimal places in the multiplier and multiplicand taken together.*

Hence, the above multiplication might have been performed as follows:

```
   3.372
    2.28
   26976
   6744
  6744
 7.68816, Product.
```


Or, in general, *multiply as in whole numbers; point off as many decimal places from the right in the product as there are decimal places in both multiplier and multiplicand taken together, prefixing zeroes to the product if necessary.*

Ex. Multiply 3.0125 by .00104.

```
   3.0125
   .00104
   120500
  30125
 .003133, Product.
```

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It is to be observed that as compared with the multiplication of common fractions, in the multiplication of decimals the multiplication of the two denominators is abbreviated into a mere placing of the decimal point in the product.

EXERCISE 55.

	1.	*2.*	*3.*	*4.*	*5.*
Multiply	8.3	3.5	7.6	10.4	.88
	5.	1.2	.05	0.15	.7

6. Multiply each of the following by 4:
1.5; 2.4; 12; 63.5; 23.14; 75.007.

7. Multiply each of the following by 3.6:
2.5; 13; .07; 1.05; 4.005; 1.0008.

8. Multiply each of the following by 2.05:
0.32; 0.036; 10.08; 200.04; 35.1; 71.09.

9. 2×0.2.	*11.* $1 \times .01$.	*13.* 100.1×1.001.
10. $20 \times .02$.	*12.* $100 \times .001$.	*14.* $3.003 \times .03003$.

15. $5 \times .4 \times 30 \times 1.8$.	*18.* $5.3 \times 2.01 \times 0.46$.
16. $1.1 \times 4 \times 5.5 \times .02$.	*19.* $27.5 \times 1.6 \times .014$.
17. $0.24 \times 25 \times .004 \times 10$.	*20.* $5.8 \times .025 \times 1.003$.

21. Find the value of 76.235 acres of land at \$51.24 an acre.

22. Find the cost of 128.4 yards of cloth at \$2.125 a yard.

23. Find the weight of 26.735 cubic yards of earth if one cubic yard weighs 0.76 of a ton.

24. 61.038 tons of hay are worth how much at $20.25 a ton?

25. What is the simplest method of multiplying by 10? By 100? By 1000?

26. What is the simplest way of multiplying by 0.1? By .01? By .001?

151. IV. **Division of Decimals** may be performed directly, but it is of advantage first to multiply the divisor and dividend by such a number (10, 100, 1000, etc.) as will remove the decimal point from the divisor. This will leave the value of the quotient unchanged (See A, Art. 112). The multiplications required are performed by shifting decimal points (See Art. 147).

Ex. 1. Divide .0221 by .013.

If we multiply both divisor and dividend by 1000, that is, shift the decimal point three places to the right in each of them, the value of the quotient will be unchanged and the divisor will be an integer. Hence, we have

OPERATION.

```
13)22.1(1.7, Quotient.
   13
    91
    91
```

Since $22.1 = \frac{221}{10}$, we really divide 221 tenths by 13; hence, the quotient is 17 tenths, or 1.7. Hence, it is necessary in each case to mark off as many decimal places in the quotient as there are decimal places in the dividend, or, in general, *move the decimal point in both divisor and dividend as many places to the right as there are decimal places in the divisor; divide as with integers; mark off as many decimal places from the right in the quotient as there are decimal places in the dividend.*

Ex. 2. Divide .004551 by 1.5.

OPERATION.

```
15).045510(.003034, Quotient.
    45
     51
     45
      60
      60
```

EXPLANATION.

As the divisor, 1.5, contains one decimal place, we move the decimal point one place to the right in both divisor and dividend. This will leave the value of the quotient unchanged. Hence, the quotient of .045510 by 15, or .003034, is the required quotient.

152. Abbreviated Cases.—The student may state for himself the abbreviated ways of dividing a number by 10, 100, 1000, etc.; also by .1, .01, etc.

If the divisor be a whole number ending in one or more zeroes, as, for instance, in dividing 16.45 by 7000, it is more convenient to divide first by 1000 by shifting the decimal point in the dividend three places *to the left*, and then dividing by 7, that being the remaining factor of the divisor. Thus,

```
7).01645
  .00235, Quotient.
```

EXERCISE 56.

Divide:

1. 7.5 by 3.	*9.* 24 by .8.	*17.* .4 by 4.
2. .075 by 5.	*10.* .27 by .9.	*18.* 5 by .5.
3. 3.24 by 18.	*11.* 3.6 by 1.2.	*19.* .06 by 6.
4. 25.6 by 32.	*12.* .42 by .14.	*20.* 4.5 by 50.
5. .0121 by 11.	*13.* .063 by .07.	*21.* 10 by .01.
6. .0513 by 27.	*14.* .084 by .12.	*22.* .01 by 100.
7. 4.185 by 15.	*15.* .096 by .008.	*23.* 25 by .05.
8. 2.4 by .3.	*16.* .007 by .025.	*24.* .02 by .005.

25. 16.8 by .021.	*35.* 1.2915 by .041.
26. .945 by 1.35.	*36.* 30.622 by 12.2.
27. 46.5 by .015.	*37.* .203412 by 2.01.
28. 70.8 by .004.	*38.* 63817.2 by .311.
29. 10.11 by .01011.	*39.* 14.17 by .325.
30. .228 by 120.	*40.* 87.098 by 4.07.
31. 700 by 6.25.	*41.* 20.202 by .025.
32. .7 by 625.	*42.* 30030.3 by .0375.
33. 1.405 by 2810.	*43.* .00123 by .075.
34. 4.64 by .145.	*44.* .0456 by .0076.

Divide correctly to four decimal places:

45. 7.101 by 19.	*47.* 101.5 by 30.7.
46. 31.76 by 23.	*48.* .0077 by .058.

The pupil should be required to solve an indefinite number of this kind of examples.

RELATION OF DECIMAL FRACTIONS TO COMMON FRACTIONS.

153. I. **To reduce a decimal fraction to an equivalent common fraction,** it is evidently sufficient to *write the decimal fraction as a common fraction and reduce it to its lowest terms.*

Ex. 1. Express .75 as a common fraction.

$$.75 = \tfrac{75}{100} = \tfrac{3}{4},\ \textit{Result.}$$

Ex. Reduce $.56\frac{1}{4}$ to a common fraction.

$$.56\tfrac{1}{4} = \frac{56\frac{1}{4}}{100} = \frac{\frac{225}{4}}{100} = \frac{225}{400} = \frac{9}{16},\ \textit{Result.}$$

Or,

$$.56\tfrac{1}{4} = .5625 = \tfrac{5625}{10000} = \tfrac{9}{16}.$$

EXERCISE 57.

Reduce each decimal to its equivalent common fraction in its lowest terms:

1. .8.	*8.* .075.	*15.* $.66\frac{2}{3}$.	*22.* 1.4.
2. .25.	*9.* .625.	*16.* $.12\frac{1}{2}$.	*23.* 2.52.
3. .32.	*10.* .875.	*17.* $.62\frac{1}{2}$.	*24.* 5.08.
4. .125.	*11.* .925.	*18.* $.06\frac{1}{4}$.	*25.* 6.4124.
5. .275.	*12.* .006.	*19.* $.16\frac{2}{3}$.	*26.* 10.11375.
6. .375.	*13.* .015.	*20.* $.14\frac{2}{7}$.	*27.* 7.00064.
7. .025.	*14.* $.33\frac{1}{3}$.	*21.* $.08\frac{1}{3}$.	*28.* 1.0875.

154. II. **To reduce a common fraction to a decimal** we may *regard the numerator of the common fraction as an integer, and divide it by the denominator.*

Ex. 1. Reduce $\frac{7}{8}$ to the form of a decimal fraction.

OPERATION.

$$8\overline{)7.000}$$
$$.875,\ \textit{Result.}$$

EXPLANATION.

7 units = 7000 thousandths of a unit; hence, $\frac{1}{8}$ of 7000 thousandths is 875 thousandths, or .875.

Ex. 2. Express $\frac{2}{3}$ as a decimal fraction.

OPERATION.

$$3\overline{)2.0000}$$
$$.6666\ldots$$
or, .6667 —, *Result.*

EXPLANATION.

In this case, no matter how far we continue the division the quotient will not terminate. As it is convenient to terminate the quotient at some place, as the fourth, and as the next figure is 6, or more than half of a unit in the fourth place, we write 7— as the last figure in the quotient, giving .6667 as the quotient.

In general, a decimal which continues to repeat the same figure, or set of figures, is called an *infinite,* or *repeating, decimal.* If, after reducing a fraction to its lowest terms, the denominator contains any factor beside 2 or 5 (which are the only exact divisors of 10, beside 10 and 1), the division of the numerator by the denominator of a common fraction will produce an infinite decimal. Since it has been shown, however (Art. 78), that all figures beyond the sixth and seventh places vanish into insignificance for all practical purposes, it is sufficient ordinarily to let the division terminate at the sixth or seventh place. If the remainder is less than a half, we reject it and annex a plus sign; if equal to or greater than a half, we increase the last figure of the quotient by 1 and annex a minus sign.

155. III. **Comparative value of decimal and common fractions.**

On comparing decimal and common fractions, it will be observed that it is sometimes more advantageous to use one, sometimes the other. In general, decimal fractions have a simpler notation, since their denominators are indicated by a decimal point merely. This leads to special advantages in adding, subtracting, multiplying, and dividing fractions, which have already been pointed out. On the other hand, in particular instances a common fraction is simpler than a decimal fraction, thus $\frac{1}{8}$ is simpler than 0.125, and $\frac{1}{7}$, than 0.142857142.

In general, we may say that as a system for standard use the decimal system of fractions is superior, but that it is advantageous to supplement it by the use of common fractions in certain special cases. Hence, the tendency in practical life is to use systems of decimals (as the metric system) wherever possible, using common fractions in a supplementary way.

EXERCISE 58.

Reduce each common fraction to an equivalent decimal:

1. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{1}{8}$, $\frac{5}{8}$, $\frac{7}{16}$, $\frac{11}{16}$, $\frac{8}{25}$, $\frac{21}{25}$, $\frac{19}{32}$, $\frac{31}{32}$.
2. $\frac{17}{20}$, $\frac{29}{20}$, $1\frac{5}{16}$, $4\frac{18}{25}$, $\frac{15}{64}$, $\frac{31}{64}$, $1\frac{45}{64}$, $3\frac{27}{80}$, $1\frac{71}{80}$.

Find correctly to five decimal places the value of:

3. $\frac{1}{6}$, $\frac{2}{7}$, $\frac{5}{9}$, $\frac{7}{11}$, $\frac{8}{13}$, $\frac{15}{17}$, $\frac{4}{19}$, $1\frac{1}{15}$, $3\frac{4}{15}$, $5\frac{10}{21}$.

4. $\frac{7}{30}$, $\frac{9}{400}$, $\frac{11}{600}$, $\frac{27}{7000}$, $\frac{37}{9000}$, $1\frac{75}{76}$, $\frac{301}{11000}$.

What decimal fraction is,

5. 20 of 80? Of 50? Of 140?
6. 16 of 48? Of 40? Of 200?
7. 24 of 36? Of 60? Of 192?
8. 36 of 54? Of 96? Of 64?

EXERCISE 59.

Perform the operations indicated:

1. $2.5 + 1\frac{3}{5} + 4.07 + 3\frac{7}{250}$.
2. $5.06 - 4.001 + \frac{7}{16} - .09 + 3.02\frac{2}{3} + \frac{1}{2}$.
3. $.0001 + 1\frac{3}{5000} - \frac{5}{32} + 1.03\frac{1}{2} - .07\frac{3}{8}$.
4. $5.4 + .05\frac{2}{5} - .005\frac{1}{4} - 5.00\frac{4}{9} + 7.125$.
5. From six hundred and seven thousandths take six hundred seven thousandths.
6. Subtract nine hundred forty and seventy-six millionths from ten hundred twenty and eight tenths.

7. Multiply 7.0032 by $5\frac{3}{20}$.
8. Multiply .00075 by $1.03\frac{1}{4}$.
9. Multiply 2.10007 by .1072.
10. Multiply 12.35 by $.005\frac{1}{3}$.

Divide

11. .7644 by .0052.
12. .00169 by 2.6.
13. 2890 by .085.
14. .002 by 500.
15. 501 by .0003.
16. 5 by .00025.
17. .0001 by 1000.
18. 1000 by .00001.
19. 7.3 by 8000.
20. 40.1 by .00125.

21. If the divisor is 4.153, the remainder .02375, and the quotient 4.25, what is the dividend?
22. If $\frac{2}{3}$ of a bushel of corn be worth $\frac{3}{7}$ of a bushel of wheat, and wheat be worth $1.40 a bushel, how many bushels of corn can be bought for $27?
23. $\frac{9}{13}$ of $\frac{17}{50}$ of 56 times what number equals 50.4?

APPLICATIONS OF THE DECIMAL SYSTEM.

156. Decimal Systems of Money.—Owing to the advantages which arise from the decimal method of representing

units and parts of a unit, decimal systems of money have come to be used in all civilized countries except Great Britain. Thus, in France, the **franc** is divided into 100 equal parts called **centimes**; in Germany, the **mark** is divided into 100 equal parts, called **pfennige**, etc.

This general adoption of decimal systems of money is due to the fact that money and its units are used more often and reckoned with more extensively than any other system of units, as, for instance, those of length, weight, etc. Hence, the aggregate of economies which result from the use of a decimal scale for units of money is greater than it would be in the case of any other class of units as those of length, weight, etc.

157. United States Money.—The primary unit in the system of money in use in the United States is the dollar. The other units used in connection with the dollar, and their relation to each other, are shown in the following table:

10 mills = 1 cent.
10 cents = 1 dime.
10 dimes = 1 dollar or $1.
10 dollars = 1 eagle.

By means of the decimal system, all the other units of United States money may be expressed as dollars or fractions of a dollar. Thus, 7 eagles, 8 dollars, 4 dimes, 5 cents, and 3 mills are most conveniently *written* as $78.453. Such sums are, however, most conveniently *read* in terms of two units, *dollars* and *cents*. Mills are used only for purposes of computation; if in any result the number of mills is less than 5, it is rejected; if it is 5 or more than 5, it is reckoned as 1 cent.

Thus, $78.453 is read as 78 dollars, 45 cents, 3 mills; or rejecting the mills, 78 dollars, 45 cents.

158. Aliquot Parts of a Dollar.—Operations with United States money are often much facilitated by remembering the number of units which form aliquot parts of a dollar (See Art. 74).

Thus, $6\frac{1}{4}$ cents $= \frac{1}{16}$ of \$1.
$8\frac{1}{3}$ cents $= \frac{1}{12}$ of \$1.
$12\frac{1}{2}$ cents $= \frac{1}{8}$ of \$1.
$16\frac{2}{3}$ cents $= \frac{1}{6}$ of \$1.

The student may supply the remaining aliquot parts of \$1.

Ex. What is the cost of 18 pounds of sugar at $6\frac{1}{4}$ cents a pound?

$$18 \times 6\tfrac{1}{4} \text{ cents} = 18 \times \$\tfrac{1}{16} = \$\tfrac{9}{8}.$$
$$= \$1.125.$$
$$= \$1.13, \textit{Result.}$$

EXERCISE 60.

Add—

1. \$27.315 + \$15.05 + \$70.145 + \$48.76.

2. Thirteen dollars and thirty-two cents, seven dollars and nineteen cents, forty dollars and ninety-six cents.

3. Twelve dollars and eighty cents, seventy-five cents, eight dollars and three cents, fifty dollars and ten cents.

4. Eleven dollars and seven cents, six cents, nineteen dollars, sixty dollars and nine cents.

5. From \$100 take \$30.25.

6. Take \$25.08 from \$30.41.

7. A man gave \$50 in payment of three items of \$10.76, \$16.13 and \$21.05. How much was due him in return?

8. A farmer bought $65\frac{3}{4}$ acres of land at \$51.50 an acre. What was the amount paid?

9. What will $5\frac{3}{8}$ yards of carpet cost at \$1.75 per yard? At \$2.05 per yard?

10. At $12\frac{1}{2}$ cents a line, what will it cost to insert in a newspaper an advertisement of 17 lines? Of 45 lines?

11. How many chairs at \$2.25 each can be bought with \$29.25? With \$83.25?

12. How many acres of land worth \$71.30 an acre can be bought with \$2032.05?

13. With pads worth 12 cents each, a boy gets all he can for \$4.92. How many did he buy?

14. A certain kind of stock is worth $\$27\frac{1}{2}$ a share. How many shares can be bought for \$1347.50?

15. A man paid \$26.25 for 5 tons of coal. What would $17\frac{3}{5}$ tons cost at the same rate?

16. There are $272\frac{1}{4}$ square feet in a square rod. What will be the cost of $20\frac{2}{3}$ square rods of land at $4\frac{1}{2}$ cents a square foot?

17. A dealer bought 3 barrels each containing 31.5 gallons of oil at the rate of $45\frac{1}{2}$ cents a gallon. He sold it at $51\frac{1}{4}$ cents a gallon. How much was his gain? (Solve this problem by two methods.)

18. A miller filled 125 barrels of flour at a cost of \$7.75 each. He then sold 90 of them for \$8.30 each, and the remainder at \$6.35 each. What was his total gain?

19. A farmer spent on a crop of grain \$55.40 for seed, \$1.75 each, for 20 days of labor, and \$62.35 for rent. How much would he gain if the crop yielded 625.30 bushels which sold for 84 cents apiece?

Solve by the method of aliquot parts of \$1.

20. 27 lbs. × \$0.25.	*29.* 80 lbs. × $\$0.66\frac{2}{3}$.
21. 48 lbs. × $\$0.37\frac{1}{2}$.	*30.* 54 lbs. × $\$0.16\frac{2}{3}$.
22. 70 oz. × $\$0.33\frac{1}{3}$.	*31.* 77 pt. × $\$0.37\frac{1}{2}$.
23. 66 oz. × \$0.60.	*32.* 46 pk. × $\$0.33\frac{1}{3}$.
24. 94 ft. × $\$0.66\frac{2}{3}$.	*33.* 48 bu. × \$0.25.
25. 68 bu. × $\$0.87\frac{1}{2}$.	*34.* 61 lbs. × $\$0.12\frac{1}{2}$.
26. 95 qt. × $\$0.12\frac{1}{2}$.	*35.* 47 gal. × \$0.20.
27. 83 gal. × \$0.75.	*36.* 38 rolls × $\$.16\frac{2}{3}$.
28. 77 in. × $\$0.66\frac{2}{3}$.	*37.* 91 ft. × \$0.75.

159. Business Forms in the Use of U. S. Money, Accounts, Bills, Etc.—By the aid of certain abbreviations, and by systematic methods of arranging items, the advantages which arise from the use of a decimal system of money are further increased.

Ex. James Smith bought of Mitchell, Fletcher & Co., 17 pounds of coffee at 38 cents a pound; 75 pounds of sugar at 4½ cents a pound; 20 pounds of oatmeal at 4 cents a pound; and 4 pounds of tea at 80 cents a pound. What is the entire cost of his purchases?

If the purchases be arranged as a *bill*, it is much easier to inspect them at a glance, to verify each item, to make corrections where necessary, and to determine of whom the purchase was made, by whom, whether it is receipted, etc. Thus,

Philadelphia, May 8, 1901.

JAMES SMITH,

BOUGHT OF MITCHELL, FLETCHER & CO.

17 lbs. coffee	@ 38¢	$6	46
75 〃 sugar	@ 4½¢	3	38
20 〃 oatmeal	@ 4¢		80
4 〃 tea	@ 80¢	3	20
		$13	84

160. Business Terms and Abbreviations.—**Price** is the value of 1 unit of quantity; **cost** is the value of the entire number of units used or bought. Thus, the *price* of 1 pound of coffee is 38 cents; the *cost* of 17 pounds is $6.46.

A **bill** is a written statement showing the price, quantity, and cost of each item, and the aggregate cost of all the items.

How is a bill **receipted**? What is a **debtor**? A **creditor**?

Let the student determine the meaning of the following abbreviations used in connection with bills and accounts:

@	Cr.	Dr.	Pay't
a/c	Bal.	Per	Rec'd
	No. or #	E. and O. E.	

EXERCISE 61.

Find the amount of each of the following bills:

1.

To 6 lbs. nails	@ 5½¢	$
〃 12 ft. zinc	@ 7¼¢	
〃 15 doz. screws	@ 4⅔¢	
〃 40 squares tin	@ 17¢	

2.

To 8½ yds. ribbon	@ 32½¢	$
〃 15 〃 flannel	@ 65¢	
〃 38 〃 calico	@ 7½¢	
〃 16 〃 cloth	@ $1.10	

3.

To 10 doz. eggs	@ 18¢	$
〃 30 lbs. sugar	@ 5½¢	
〃 5 〃 tea	@ 76¢	
〃 18½ 〃 cheese	@ 16¢	
〃 ¾ 〃 pepper	@ 30¢	

4.

To 7 bushels potatoes	@ 63¢	$
〃 8 qts. beans	@ 10½¢	
〃 15 〃 tomatoes	@ 3½¢	
〃 25 gal. oil	@ 9½¢	
〃 11 loaves bread	@ 8¢	

5.

37 yds. Brussels carpet	@ $1.65	$
14 〃 Axminster 〃	@ 2.85	
41½ 〃 Ingrain filling	@ .75	
9 small rugs	@ 2.75	
45 step pads	@ .27½	

6. *Philadelphia, Sept. 18, 1900.*

MRS. FLETCHER EDWARDS,

BOUGHT OF FINLEY ACKER & CO.

		$
26 jugs syrup	@ 30¢	
7 lbs. coffee	@ 28¢	
9 " rice	@ $9\frac{1}{2}$¢	
45 " oatmeal	@ $4\frac{1}{2}$¢	
30 bu. potatoes	@ 48¢	

7. *New York City, Dec. 5, 1900.*

MRS. MARION RUTLEDGE,

BOUGHT OF JOHN WANAMAKER.

		$
61 yds. muslin	@ $11\frac{1}{2}$¢	
55 " table linen	@ $1.25	
23 " silk	@ 1.15	
10 " lining	@ 23¢	
31 " carpet	@ $1.50	

8. MR. THOMAS D. KEYSER,

BOUGHT OF MESSRS. R. L. MYERS & CO.

		$
76 algebras	@ $1.25	
68 grammars	@ 1.40	
83 readers	@ 1.35	
95 Bibles	@ .90	
57 arithmetics	@ .85	

161. Articles Bought and Sold by the Hundred or Thousand.—Some of the advantages which come from the use of the decimal scale are obtained by selling articles by the hundred (or **C**), or thousand (or **M**).

Ex. 1. What will be the cost of 3760 shingles at $7 a thousand (or $7 per M).

The number of thousands is determined by moving the decimal point of 3760 three places to the left, giving 3.76 thousands in the above example.

If one thousand shingles cost $7, the cost of 3.76 thousands will be 3.76 times $7, or $26.32, *Cost.*

Similarly, the number of *tons* in a given number of pounds may be obtained by moving the decimal point three places to the left and dividing by 2.

Ex. 2. Find the cost of 13567 pounds of coal at $6 a ton.

$$13567 \text{ pounds} = \frac{13.567}{2} \text{ tons.}$$

$$\text{If 1 ton cost \$6, the cost of } \frac{13.567}{2} \text{ tons } = \$\frac{13.567}{\cancel{2}} \times \overset{3}{\cancel{6}} = \$40.701, \textit{ Cost.}$$

EXERCISE 62.

1. Find the cost of 55260 cubic feet of gas at $1.40 per M.

2. Find the cost of 75490 bricks at $8.25 a thousand.

3. A coal dealer supplies a tinsmith with 7565 pounds of coal at $5.75 a ton, and the tinsmith roofs the coal-dealer's house with 156 pounds of tin at $14.15 a C. Which owes the other, and how much?

4. What will it cost to set the type of a book containing 560 pages of 1115 ems each, at 60 cents per thousand ems?

5. Required the cost of 83410 pounds of coal at $5.38 a ton, and 47380 shingles at $5.55 a thousand.

6. I borrowed $7500 for a year, at the rate of $4.50 per hundred. What must I pay for the loan?

7. I lent $31500 for a year at the rate of $5.25 per hundred. What do I receive for the loan?

162. Further Use of Base 100 in Percentage, Interest, Etc.—So great are the advantages of the decimal base in business and other computations and comparisons that the use of this base is developed into special subjects called Percentage, Interest, etc.

CHAPTER XI.

COMPOUND NUMBERS.

163. Use of Different Units for the same kind of Quantity.—In measuring a great variety of distances, it is an advantage to have different units of distance, some large, some small. Thus, we measure the dimensions of a window pane in *inches*, the length of a man's jump in *feet*, the distance between two cities, as between New York and Philadelphia, in *miles*. Similarly, in weighing objects, it is convenient to have different units of weight. It would be inconvenient, if not impossible, to weigh gold by the ton, and coal by the ounce.

164. Measurement may be defined as the process of finding how many times a given quantity contains another given object or quantity of the same kind, taken as a unit.

Thus, to measure a mass of sugar, is to find how many times a certain mass of sugar, called a pound, must be repeated in order to make up the given mass.

Hence, it is evident that, in measuring large objects, it is convenient to have large units; in measuring small objects, it is convenient to have small units.

165. Compound Numbers.—In measuring a quantity it is often useful to use two or more units of different sizes, but of the same general class. Thus, in measuring the distance which an athlete jumps, we first measure the number of feet in the jump, then the number of inches, if any, in the remainder of the jump, obtaining 19 feet 7 inches say, as the entire jump.

Similarly, the length of a man's life is expressed in terms

of several units of time, as 59 years 8 months and 12 days, for instance.

A **compound number** is a number expressed in terms of several units of the same class. Exs. 19 feet 7 inches; 59 years 8 months 12 days.

A **simple number** is a number expressed in terms of a single unit, as 138 inches.

166. Relative Value of Compound Numbers and Simple Numbers. Reductions.—When a given magnitude is expressed as a compound number it is often easier to form a definite conception of it than when it is expressed as a simple number. Thus, it is much easier to form a definite conception of 19 feet 7 inches, than of 235 inches; similarly, of 59 years 8 months 12 days, than of 21787 days.

On the other hand, if a magnitude be expressed as a simple number, it is much easier to operate with it, that is, to multiply, divide, etc. Thus, if it be required to find the area of a room that is 14 feet 8 inches long and 12 feet 3 inches wide, it is best to reduce the length and breadth of the room either to feet or to inches, before multiplying them.

Reduction descending is the process of reducing a number expressed in several units to an equivalent number expressed in a single small unit, as reducing 19 feet 7 inches to 235 inches.

Reduction ascending is the process of reducing a simple number to an equivalent number expressed in terms of higher units, as reducing 21787 days to 59 years 8 months 12 days.

These processes will be illustrated in connection with each table of units given in this chapter.

167. The **different classes of units** in common use are those of:

1. **Weight.**
2. **Length.**
3. **Area.**
4. **Volume (and capacity).**
5. **Value.**
6. **Time.**
7. **Angular magnitude (longitude).**
8. **Miscellaneous units.**

EXERCISE 63.

ORAL.

1. What unit would naturally be used in weighing hay? Tea? Flour? A rock? Ginger? Coffee?

2. What unit of length would be used in measuring the distance between two houses on same street? Between the floor and ceiling of a room? Between the walls of a room? Distance around your waist? Your height?

3. What unit of length is used in measuring the length of a railroad? Of a flag-pole? Of a roll of carpet? Of a man's arm?

4. What unit of area is employed in measuring land? Plastering?

5. What unit of capacity is used in the measurement of milk? Molasses? Potatoes? Strawberries? Beans? Apples?

6. What unit in measuring illuminating gas?

7. What unit of money is used in paying for a horse? A newspaper? A farm? A pencil?

8. What units of time are used in expressing your age? The age of the world? Of a baby? In stating time of an eclipse?

9. What unit is first used in measuring the length of a board? In completing the measurement?

10. What combination of units is used in expressing the time required to run a mile? To run a hundred yards? What units, in hiring a laborer?

11. In paying a bill of $28.75, what combination of units is used (bills and coins)?

WEIGHT.

168. I. **Avoirdupois weight** is used in weighing all ordinary objects, the exceptions being the precious metals, jewels, and drugs when retailed.

The primary unit in avoirdupois weight is **pound avoirdupois**, which is determined by the weight of a certain piece of metal kept in the government archives (see Art. 171).

AVOIRDUPOIS WEIGHT.

16 drams (dr.)	= 1 ounce (oz.).
16 oz.	= 1 pound (lb.).
100 lbs.	= 1 hundred-weight (cwt.).
20 cwt.	= 1 ton (T.).

The pound avoirdupois is also regarded as made up of 7000 grains.

The long ton, or 2240 pounds, is used in weighing objects in the United States Custom Houses, and in wholesale transactions in coal and iron.

Ex. 1. How many oz. in 7 T. 8 cwt. 63 lb. 14 oz.?

SOLUTION.

7 T. = 20 cwt. × 7 = 140 cwt.
140 cwt. + 8 cwt. = 148 cwt.
148 cwt. = 100 lb. × 148 = 14800 lb.
14800 lb. + 63 lb. = 14863 lb.
14863 lb. = 16 oz. × 14863 = 237808 oz.
237808 oz. + 14 oz. = 237822 oz., *Result.*

This is an example of *reduction descending*, or of reducing a compound number to an equivalent simple number in terms of the lowest unit in the given compound number.

In general, to reduce a compound number by reduction descending, *in the given compound number take the number of highest denomination; multiply it by the number of units of the next lower kind which equal one of the higher units, and add to the product the given number of units of the second kind; proceed similarly till all the units have been reduced to the lowest unit.*

Ex. 2. How many tons, hundred-weight, and pounds in 9382 pounds of coal?

SOLUTION.

$$9382 \text{ lb.} = \frac{9382}{100} \text{ cwt.} = 93 \text{ cwt.} + 82 \text{ lb.}$$

$$93 \text{ cwt.} = \frac{93}{20} \text{ T.} = 4 \text{ T.} + 13 \text{ cwt.}$$

4 T. + 13 cwt. + 82 lb., *Result.*

This is an example of *reduction ascending*, or of converting a simple number into a compound number containing higher denominations.

In general, in reduction ascending, *divide the given number by the number of units of the same kind which equals a unit of the next highest kind, and set aside the remainder; divide the quotient in a like manner and so proceed till no further division is possible*

EXERCISE 64.

Reduce:

1. 3 t. 13 cwt. 70 lbs. 10 oz. to ounces.
2. 5 t. 80 lbs. to pounds.
3. 6 t. 1 cwt. 15 oz. to ounces.
4. 15 cwt. 75 lbs. 11 oz. to ounces.
5. 45 t. 17 cwt. 90 lbs. to pounds.
6. 120650 oz. to higher denominations.
7. 236949 oz. to higher denominations.
8. 13280 lbs. to higher denominations.
9. 332805 oz. to higher denominations.
10. 289135 oz. to higher denominations.

Change:

11. 70 t. 48 cwt. 60 lbs. 6 oz. to ounces.
12. 30854 oz. to higher denominations.

13. How many firkins of butter each weighing 31 lbs. 4 oz. will be required to weigh a ton?

14. If there are 9 oz. of iron in a man's blood, how many men would supply iron enough to make a 27-lb. ball?

15. A cook uses 7 pounds 8 ounces of flour at every baking. How many bakings can she get out of 3 cwt. 30 lbs. of flour?

16. The average weight of each book in a library is 2 lbs. 5 oz. What would be the entire weight of the 8560 volumes?

EXERCISE 65.

ORAL.

1. How many ounces in 5 pounds? In $10\frac{1}{2}$ pounds?

2. How many tons in 1000 pounds? In 60000 lbs.?

3. How many pounds in 6 tons? In 15 tons? In 80 ounces?

4. If butter sells for 2 cents an ounce, what will be the price per pound?

5. When flour is worth 2 cents per pound, what will 2 hundred-weight cost? What will 8 ounces cost?

6. How many pounds in $\frac{1}{4}$ ton? In $\frac{2}{3}$ ton? In $\frac{3}{4}$ hundred-weight? In $1\frac{1}{2}$ hundred-weight? In 40 ounces? In 100 ounces?

169. II. **Troy weight** is used in weighing gold, silver, and jewels.

The primary unit is a *pound Troy*, or 5760 grains.

TROY WEIGHT.

24 grains (gr.)	= 1 pennyweight (pwt. or dwt.).
20 pwt.	= 1 ounce (oz.).
12 oz.	= 1 pound (lb.).

1 pound = 5760 gr.

Diamonds and other jewels are also weighed in terms of another unit, the *carat*. The carat = $3\frac{1}{6}$ grains. (The term carat is also used in another sense, viz.: to express the parts of gold contained by a given metal alloy. In this sense one carat means one twenty-fourth part. Gold 18 carats fine contains 18 parts of pure gold and 6 parts of some other metal.)

EXERCISE 66.

Reduce to grains:

1. 3 oz. 4 pwt. 20 gr.
2. 7 oz. 15 pwt. 18 gr.
3. 5 lb. 8 oz. 10 pwt. 9 gr.
4. 8 lb. 9 oz. 12 gr.
5. 8 lb. 5 pwt. 22 gr.
6. 12 lb. 6 oz. 18 pwt.

Reduce to higher denominations:

7. 15136 gr.
8. 5117 gr.
9. 954 pwt.
10. 31701 gr.
11. 46474 gr.
12. 27009 gr.

13. If a silver medal weigh 6 oz. 18 pwt., what will be the weight of 5 such medals? How many will it take to weigh 6 lbs. 3 oz. 18 pwt.?

170. III. **Apothecaries' Weight** is used in mixing and selling drugs and medicines at retail (these being purchased by druggists at wholesale by avoirdupois weight).

The primary unit is the same as in Troy weight (5760 grains), but it is divided somewhat differently.

APOTHECARIES' WEIGHT.

20 grains (gr.)	= 1 scruple (℈).
3 scruples	= 1 dram (ʒ).
8 drams	= 1 ounce (℥).
12 ounces	= 1 pound (lb.).

EXERCISE 67.

Reduce to grains:

1. 5 ℥ 3 ʒ 2 ℈ 8 gr.	*4.* 6 ℥ 1 ℈ 16 gr.
2. 3 lb. 8 ℥ 6 ʒ 10 gr.	*5.* 5 lb. 10 ℥ 5 ʒ 2 ℈.
3. 2 lb. 5 ʒ 1 ℈ 15 gr.	*6.* 10 lb. 3 ʒ 12 gr.

Change to higher denominations:

7. 5148 gr.	*9.* 8665 gr.	*11.* 23897 gr.
8. 1691 gr.	*10.* 875 ℈.	*12.* 40370 gr.

171. Relation of the Different Systems of Units of Weight.—In the United States the fundamental unit of weight is the weight of a certain piece of brass in the custody of the United States Mint.

This is the Troy (and apothecaries') pound, and is regarded as made up of 5760 grains. From this the *avoirdupois* pound is derived by taking the weight of 7000 grains. Hence, the only unit in common to avoirdupois weight and Troy weight is the *grain*. In Troy weight and apothecaries' weight the *pound*, *ounce*, and *grain* are the same, but the other units are different.

In Great Britain the standard of weight is the weight of a certain piece of platinum kept in the Exchequer Office. This weight is called the Imperial Pound. The Troy pound is derived from it.

EXERCISE 68.

1. How many grains in an ounce avoirdupois? Ounce Troy? Ounce apothecaries'?

2. How many grains in a dram apothecaries'? Dram avoirdupois?

3. Which is heavier, and how much, a pound avoirdupois or a pound Troy? An ounce avoirdupois or an ounce Troy?

4. What is the gain in buying drugs by avoirdupois and selling them at the same rate apothecaries' weight? Buying gold by avoirdupois and selling by Troy weight?

5. How many avoirdupois pounds in 175 Troy pounds?

MEASURES OF LENGTH.

172. I. Long Measure.—The primary unit of length in the United States and Great Britain is the *yard*. This is the distance between two marks on a bar of metal kept in the Exchequer Office of Great Britain.

From the yard are derived the following units of length:

LONG MEASURE.

12 inches (in.)	= 1 foot (ft.).
3 ft.	= 1 yard (yd.).
$5\frac{1}{2}$ yd. or $16\frac{1}{2}$ ft.	= 1 rod (rd.).
320 rds.	= 1 mile (mi.).

Rods are also called *poles* or *perches*.

A *furlong* = 40 rods; hence, 8 furlongs = 1 mile.

Civil engineers often divide the foot into tenths instead of inches.

Ex. 1. Reduce 5 mi. 208 rd. 2 yd. 1 ft.

SOLUTION.

$$5 \text{ mi.} = 320 \text{ rd.} \times 5 = 1600 \text{ rd.}$$
$$1600 \text{ rd.} + 208 \text{ rd.} = 1808 \text{ rd.}$$
$$1808 \text{ rd.} = 5\tfrac{1}{2} \text{ yd.} \times 1808 = 9944 \text{ yd.}$$
$$9944 \text{ yd.} + 2 \text{ yd.} = 9946 \text{ yd.}$$
$$9946 \text{ yd.} = 3 \text{ ft.} \times 9946 = 29838 \text{ ft.}$$
$$29838 \text{ ft.} + 1 \text{ ft.} = 29839 \text{ ft., } \textit{Result.}$$

Ex. 2. Reduce 29839 feet to higher denominations.

SOLUTION.

$$29839 \text{ ft.} = \frac{29839}{3} \text{ yd.} = 9946 \text{ yd.} + 1 \text{ ft.}$$
$$9946 \text{ yd.} = \frac{9946 \times 2}{11} \text{ rd.} = 1808 \text{ rd.} + 4 \text{ half yd. (or 2 yd.).}$$
$$1808 \text{ rd.} = \frac{1808}{320} \text{ mi.} = 5 \text{ mi.} + 208 \text{ rd.}$$

5 mi. 208 rd. 2 yd. 1 ft., *Result.*

Dividing 9946 yards by $5\frac{1}{2}$ is the same as dividing it by $\frac{11}{2}$. This, in effect, consists in reducing 9946 yards to 19892 half yards, and dividing by 11. This gives 1808 rods with 4 half yards, or 2 yards, as a remainder.

173. II. **Surveyors' linear measure** is used by surveyors in measuring the dimensions of tracts of land. In it the units are chosen with a view to determining the area of the tract measured, in an advantageous way. See Art. 177.

SURVEYORS' LINEAR MEASURE.

7.92 inches	=	1 link (li.).
100 li.	=	1 chain (ch.).
80 ch.	=	1 mile.

Hence, 1 ch. = 4 rds. = 66 ft. = 792 in.

EXERCISE 69.

Reduce to feet:

1. 44 rds. 3 yds. 2 ft.
2. 2 mi. 125 rds. 5 yds.
3. 3 mi. 250 rds. 4 yds. 1 ft.
4. 4 mi. 3 yds. 2 ft.

Reduce to inches:

5. 36 rds. 4 yds. 2 ft. 10 in.
6. 3 rds. 3 yds. 1 ft. 7 in.
7. 2 mi. 170 rds. 5 yds. 6 in.
8. 3 mi. 240 rds. 1 ft. 8 in.

Reduce to higher denominations:

9. 214 in.
10. 1710 in.
11. 14198 ft.
12. 380341 in.
13. 278222 in.
14. 504009 in.

Reduce to links:

15. 45 rds.
16. 3 miles.
17. 1 yd. 3.6 in.

18. How many rails 30 feet long will be required to build 7 miles of railroad (single track)?

19. What is the cost of building 30 miles 250 rods of road at $2.25 a yard?

20. How many panels of fence, each 2 yds. 1 ft. 6 in., will be required along the sides of a lane 1 mi. 202 rds. 4 yds. long?

21. Along the side of a room mark off 1 ft., 3 ft., 8 ft., 1 yd., 3 yds., 1 rd., 3 yds. 2 ft., 2 yds. 1 ft. 8 in.

EXERCISE 70.

ORAL.

1. How many inches in 6 ft.? In $7\frac{1}{2}$ ft.? In 1 yd.?
2. How many feet in 5 yds.? In 2 rds.? In a mile? In 60 in.?
3. How many feet in 125 in.? In $3\frac{1}{2}$ yds.? In 1 rd. 2 yds.?
4. What part of a foot is 3 in.? 4 in.? 6 in.? 8 in.? 9 in.? 10 in.?
5. What part of a yard is 9 in.? 12 in.? 18 in.? 24 in.? 27 in.?
6. What part of a mile is 40 rds.? 80 rds.? 200 rds.? 280 rds.?
7. How many chains are there in 400 rds.? In 110 rds.?
8. How many inches are there in 50 links? In 80 li.? In 6 li.?

MEASURES OF SURFACE.

174. A **surface** has two dimensions, length and breadth. In measuring surfaces, or square measure, the primary unit is a flat square, each of whose sides is 1 yard. It is sometimes more convenient to use other units of surface, as a square, each of whose sides is 1 inch, or 1 foot. These units may be regarded as derived from the primary unit, the square yard.

A surface is measured by determining the number of times the unit of surface must be used to make up the given surface.

175. The **area** of a surface is the number of times the unit of surface is contained in the given surface.

Thus, if a rectangle be 7 inches long and 5 inches wide, it contains 35 square inches, and its area is said to be 35 square inches.

It is evident that in a rectangle, whose sides contain an exact number of linear units, the number of units in the area will equal the number of linear units in one dimension multiplied by the number of linear units in the other dimension (see Figure, p. 53). For in such a rectangle the entire number of small unit squares is equal to the number of them in each row multiplied by the number of rows, which equals the number of linear units in the length multiplied by the number in the breadth.

Hence, to find the area of a rectangle, *multiply the length by the breadth.*

176. I. **Surface or square measure.**

SQUARE MEASURE.

144 square inches (sq. in.)	=	1 square foot (sq. ft.).
9 sq. ft.	=	1 square yard (sq. yd.).
$30\frac{1}{4}$ sq. yds.	=	1 square rod (sq. rd.).
160 sq. rds.	=	1 Acre (A.).
640 A.	=	1 square mile.

177. II. In **Surveyors' square measure** the primary unit of surface is the square chain, that is, a square piece of land, each side of which is 1 chain, or 66 feet.

SURVEYORS' SQUARE MEASURE.

16 sq. rds. = 1 sq. ch.
10 sq. ch. = 1 acre.

Hence, 1 acre = 10 sq. ch. = 160 sq. rds.

The advantage in using surveyor's measure in measuring land lies in the fact that, after the number of square chains in a piece of land have been determined, the *number of acres can be determined by simply dividing the number of square chains by 10;* that is, by moving the decimal point one place to the left.

EXERCISE 71.

Reduce:

1. 6 sq. rds. 4 sq. yds. 5 sq. ft. to sq. ft.
2. 2 A. 56 sq. rds. 10 sq. yds. to sq. ft.
3. 3 A. 24 sq. yds. 7 sq. ft. to sq. ft.
4. 8 sq. yds. 3 sq. ft. 100 sq. in. to sq. in.
5. 3 sq. rds. 16 sq. yds. 25 sq. in. to sq. in.
6. 4 A. 3 sq. yds. 4 sq. ft. to sq. in.

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Reduce to higher denominations:

7. 17188 sq. yd.	*9.* 944720 sq. in.	*11.* 117804 sq. ft.
8. 16567 sq. ft.	*10.* 39725 sq. in.	*12.* 3657930 sq. in.

13. How many sq. ch. in one sq. mi.? In 800 sq. rds.?
14. Change 15000 sq. ch. to higher denominations.
15. Draw upon the blackboard a sq. ft. and subdivide it into sq. in. How many are there?
16. Draw a sq. yd. and subdivide it into sq. ft. How many?

EXERCISE 72.

ORAL.

1. How many square feet in 3 sq. yds.? In $5\frac{1}{3}$ sq. yds.? In 2 sq. rds.?
2. How many square inches in 4 sq. ft.? In $2\frac{1}{4}$ sq. ft.? In 1 sq. yard?
3. How many acres in 2 sq. miles? In 800 sq. rds.?
4. What is the difference between a square foot and a square yard?
5. What is meant by 6 in. square? 10 ft. square? 3 yards square?
6. What is the difference between 3 sq. ft. and 3 feet square? Between 6 sq. yds. and 6 yards square?
7. How many square inches in the lid of a box 1 ft. 3 in. square? In another, 1 ft. 4 in. square? In another $\frac{2}{3}$ ft. square?

MEASURES OF VOLUME AND CAPACITY.

178. A solid has three dimensions: length, breadth, and thickness.

A **cube** is a solid bounded by six equal squares.

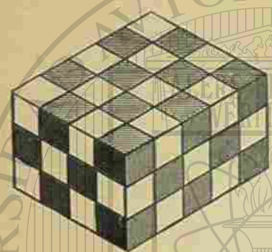
The **primary unit of volume** is a cube, each of whose edges is 1 yard, and is called a **cubic yard**.

It is sometimes more convenient to use other derived units of volume, as 1 cubic inch, or 1 cubic foot.

A solid is measured by determining how many times the unit of volume must be taken to make up the given solid.

The **cubic contents**, or volume, of a solid is the number of times the unit of volume is contained in the given solid.

It is evident that in a box-shaped, or rectangular, solid, each of whose edges contains an exact number of linear units, the number of units of volume may be readily obtained from the number of linear units in the edges. Thus, if we have such a solid whose edges are 3, 4, and 5 inches, and the solid be divided into small unit cubes, each edge of which is 1 inch, each layer will contain 4×5 cubic inches; and as there are 3 layers the entire solid will contain $3 \times 4 \times 5$ (or 60) cubic inches.



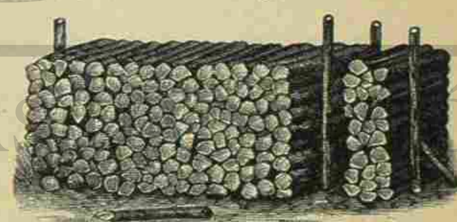
Hence, in order to determine the *volume of a rectangular solid*, instead of cutting the solid up into little cubes, and counting them, we substitute the less labor of making linear measurements of the three edges, and taking the *product of the length by the breadth by the thickness.*

179. I. **Cubic Measure** in general.

CUBIC MEASURE.

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.).
27 cu. ft. = 1 cubic yard (cu. yd.).

180. II. **Wood Measure.**—In measuring wood the *cord* is the primary unit used. A cord is a pile of wood 8 feet long, 4 feet wide, and 4 feet high. A cord foot is that part of a cord which is 1 foot long.



WOOD MEASURE.

16 cubic feet = 1 cord foot.
8 cord feet = 1 cord.
Or 128 cubic feet = 1 cord.

EXERCISE 73.

Reduce:

1. 2 cu. yds. 5 cu. ft. 200 cu. in. to cu. in.
2. 5 cu. yds. 20 cu. ft. to cu. ft.
3. 7 cd. 4 cd. ft. to cu. ft.
4. 3 cd. 5 cd. ft. 10 cu. ft. to cu. ft.

Reduce to higher denominations:

5. 166413 cu. in. | *6.* 2046 cu. ft. | *7.* 3455 cu. ft.

8. How many cubic yards in a cord? How many cords in 432 cu. yds.?

9. Measure the edges of a crayon box and compute its volume.

10. How many cords in a pile of wood whose dimensions are 6, 10 and 32 feet?

EXERCISE 74.

ORAL.

1. How many cu. in. in half a cubic foot? In $\frac{3}{4}$ cu. ft.? In 2 cu. ft.?
2. How many cu. ft. in $\frac{1}{3}$ cu. yd.? In 3 cu. yds.? In $2\frac{2}{3}$ cu. yds.?
3. How many cu. ft. in $\frac{1}{2}$ cord? In $\frac{1}{4}$ cord? In $\frac{3}{8}$ cord?
4. What is the volume of a box whose dimensions are 3, 5 and 6 inches? Or another 4, 6 and 7 feet?
5. What is the volume of a 3-in. cube? Of a 4-ft. cube?
6. What is the difference between a 5-in. cube and 5 cu. in.? Between a 2-ft. cube and 2 cu. ft.?

181. Measure of Capacity.—For fluids and loose objects, as grain, fruit, etc., it is found convenient to use other units of volume or capacity, as the pint, gallon, bushel, etc., each of which, however, can be expressed as a certain number of cubic inches.

In dealing with such materials it is usually not convenient to make linear measurements, and a given material is measured directly by counting the number of times it will fill a unit vessel. Wherever possible, however, the method of linear measurements and of computations from these is to be preferred.

182. I. **In Dry Measure** the fundamental unit is the bushel, which contains 2150.42 cu. in.

DRY MEASURE.

2 pints (pt.)	= 1 quart (qt.).
8 qts.	= 1 peck (pk.).
4 pks.	= 1 bushel (bu.).

183. II. **In Liquid Measure** the fundamental unit is the gallon, containing 231 cu. in.

LIQUID MEASURE.

4 gills (gi.)	= 1 pint (pt.).
2 pts.	= 1 quart (qt.).
4 qts.	= 1 gallon (gal.).
$31\frac{1}{2}$ gals.	= 1 barrel (bbl.).
63 gals.	= 1 hogshead (hhd.).

Barrels and hogsheads in common use often vary greatly from the standard size.

184. III. **Apothecaries' Fluid Measure.**

60 minims, or drops (♏)	= 1 fluid dram (fʒ).
8 fluid drams	= 1 fluid ounce (f℥).
16 fluid ounces	= 1 pint (O).
8 fluid pints	= 1 gallon (Cong.).

A common glass or teacup contains 8 fluid ounces.
What is called a *tablespoonful* contains about $\frac{1}{2}$ a fluid ounce.
What is called a *teaspoonful* contains about $\frac{1}{8}$ a fluid ounce.

EXERCISE 75.

Reduce to pints:

1. 5 bu. 3 pks. 6 qts.
2. 3 bbls. 2 gal. 3 qt. 1 pt.
3. 2 bbl. 23 gal. 1 pt.
4. 10 bu. 2 pk. 6 qt. 1 pt.

Reduce to quarts:

5. 8 bu. 2 pk. 5 qt.
6. 3 hhd. 18 gal. 3 qt.

Reduce to higher denominations:

7. 894 gi.
8. 233 pt. d.-m.
9. 712 qt. l.-m.
10. 319 pt. d.-m.
11. 3291 gi.
12. 1629 pt. d.-m.

13. How many cu. in. in a standard barrel? In 5 bushels?
14. How many cu. ft. in 12 bushels?
15. How many cu. in. in a dry qt.? In a liquid qt.?
16. How many gallons have the same capacity as one bushel?
17. What will be the cost of 5 gal. 3 qt. 1 pt. of vinegar at 4 cents a pt.?
18. What will 10 bu. 3 pk. 6 qt. of grain cost at 3 cents a qt.?
19. How much milk at a cent a gill must be given in exchange for 2 bu. 1 pk. 5 qt. of grass seed at 10 cents a pt.?

EXERCISE 76.

ORAL.

How many:

1. Pints in a gallon? In a peck? In a bushel?
2. Quarts in a bushel? In a barrel? In 4 gal. 3 qt.?
3. Gallons in 36 qts.? In 48 qts.? In 3 bbl.?
4. Bushels in 12 pk.? In 64 qt.? In 64 pk.?
5. What part of a gallon is a qt.? A pt.? A gill?
6. What part of a bushel is a pk.? A qt.? 4 qts.? 24 qts.?
7. Which is greater, a dry pint or a liquid pint?

MEASURES OF VALUE.

185. **Units of money** are unit quantities of the precious metals, as gold and silver, which are used to measure the values of things.

Coin is the actual metal itself as weighed, shaped, and stamped by the government to form single units or combinations of units of value.

Paper money consists of engraved and printed promises to pay a certain number of units of coin to the bearer.

Currency is a general name for both coin and paper money.

186. I. **In United States money** the primary unit is the gold dollar.

10 mills	= 1 cent (ct. or ¢).
10 cents	= 1 dime (d.).
10 dimes	= 1 dollar ($).
10 dollars	= 1 eagle.

The coins in use in the United States are as follows:

Bronze: the cent.

Nickel: the five-cent piece.

Silver: the dime, quarter dollar, half dollar, and dollar.

Gold: the dollar, quarter eagle, half eagle, eagle, and double eagle.

187. II. In **English money** the fundamental unit is 1 pound, whose value in United States money is $4.8665.

4 farthings (far.)	= 1 penny (d.).
12 pence	= 1 shilling (s.).
20 shillings	= 1 pound (£).
21 shillings	= 1 guinea (G.).

The coins used in Great Britain are as follows:

Copper: penny, half-penny, farthing.

Silver: three pence, six pence, shilling, florin (2 s.), double florin (4 s.), half-crown ($2\frac{1}{2}$ s.), crown (5 s.).

Gold: half-sovereign (10 s.), sovereign (20 s.).

EXERCISE 77.

Reduce to farthings:

1. 6 s. 9 d. 2 far.
2. £10 7 s. 3 d. 1 far.
3. £16 15 s. 8 d. 3 far.
4. £73 13 s. 7 d. 3 far.

Reduce to higher denominations:

5. 728 cents.
6. 3452 cents.
7. 605 far.
8. 608 d.
9. 755 far.
10. 3573 far.
11. 8493 far.
12. 9614 far.
13. 15987 far.

Find the value in U. S. money of:

14. £13.
15. 15 s.
16. £10 6 s.
17. £65 18 s. 6 d.
18. £5.8.
19. £3.45.

Find the value in English money of:

20. $486.65. | *21.* $116.796. | *22.* $583.98. | *23.* $47.6917.

EXERCISE 78.

ORAL.

1. How many cents in $\$\frac{1}{4}$? In $\$\frac{1}{2}$? In $\$\frac{3}{4}$? In $\$\frac{5}{8}$? In $\$\frac{4}{5}$? In $6?

2. What part of a dollar is 10 cts.? 20 cts.? $33\frac{1}{3}$ cts.? $37\frac{1}{2}$ cts.? 50 cts.? $66\frac{2}{3}$ cts.? 60 cts.?

3. How many shillings in 36 d.? In £5? In 8 guineas?

4. How many farthings in 8 d.? In 1 shilling? In £1? In 3 crowns?

5. What part of a pound is a crown? Of $5 is 50 cents?

6. How many pence in 3 s. 3 d.? In a pound?

188. III. In **French money** the unit is the **franc,** the value of which in U. S. money is $0.193.

100 centimes (c.) = 1 franc (f.).

In Belgium and Switzerland the unit of money is also the franc. In Italy it is the **lira,** in Spain the **peseta,** each of which has the same value as the franc.

189. IV. In **German money** the unit is the **mark,** the value of which in U. S. money is $0.238.

100 pfennige (pf.) = 1 mark (m.).

In **Austria** the unit of money is the **crown,** the value of which is $0.203.

EXERCISE 79.

Reduce to dollars:

1. 200 francs.
2. 155 marks.
3. 60 francs 45 centimes.
4. 92.65 marks.

Change to French money and to German money:

5. $75. | *6.* $123.45. | *7.* $976.80.

8. A pair of gloves cost 12 francs in Paris. How much was that in U. S. money?

9. A book sold for 32 marks in Berlin. What was its equivalent in American money?

MEASURES OF TIME.

190. The primary unit of time is the **mean solar day.** This is the average or mean interval between two successive passages of the sun across the same meridian. It is customary to make the day at any place begin at midnight.

Another natural unit of time is the *solar year,* or the time it takes the earth to make a single revolution about the sun.

TABLE OF TIME.

60 seconds (sec.)	=	1 minute (min.).
60 minutes	=	1 hour (h.).
24 hours	=	1 day (da.).
365 days	=	1 common year (yr.).
366 days	=	1 leap year.
100 years	=	1 century (cen.).

7 days make one week; 4 weeks make one month for many purposes. 30 days = 1 month for certain purposes.

191. The calendar. The solar year is divided into twelve months called *calendar* months. The names of these months with the number of days in each are as follows:

	Days.		Days.
1. January (Jan.)	31	*7.* July	31
2. February (Feb.)	28 or 29	*8.* August (Aug.)	31
3. March (Mar.)	31	*9.* September (Sept.)	30
4. April (Apr.)	30	*10.* October (Oct.)	31
5. May	31	*11.* November (Nov.)	30
6. June	30	*12.* December (Dec.)	31

The calendar year begins with January 1. The calendar year is also divided into 4 seasons of three months each.

The following lines, if committed to memory, will enable the pupil to recall readily the number of days in each calendar month:

Thirty days hath September,
April, June, and November.
All the rest have thirty-one,
Except the second month alone,
To which we twenty-eight assign,
Till leap year gives it twenty-nine.

The explanation of the fact that February sometimes has 28 days and sometimes 29 days is as follows: The exact length of the solar year is 365 dys. 5 hrs. 48 min. and 48 sec., which is a little less than $365\frac{1}{4}$ days. It is convenient to have each calendar year contain an exact number of days. This end is obtained by having three years in succession each containing 365 days (called common years), followed by a fourth year containing 366 days (called a leap year). Since, however, the true or solar year is a little less than $365\frac{1}{4}$ days, it is necessary to omit 3 leap years in every 400 years, in order to keep an exact correspondence between the average calendar year and the solar year. This is done by regarding only *those century years which are divisible by four hundred as leap years.*

Thus the years 1700, 1800, 1900 are not leap years, but 2000 is a leap year.

EXERCISE 80.

Reduce to minutes:

1. 4 da. 17 hr. 50 min.
2. 1 yr. 250 da. 6 hr. 30 min.
3. 2 yr. 175 da. 18 hr. 40 min.
4. 5 yr. 60 da. 9 hr. 51 min.

Reduce to seconds:

5. 220 da. 25 min. 45 sec.
6. 315 da. 21 hr. 38 sec.

Reduce to higher denominations:

7. 57330 sec.
8. 93324 min.
9. 64240 hr.
10. 128140 sec.
11. 1052348 min.
12. 86215 hrs.

13. How many seconds in 4 wk. 5 da. 17 hr. 30 sec.?

14. How many hours in 41 wk. 6 da. 21 hr.?

How many days from:

15. May 1 to July 18?
16. June 2 to Dec. 15?
17. July 19 to Dec. 25?
18. Nov. 21 to Feb. 14?
19. Aug. 16 to Feb. 20?
20. Mar. 7 to Jan. 17?
21. Apr. 14 to Oct. 18?
22. Apr. 23 to Feb. 25?

23. Which has the least number of days, the spring months, the summer months, or the fall months?

EXERCISE 81.

ORAL.

1. How many days in 6 weeks? In 96 hours? In 3 years?

2. How many hours in 3 days? In 480 minutes? In 2 weeks?

3. Which has the greater number of days, summer or winter of 1904?

4. Which of the following years are leap years, and why?

1775,	1805,	1826,	1836,	1866,	1884,	1898,	3000.
1600,	2001,	1640,	1920,	1970,	1954,	1900,	3456.

5. What improvements could you suggest in the distribution of days in the calendar?

6. Why is the extra day in leap year added to February and not to some other month?

7. How long is the day when the sun rises at 7 a. m. and sets at 4 p. m.? How long, when it rises at 4 a. m. and sets at 7 p. m.?

8. What instant is the exact middle of the week? Of July? Of November?

CIRCULAR AND ANGULAR MEASURE. LONGITUDE.

192. A circle is a plane figure bounded by a curved line, every point of which is equidistant from a point within called the **center.**

The **circumference** is the line which bounds the circle. A **quadrant** is one of the four equal parts into which a circumference is divided. Thus, if *BC* is one-fourth of the circumference *ACBD*, it is called a quadrant. The angle *BOC* is then a right angle. Each quadrant is subdivided into 90 equal parts called **degrees.**

An **angle** is the amount of opening between two lines which meet, as *COP*, or *POB*.

TABLE OF CIRCULAR MEASURE.

60 seconds (″)	= 1 minute (′).
60 minutes	= 1 degree (°).
360 degrees	= 1 circumference (c.).

The angle at the center of a circle is regarded as containing the same number of degrees as the part of the circumference (arc) corresponding to it. Thus, if the arc *PB* contains 13°, the angle *POB* is also spoken of as an angle of 13°. Hence, a right angle, which corresponds to a quadrant, contains 90°.

193. Longitude.—Each great circle on the surface of the earth, as, for instance, the equator, is divided into 360 degrees. A degree of the earth's equator is called a degree of longitude. $\frac{1}{60}$ of a degree of longitude is called a geographical unit or *knot.*

Longitude is measured east or west from a fixed point or meridian (usually the meridian of Greenwich).

EXERCISE 82.

Reduce to seconds:

1. 35° 17′ 25″.
2. 150° 50″.
3. 205° 10′ 40″.
4. 330° 3′ 6″.

Reduce to higher denominations:

5. 21026″. *6.* 270040″. *7.* 398234″.

8. What part of the circumference is 30°? 45°? 60°? 90°? 120°? 150°? 225°? 300°? 330°?

9. What part of a semi-circumference is 30°? 60°? 45°? 90°? 120°? 135°? 150°?

10. Between two cities the longitude is 3° 47′ 15″. How many seconds are they apart?

11. If the earth's equator contains 24902.302 miles, how many miles in a degree? In 1″?

12. How many miles in one geographical mile (1′ on equator)?

ORAL.

1. How many minutes in 5°? In 80°? In 360″?

2. What is the difference between a quadrant and a right angle? How many degrees in 3 right angles? In $\frac{3}{2}$ rt. angles? In $\frac{2}{3}$ rt. angle? In $\frac{1}{2}$ rt. angle? In $\frac{1}{4}$ rt. angle? In $\frac{4}{3}$ rt. angles?

MISCELLANEOUS UNITS.

194. I. **Lengths.**

4 inches = 1 hand.	6 feet = 1 fathom.
9 inches = 1 span.	120 fathoms = 1 cable length.
3 feet = 1 pace.	

195. II. **Numbers in General.**

12 units = 1 dozen (doz.).
12 dozen = 1 gross.
12 gross = 1 great gross.
20 units = 1 score.

III. **Sheets of Paper.**

24 sheets = 1 quire.
20 quires = 1 ream.
2 reams = 1 bundle.
5 bundles = 1 bale.

196. Capacity determined by Weight.—It is often more convenient to determine the number of bushels, or barrels, in a large quantity of material by weight than by direct measurement.

Some of the different equivalents vary in different states, but the following are representative:

1 bush. of wheat	= 60 lbs.	1 bush. of barley	= 48 lbs.
1 bush. of potatoes	= 60 lbs.	1 bush. of oats	= 32 lbs.
1 bush. of beans	= 60 lbs.	1 bush. of coarse salt	= 56 lbs.
1 bush. of clover-seed	= 60 lbs.	1 bbl. of flour	= 196 lbs.
1 bush. of shelled corn	= 56 lbs.	1 bbl. of pork or beef	= 200 lbs.
1 bush. of rye	= 56 lbs.	1 cental of grain	= 100 lbs.

EXERCISE 83.

1. How many fathoms in a mile? How many paces?

2. How many feet in 16 hands? In 5 spans? In 6 fathoms?

3. For what, and by whom, are the following units used: hand, span, fathom, dozen, score, quire, league?

4. How many pounds does a peck of wheat weigh? A peck of oats? Of corn? A quart of oats? 4 quarts of wheat?

5. How many units in 1 great gross? How many dozens in 9 scores?

6. Bought eggs at 45 cents a score and sold them at 30 cents a dozen. What is the gain on a great gross?

7. A box of pencils contains $\frac{1}{2}$ gross. How many pencils in a score of boxes? In 5 dozen boxes?

8. How many sheets of paper in 1 ream? In one bundle?

9. Reduce 17904 sheets to higher denominations.

10. A dealer bought a bale of paper @ 30 cents a quire and sold it by the sheet so as to make $52. Find the rate of sale per sheet.

11. How many bushels of wheat worth $1\frac{1}{2}$ cents per pound would equal in value 27 bbl. flour @ $2\frac{1}{4}$ cents a pound?

12. What part of the weight of a bushel of corn is the weight of a bushel of oats?

13. Which is heaviest, 7 bu. barley, $5\frac{1}{2}$ bu. wheat, 10 bu. oats, or $5\frac{6}{7}$ bu. rye? What is the combined value of the lot at a cent and a half a pound?

OPERATIONS WITH COMPOUND NUMBERS.

197. I. Reduction Ascending and Descending.—These processes have been considered in connection with the individual tables, but they may be conveniently renewed in connection with the following exercise of miscellaneous examples.

EXERCISE 84.

REVIEW.

~~in class~~
home work
paper to class

Reduce:

1. 5 cwt. 46 lb. 12 oz. to ounces.
2. 7 yr. 261 da. 19 hr. 51 min. to minutes.
3. 2 mi. 100 rds. 4 yds. 2 ft. 8 in. to inches.
4. 8° 44′ 19″ to seconds.
5. 3 A. 75 sq. rds. 14 sq. yds. 7 sq. ft. to square feet.
6. 2 ba. 3 bun. 1 r. 17 qu. 15 sh. to sheets.
7. 7 t. 19 cwt. 56 lb. to pounds.
8. 43 da. 15 hr. 14 min. 55 sec. to seconds.
9. 25 bu. 3 pk. 6 qt. 1 pt. to pints.
10. 7 bbl. 14 gal. 3 qt. 1 pt. to pints.
11. 3 cu. yd. 18 cu. ft. 560 cu. in. to cubic inches.
12. 5 A. 104 sq. rd. 26 sq. yd. to square yards.
13. 135 rd. 3 yd. 1 ft. 11 in. to inches.
14. 7 lb. 9 oz. 15 pwt. 10 gr. to grains.
15. 1 mi. 141 rd. 5 yd. 2 ft. 9 in. to inches.
16. 9 lb. 7 ℥ 5 ʒ 2 ℈ 16 gr. to grains.

March [illegible]

Change to higher denominations:

17. 17058 ft.	*23.* 44456 gr. Ap.
18. $16501\frac{1}{4}$ sq. yds.	*24.* 175036 cu. in.
19. 254020 oz. Av.	*25.* 239728″.
20. 565 pts. d.-m.	*26.* 51764 gr. T.
21. 1334380 sec.	*27.* 60855 in.
22. 2441 pts. l.-m.	*28.* $39997\frac{1}{2}$ sq. ft.

29. How many pills of 2 gr. each can be made from 2 lb. 6 ℥ 3 ʒ 2 ℈ of quinine?

30. How many times will a wagon-wheel 12 ft. 10 in. in circumference, revolve in going 4 mi. 50 rds.?

31. How long, working 8 hrs. a day, will it require to count $1000000 at the rate of $50 a minute?

32. How many half-gill ink wells can be filled from 7 gallons of ink?

33. If the income from a store averages 5 cents a minute, what will it amount to during the three summer months?

34. Sound travels about 1100 feet per second. How far away was a flash of lightning when the sound of the thunder reached me $6\frac{3}{4}$ sec. later than I saw the flash?

35. A farm 230 rds. long and 180 rds. wide is worth a cent per. sq. yd. What is its total value?

36. From 3 T. 18 cwt. of grain a dealer sells sacks containing 16 lbs. 4 oz. How many such sacks will there be?

37. If the grain in the last example were oats, and the sacks contained 1 bushel 2 pecks 4 quarts each, how many would there be?

198. II. In the **addition of compound numbers** it is necessary to set *similar units in the same column; add each column, beginning at the right; simplify the sum of each column by reduction ascending.*

Ex. 1. Add:

	mos.	das.	hrs.	min.
	5	12	15	22
	6	5	17	48
	3	18	16	17
		6	0	8
1 yr.	3	13	1	35

EXPLANATION.—The sum of the minutes is 95, which reduces to 1 hour and 35 minutes. We set down the 35 min. and add 1 hr. with the hrs. column. The sum of the hours is 49, which reduces to 2 das. and 1 hr. We set down the 1 hr. and carry the 2 das. to the column of das. Proceeding in like manner, the entire sum is 1 yr. 3 mos. 13 das. 1 hr. 35 min.

Ex. 2. Add:

This process is the same as in Ex. 1, except that the $\frac{1}{2}$ yd. obtained as part of the sum of the yds. column is reduced to 1 ft. 6 in. and added with the ft. and in.

mi.	rds.	yds.	ft.	in.
4	120	3	2	2
12	18	1	1	6
5	212	2	2	8
22	31	$2\frac{1}{2}$	0	4
		($\frac{1}{2}$ =)	1	6
22	31	2	1	10

EXERCISE 85.

Add:

1.
5 cwt. 81 lbs. 14 oz.
9 cwt. 70 lbs. 8 oz.
4 cwt. 97 lbs. 12 oz.

2.
7 yr. 123 da. 9 hr. 17 min. 40 sec.
3 yr. 96 da. 13 hr. 44 min. 53 sec.
5 yr. 215 da. 21 hr. 52 min. 28 sec.

3.

lb.	oz.	pwt.	gr.
15	10	18	14
9	4	13	21
23	9	7	6
1	11	15	22

4.

lb.	℥	ʒ	℈	gr.
3	7	5	2	15
4	10	6	1	17
11	9	4	2	8
7	0	3	0	19

5.

bu.	pk.	qt.	pt.
7	3	5	1
4	2	6	0
10	3	7	1
15	0	4	0

6.

bbl.	gal.	qt.	pt.	gi.
5	27	3	1	2
4	19	1	0	3
16	21	2	1	0
20	5	0	1	2

7.

£	s.	d.	far.
5	10	6	1
3	15	11	3
7	14	8	2
9	5	9	0

8.

cu. yd.	cu. ft.	cu. in.
7	20	1115
3	19	1076
10	13	263
17	26	709

9.

mi.	rd.	yd.	ft.	in.
10	255	4	1	10
33	163	3	2	7
28	75	0	0	9
17	239	5	2	5
22	190	4	1	6

10.

A.	sq. rd.	sq. yd.	sq. ft.	sq. in.
5	145	26	6	109
3	108	30	7	128
7	95	17	3	105
4	82	21	8	96
10	150	11	7	117

11. 7 T. 5 cwt. 63 lb. 4 oz.; 10 T. 16 cwt. 90 lb. 13 oz.; 17 T. 8 cwt. 48 lb. 9 oz.; 8 T. 14 cwt. 56 lb. 10 oz.

12. 8 mi. 156 rd. 4 yd. 2 ft. 6 in.; 7 mi. 97 rd. 3 yd. 10 in.; 5 mi. 2 yd. 1 ft. 9 in.; 296 rd. 4 yd. 2 ft.; 3 mi. 4 yd. 2 ft. 8 in.

13. 5 yr. 153 da. 9 min. 59 sec.; 24 yr. 260 da. 8 hr. 45 sec.; 270 da. 15 hr. 58 min.; 13 yr. 21 hr. 43 min. 28 sec.; 60 da. 55 min.

14. 23° 14′ 15″; 68° 23′ 44″; 13° 46′ 35″; 9° 1′ 7″.

15. 1 A. 30 sq. yd. 5 sq. ft. 112 sq. in.; 9 A. 80 sq. rd. 7 sq. ft.

38 sq. in.; 31 A. 136 sq. rd. 8 sq. ft. 100 sq. in.; 75 sq. rd. 47 sq. in.

16. 7 lb. 9 ℥ 7 ʒ 2 ℈ 16 gr.; 3 lb. 5 ʒ 9 gr.; 8 ℥ 3 ʒ 1 ℈; 30 lb. 11 ℥ 1 ℈ 11 gr.

199. III. Subtraction of compound numbers.

Ex. From 5 mi. 32 rds. 4 yds. 2 ft.
Subtract 3 mi. 125 rds. 5 yds. 1 ft.

OPERATION.

mi.	rds.	yds.	ft.	
5	32	4	2	
3	125	5	1	
1	226	4½	1	
		(½ =)	1	6 in.
1	226	4	2	6 in.

EXPLANATION.

We write similar units in the same column, and begin with the right-hand column. 1 ft. from 2 ft. leaves 1 ft.; 5 yds. is more than 4 yds.; hence, it is necessary to borrow 1 rd. (or 5½ yds.) from 32 rds. Adding 5½ yds. to 4 yds., we get 9½ yds.; 5 yds. subtracted from 9½ yds. leaves 4½ yds. Similarly, we borrow 1 mile, or 320 rds., from 5 miles, and add it to 31 rds., and then subtract 125 rds. from 351 rds., giving 226 rds. as a remainder. Hence, we obtain, 1 mi. 226 rds. 4½ yds. 1 ft. as the difference; it is necessary, however, to reduce ½ yd. to 1 ft. 6 in. to get the result in the most convenient shape, which gives as a final result

1 mi. 226 rds. 4 yds. 2 ft. 6 in.

200. Difference between Two Dates.—In finding the interval of time between two dates, 30 days are usually reckoned as 1 month, and 12 months as 1 year. If hours are included, the reckoning is made to begin at 12 o'clock midnight.

Ex. How many years, months, and days between Oct. 19, 1895, and June 6, 1898?

yrs.	mos.	das.
1898	6	6
1895	10	19
2	7	17, *Result.*

EXERCISE 86.

Subtract:

1.

bbl.	gal.	qt.	pt.
9	24	2	1
1	19	3	0

2.

T.	cwt.	lb.	oz.
8	11	47	10
4	15	50	8

3.

lb.	℥	ʒ	℈	gr.
5	8	5	2	7
2	10	6	0	15

4.

cu. yd.	cu. ft.	cu. in.
55	14	328
41	23	1518

5.

gal.	qt.	pt.	gi.
27	2	0	3
18	3	1	2

6.

48°	19′	7″
39°	41′	32″

7.

mi.	rd.	yd.	ft.	in.
38	111	3	2	5
26	244	4	2	8

8.

A.	sq. rd.	sq. yd.	sq. ft.	sq. in.
75	108	21	5	46
38	150	6	7	125

9.

180°	0′	5″
76°	34′	48″

10.

180°	0′	0″
125°	39′	46″

11.

mi.	rd.	yd.	ft.	in.
120	251	0	2	0
89	300	4	2	9

12. From 7 bbl. 9 gal. 1 qt. take 3 bbl. 25 gal. 1 pt. 2 gi.

13. From 99 mi. 4 yd. 6 in. take 30 mi. 166 rd. 5 yd. 2 ft. 10 in.

14. From 83 A. 115 sq. rd. take 76 A. 139 sq. rd. 25 sq. yd. 118 sq. in.

15. From 17 T. take 3 T. 16 cwt. 49 lb. 15 oz.

16. From 360° take 315° 46′ 50″.

17. From the sum of 9 mi. 4 yd. 2 ft. 8 in. and 18 mi. 130 rd. 1 ft. 10 in. take 25 mi. 275 rd. 5 yd. 2 ft. 11 in.

18. From the difference between 5 A. and 85 sq. rd. 19 sq. yd. 108 sq. in. take the sum of 1 A. 99 sq. rd. 130 sq. in. and 2 A. 83 sq. rd. 19 sq. yd. 8 sq. ft. 116 sq. in.

Find the difference in years, months, and days between the following pairs of dates.

19. Feb. 18—Nov. 30. | *20.* Mch. 25—Dec. 5.

21. June 8, 1875—Oct. 15, 1879.
22. Feb. 22, 1732—Dec. 14, 1799.
23. Dec. 14, 1834—Jan. 13, 1858.
24. Dec. 5, 1870—June 23, 1897.
25. Oct. 12, 1887—Aug. 9, 1892.
26. Apr. 30, 1817—Feb. 2, 1903.

27. Add to Aug. 25, 1900, 7 yrs. 4 mo. 15 da.

28. When was a man born who died Oct. 18, 1875, aged 92 yrs. 11 mo. 26 da.?

29. A man was born Jan. 23, 1810, and lived 71 yrs. 3 mo. 24 da. What was the date of his death?

30. Mr. Smith was born Nov. 8, 1850, and his son, Jan. 17, 1877. On what day is the son half as old as his father? How old was each Jan. 1, 1901?

201. III. **Multiplication of compound numbers.**

Ex. If one lot is 5 rds. 2 yds. 2 ft. wide, how wide are 7 lots?

OPERATION.

rds.	yds.	ft.	
5	2	2	
		7	
38	$1\frac{1}{2}$	2	
	$(\frac{1}{2}=)$	1	6 in.
38	2	0	6

EXPLANATION.

We write the multiplier under the lowest unit of the multiplicand. 7 times 2 ft. are 14 ft., or 4 yds. and 2 ft. We set down the 2 ft. and reserve the 4 yds. to be added to the next product.

7 times 2 yds. gives 14 yds.; 14 yds. + 4 yds. = 18 yds., or 3 rds. and $1\frac{1}{2}$ yds. We set down $1\frac{1}{2}$ yds. and add 3 rds. to the product of 5 rds. by 7, obtaining 35 rds. + 3 rds., or 38 rds. Hence, the product in its first form is 38 rds. $1\frac{1}{2}$ yds. 2 ft. Reducing $\frac{1}{2}$ yd. to 1 ft. 6 in., this result simplifies into 38 rds. 2 yds. 0 ft. 6 in.

EXERCISE 87.

Multiply:

1. £8 15 s. 9 d. 3 far. by 6.
2. 11 T. 13 cwt. 95 lb. 12 oz. by 5.
3. 5 bbl. 25 gal. 3 qt. 1 pt. 2 gi. by 3.
4. 15 yr. 247 da. 19 hr. 25 min. 40 sec. by 8.
5. 27 lb. 8 ℥ 5 ʒ 1 ℈ 9 gr. by 13.
6. 12 mi. 45 rd. 3 yd. 2 ft. 8 in. by 10.
7. 9 mi. 156 rd. 2 ft. 10 in. by 15.
8. 8 A. 125 sq. rd. 26 sq. yd. 7 sq. ft. 131 sq. in. by 9.
9. 12 A. 130 sq. rd. 18 sq. yd. 5 sq. ft. 88 sq. in. by 7.
10. 7° 17′ 45″ by 15. | *11.* 82 bu. 3 pk. 7 qt. 1 pt. by 13.

For April 14th memorize the squares

12. If a dealer cart 27 T. 5 cwt. 85 lb. of coal one day, how much will he cart in 3 weeks?

13. If a farmer plow 3 A. 107 sq. rd. 3 sq. yd. 5 sq. ft. in one day, how much will he plow in 8 days?

14. A certain coil of wire contains 280 rds. 4 yd. 1 ft. 3 in., how much will 25 such coils contain?

202. IV. **The division of compound numbers** may be of two kinds:

(1) The division of a compound number *by an abstract number*, that is, into a number of equal parts;

(2) The division of one compound number *by another compound number.*

Ex. 1. Divide 52 gal. 3 qt. 1 pt. by 9.

SOLUTION.

	gal.	qt.	pt.
9)	52	3	1
	5	3	1

EXPLANATION.

We write the divisor to the left and divide the highest denomination first. 9 is contained in 52 gals. 5 times, with a remainder of 7 gal. We set down the 5 and convert the 7 gals. into quarts, giving 28 qts. 28 qts. + 3 qts. = 31 qts. 9 is contained in 31 qts. 3 times, with a remainder of 4 qts. We set down the 3 and convert 4 qts. into 8 pts. 8 pts. + 1 pt. = 9 pts. 9 is contained in 9 pts., once. Hence, the quotient is 5 gals. 3 qts. 1 pt.

Ex. 2. Divide 56 lbs. 9 oz. 12 pwt. by 9 lbs. 5 oz. 12 pwt.

Reducing each compound number to the same lowest denomination.

56 lb. 9 oz. 12 pwt. = 13632 pwt.
9 lb. 5 oz. 12 pwt. = 2272 pwt.

2272)13632(6, *Quotient.*
13632

EXERCISE 88.

Divide:

1. 69 bu. 3 pk. 5 qt. 1 pt. by 5.
2. 14 yr. 128 da. 20 hr. 34 min. 44 sec. by 4.
3. 59 T. 4 cwt. 93 lb. 10 oz. by 6. | *4.* 228 gal. 1 qt. 1 gi. by 15.
5. 43 mi. 11 rd. 3 yd. 2 ft. 2 in. by 8.
6. 90 mi. 186 rd. 4 yd. 1 ft. 9 in. by 11.

7. 33 A. 116 sq. rd. 13 sq. yd. 6 sq. ft. 107 sq. in. by 7.

8. 53 A. 115 sq. rd. 8 sq. yd. 8 sq. ft. 108 sq. in. by 9.

9. 84 lb. 1 ℥ 6 ʒ 2 ℈ 11 gr. by 13.

10. From a bin of grain containing 325 bu. 3 pk. 7 qt., how many sacks may be filled, each holding 1 bu. 3 pk. 5 qt.?

11. From a lot of wine amounting to 16 bbl. 2 gal. 1 qt., bottles containing 3 qt. 1 pt. 2 gi. are filled. How many are there?

12. If a lumberman get out 3 cd. 7 cu. ft. 712 cu. in. of wood a day, how many days will he require to prepare 55 cd. 5 cu. ft. 720 cu. in.?

13. A farm containing 92 A. 50 sq. rd. $7\frac{1}{2}$ sq. yd. is divided into house lots, each having an area of 20 sq. rd. 15 sq. yd. 5 sq. ft. How many will there be?

14. A lad walks 2 mi. 275 rd. 4 yd. an hour, how long will it take him to walk 22 mi. 285 rd. 4 yd. $1\frac{1}{2}$ ft.?

15. How many prescriptions, each weighing 1 ʒ 2 ℈ 10 gr., can a druggist make from 7 lb. 3 ℥ 7 ʒ 2 ℈ 8 gr. of quinine?

16. If a nugget weighs 1 lb. 1 oz. 1 pwt. 1 gr., how many similar nuggets will be required to weigh 522 lbs. 1 oz.?

APPLICATION TO LONGITUDE AND TIME.

203. Relation of Longitude and Time.—The earth revolves on its axis from West to East once in every 24 hours. As a result the sun appears to go round the earth from East to West in the same time. Hence, if we take a station on the earth at a given place, at all places east of that place any particular time, as noon, is earlier, since the sun arrives there earlier; at all places to the west time is later, since the sun arrives there later.

Since the sun passes over 360° of longitude in 24 hours, in 1 hour it passes over $\frac{1}{24}$ of 360°, or 15° of longitude. In 1 minute of time it passes over $\frac{1}{60}$ of 15°, or 15′ of longitude; in 1 second of time it passes over $\frac{1}{60}$ of 15′, or 15″ of longitude.

Stating these relations as a table:

15° of longitude = 1 hour of time.
15′ of longitude = 1 minute of time.
15″ of longitude = 1 second of time.

By means of this table, if we know the difference of longitude between two places, we may determine this difference in time; and, *vice versa*, if we know their difference in time, we may determine their difference in longitude.

In an old, well-settled country, of which maps have been made, the former relation is likely to be of use to the traveler, since he can obtain the difference in longitude between two places from a map or table, and then compute the difference in time. On the other hand, in exploring a new country the difference in time between places is known by the aid of chronometers, and it is necessary to determine the difference in longitude in order to make a map of the country, to determine distances in miles between places, etc.

EXERCISE 89.

ORAL.

1. On what and from what is latitude reckoned?

2. On what and from what is longitude reckoned?

3. What is the greatest latitude a place may have? Where is that place?

4. What is the greatest longitude a place may have?

5. What is the least latitude a place may have? Where are such places?

6. What is the least longitude a place may have? Where are such places?

7. What point on the earth has neither latitude nor longitude?

8. What class of men use latitude and longitude the most? For what do they use it?

9. What difference in longitude corresponds to a difference in time of 3 hours? Of 5 hours? Of 2 hrs. 3 min.? Of 40 min.? Of 50 min.? Of 1 hr. 10 min.? Of 30 sec.? Of 10 min. 40 sec.?

10. What difference in time corresponds to a difference in longitude of 30°? Of 60°? Of 45′? Of 1° 30′? Of 2° 30′? Of 4° 30′? Of 75°? Of 135°?

11. In which direction is the earth revolving on its axis? Which direction does the sun appear to be moving?

12. When it is noon at Chicago, is it morning or afternoon at Boston?

At Denver? At New York? At Omaha? At San Francisco? At London? At Paris? At Washington, D. C.? At Galveston? At St. Louis? At Montreal? At Havana? At your home?

13. Arrange the places of Example 12 in a column, putting the city at which the sun rises first at the top and the city at which the sun rises last at the bottom, and the rest in order between.

204. I. **Given the difference in time of two places to determine their difference in longitude.**

Ex. The difference of time between Boston and Washington is 23 min. 47 sec. What is the difference of longitude?

SOLUTION.

23 min. difference in time corresponds to 15′ × 23,
or 5° 45′ difference in longitude.
47 sec. difference in time corresponds to 15″ × 47,
or 11′ 45″ difference in longitude.

Adding, 5° 45′
11′ 45″
5° 56′ 45″, *Difference in longitude.*

205. II. **Given the difference in longitude of two places, to determine the difference of time.**

Ex. The difference in longitude between New York and San Francisco is 48° 23′ 45″. Find the difference in time.

SOLUTION.

48° difference in longitude corresponds to $\frac{48}{15}$ hr.,
or 3 hr. 12 min. difference in time.
23′ difference in longitude corresponds to $\frac{23}{15}$ min.,
or 1 min. 32 sec. difference in time.
45″ difference in longitude corresponds to $\frac{45}{15}$ sec.,
or 3 sec. difference in time.
Adding 3 hr. 12 min., 1 min. 32 sec., and 3 sec.,
we obtain 3 hr. 13 min. 35 sec., *Difference in time.*

If two places are both in east longitude, or both in west longitude, subtract in order to get their difference of longitude; if one is in east, the other in west longitude, add their longitudes in order to get the difference in longitude.

EXERCISE 90.

Determine the difference of longitude, having given the difference of time, as follows:

1. 1 hr. 25 min. 10 sec.
2. 5 hr. 0 min. 42 sec.
3. 7 hr. 55 min. 49 sec.
4. 9 hr. 31 min. 59 sec.

Find the difference of time between two places when the difference of longitude is as follows:

5. 43° 10′.
6. 77° 40′ 15″.
7. 12° 7′ 45″.
8. 69° 33′ 30.″

9. The difference of time between New York and Paris is 5 hr. 5 min. 20 sec. What is the difference of longitude? At which city is it noon first?

10. The difference of time between Canton, China, and Cincinnati is 10 hr. 49 min. 52 sec. What is the difference of longitude?

11. The difference of longitude between two cities is 42° 8′ 30″, what is the difference in time? When it is noon at the Western city, what is the time at the Eastern? When it is 10 A. M. at the Eastern city, what time is it at the Western?

12. When it is noon at the western of two points, whose difference of longitude is 75° 4′ 45″, what is the time at the other? When it is 6.30 P. M. at the Eastern city, what is the time at the other?

13. When it is 12 o'clock at San Francisco it is 2 hr. 58 min. 23$\frac{1}{3}$ sec. P. M. at Rochester. What is the difference of longitude?

14. A ship's chronometer set at Greenwich points to 8 hr. 14 min. 56 sec. P. M. when the sun is on the meridian. What is the longitude of the ship?

15. Longitude of Galveston is 94° 46′ 34″ W. and of Mobile is 88° 1′ 19″ W. When it is 10 A. M. at Mobile, what time is it at Galveston?

16. When it is noon at San Francisco (long. 122° 26′ 45″ W.)

it is 3 hr. 9 min. 7 sec. P. M. at Philadelphia. What is the longitude of Philadelphia?

[NOTE.—First find difference of longitude.]

17. Chicago is in long. 87° 37′ 30″ W. and Calcutta is 88° 23′ 15″ E. At 7 P. M. in Chicago what is the time in Calcutta? At 6 A. M. in Calcutta, what is the time in Chicago?

18. When it is 8 hr. 12 min. 48 sec. A. M. at Jerusalem (long. 35° 32′ E.) it is 6 A. M. at Paris. Find long. of Paris.

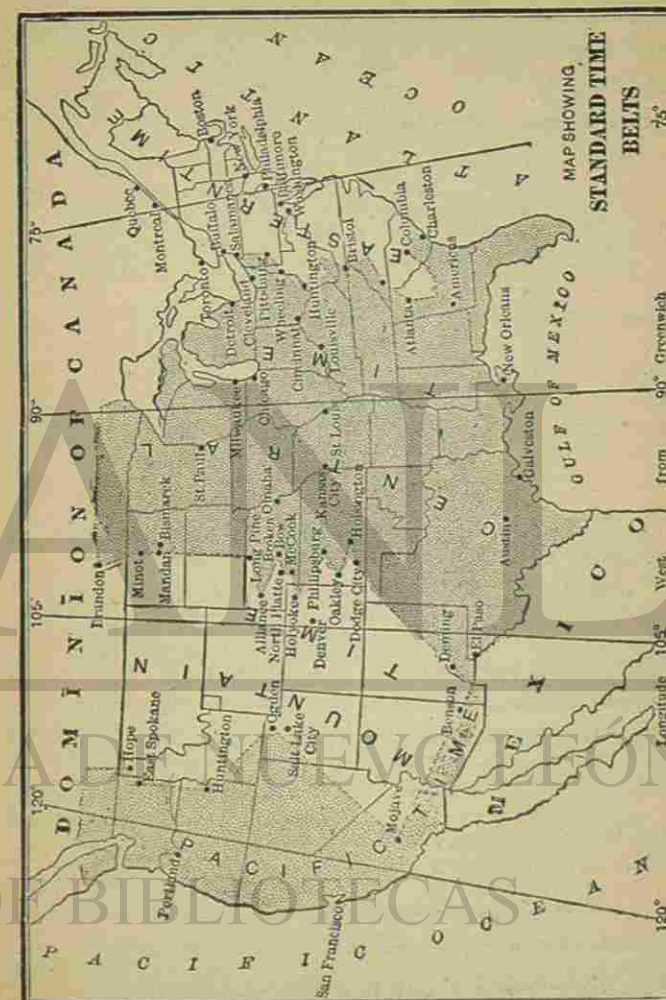
19. When it is noon at Rome (long. 12° 27′ 15″ E.) it is 7 hr. 20 min. 59 sec. P. M. in Manila Bay. What is the longitude of Manila Bay?

20. The longitude of St. Joseph is 109° 41′ 44″ W. and of Canton is 113° 14′ 1″ E. What is their difference of longitude? Of time?

21. The difference of time between St. Paul (long. 93° 5′ W.) and Havana is 42 min. 45 sec. What is Havana's longitude?

22. The difference of time between Boston (long. 71° 3′ 30″ W.) and Stockholm is 5 hr. 56 min. 28 sec. Find long. of Stockholm.

206. Standard Time.—Since 15° of longitude correspond to 1 hour of time, it has been found convenient to divide the territory of a country into belts 15° wide, the time in each belt being determined by a meridian approximately central in the belt. Time determined in this way is called Standard or Railway time. The standard meridians in the United States and Canada are the 75th, 90th, 105th, and 120th (west from Greenwich), and the corresponding belts are said to have Eastern, Central, Mountain, and Pacific time. Standard time in the eastern part of a belt may thus be as much as a half hour ahead of true local time; and in the western part of a belt may be a half hour behind. Travelers in passing from one belt to another must change their time by one hour. The boundaries of belts have been made somewhat irregular, owing to the configuration of the country, local conveniences, etc. On the opposite page is a map showing the standard time belts in the United States.



COMMON FRACTIONS AND DENOMINATE NUMBERS.

207. I. **To reduce a fraction of a denominate unit to lower units.**

Ex. 1. Reduce $\frac{2}{3}$ mile to lower units.

SOLUTION.

$$\tfrac{2}{3} \text{ of a mi.} = \tfrac{2}{3} \times 320 \text{ rods.} = 213\tfrac{1}{3} \text{ rds.}$$
$$\tfrac{1}{3} \text{ of a rd.} = \tfrac{1}{3} \times \tfrac{11}{2} \text{ yds.} = 1\tfrac{5}{6} \text{ yds.}$$
$$\tfrac{5}{6} \text{ of a yd.} = \tfrac{5}{6} \times 3 \text{ ft.} = 2\tfrac{1}{2} \text{ ft.}$$
$$\tfrac{1}{2} \text{ of a ft.} = 6 \text{ in.}$$

Hence, we have 213 rds. 1 yd. 2 ft. 6 in., *Result.*

Or the work may be expressed as an example in division.

OPERATION.

	mi.	rd.	yd.	ft.	in.
3)	2	0	0	0	0
	0	213	1	2	6, *Result.*

208. II. **To express a denominate number as a fraction of a higher unit.**

Ex. 1. Express 2 ft. 8 in. as the fraction of a yard.

SOLUTION.

$$2 \text{ ft. } 8 \text{ in.} = 32 \text{ in.} = \tfrac{32}{36} \text{ yd.} = \tfrac{8}{9} \text{ yd., } \textit{Result.}$$

Or we may proceed as follows:

$$8 \text{ in.} = \tfrac{2}{3} \text{ ft.}$$
$$2\tfrac{2}{3} \text{ ft.} = \frac{2\frac{2}{3}}{3} \text{ yd.} = \tfrac{8}{9} \text{ yd., } \textit{Result.}$$

209. It is sometimes required to express the **fraction of one unit as the fraction of a lower or higher unit.**

Ex. 1. Reduce $\frac{1}{720}$ yd. to a fraction of an inch.

$$\tfrac{1}{720} \text{ of 1 yd.} = \tfrac{1}{720} \times 3 \times 12 \text{ of 1 in.} = \tfrac{1}{20} \text{ in., } \textit{Result.}$$

Ex. 2. Reduce $\frac{8}{9}$ of a pint to a fraction of a gallon.

$$\tfrac{8}{9} \text{ pt.} = \tfrac{8}{9} \times \tfrac{1}{2} \times \tfrac{1}{4} \text{ of a gal.} = \tfrac{1}{9} \text{ gal., } \textit{Result.}$$

EXERCISE 91.

Reduce to lower denominations:

1. $\frac{7}{8}$ mi.
2. $\frac{2}{3}$ t.
3. $\frac{8}{9}$ lb. T.
4. £$\frac{19}{64}$.
5. $\frac{8}{27}$ yr.
6. $\frac{25}{72}$ degree.
7. $\frac{7}{36}$ sq. rd.
8. $\frac{7}{96}$ bbl.
9. $\frac{11}{15}$ lb. Ap.
10. $\frac{71}{72}$ cu. yd.
11. $\frac{49}{64}$ bu.
12. $\frac{25}{29}$ acre.
13. $\frac{7}{3000}$ day to min.
14. $\frac{9}{1600}$ bu. to pts.
15. $\frac{13}{17820}$ sq. rd. to sq. in.
16. $\frac{19}{158400}$ mi. to in.
17. $\frac{17}{5840}$ yr. to hr.
18. $\frac{31}{1260}$ bbl. to pt.

Reduce:

19. 4 hr. 30 min. to the fraction of a day.
20. 3 pk. 4 qt. 1 pt. to the fraction of a bushel.
21. 8 oz. 13 pwt. 8 gr. to lb.
22. 248 rd. 4 yd. 2 ft. 8 in. to mi.
23. 47 sq. rd. 12 sq. yd. 2 sq. ft. 132 sq. in. to A.
24. 2 qt. 1 pt. 2 gi. to gallons.
25. 6 da. 17 hr. 16 min. 48 sec. to weeks.
26. 6 cwt. 43 lb. $9\frac{3}{5}$ oz. to ton.

Add:

27. $\frac{1}{3}$ mi. $\frac{1}{2}$ rd. $\frac{3}{4}$ ft.
28. $\frac{5}{9}$ yr. $\frac{3}{5}$ da. $\frac{7}{8}$ hr.
29. $\frac{2}{5}$ lb. $\frac{1}{3}$ oz. $\frac{1}{4}$ pwt.
30. $\frac{2}{3}$ A. $\frac{1}{4}$ sq. rd. $\frac{15}{16}$ sq. yd.

Find the difference between:

31. $\frac{10}{27}$ A. and $\frac{7}{8}$ sq. rd.
32. $\frac{1}{3}$ mi. and $\frac{10}{11}$ rd.

DECIMAL FRACTIONS AND DENOMINATE NUMBERS.

210. I. **To reduce the decimal of a denominate unit to lower units.**

The method of this reduction is best shown by an example.

Ex. Express 0.425 gal. as quarts and pints.

SOLUTION.

$$0.425 \text{ gal.} = 4 \text{ qt.} \times 0.425 = 1.7 \text{ qt.}$$
$$0.7 \text{ qt.} = 2 \text{ pt.} \times 0.7 = 1.4 \text{ pt.}$$
$$1 \text{ qt. } 1.4 \text{ pt., } \textit{Result.}$$

211. II. **To express a denominate number as the decimal of a higher unit.**

Ex. Express 5 mo. 12 da. as the decimal of a year.

SOLUTION.

$$12 \text{ da.} = \frac{12}{30} \text{ mo.} = 0.4 \text{ mo.}$$

$$5.4 \text{ mo.} = \frac{5.4}{12} \text{ yr.} = 0.45 \text{ yr.}, \textit{ Result.}$$

EXERCISE 92.

Reduce to integral values in lower denominations:

1. .875 wk.	*5.* .842 mi.	*9.* .375 cu. yd.
2. .925 lb. T.	*6.* .423 A.	*10.* .046 mi.
3. .8324 T.	*7.* .576 lb. Ap.	*11.* .45 bbl.
4. .575 bu.	*8.* .0813 yr.	*12.* .175 A.

13. Find the value of 2.1365 months.
14. What is the sum of .14 mi. and .26 rd.?
15. What is the difference between .35 yr. and .48 mo.?

Reduce to the decimal of the next higher unit:

16. 7′ 50″.	*19.* 3 oz. 8 pwt. 12 gr.
17. 204 rd. 4 yd. 2 ft.	*20.* 3 da. 22 h. 4 min. 48 sec.
18. 16 cwt. 55 lb. 5 oz.	*21.* 2 pk. 4 qt. $1\frac{3}{5}$ pt.

EXERCISE 93.

GENERAL. ORAL.

1. What will a rod of wire cost at a cent an inch?

2. Bought a peck of nuts at 10 ct. a pt. Find the cost.

3. A grocer paid 18 ct. a doz. for some eggs and sold them at 35 ct. a score. What was his gain on each egg? What was his gain on a dozen? On a score? On a hundred?

4. How many pint bottles can be filled from 25 half-gal. jars of wine? From 20 gallons?

5. I bought calico at half a cent an inch, and sold it at 6 yards for a dollar. Did I gain or lose? How much on a yard?

6. A grocer buys tomatoes at 25 ct. a bushel and retails them at the rate of 2 qts. for 5 cents. How much does he gain on a bushel?

7. How many dozen in 7 score and 10?

8. How many square inches on a surface 3 inches square? On one 8 in. square? 2 ft. square?

9. How many cubic inches in a 4-in. cube? In a 6-in. cube? In a 1-ft. cube? In half a cubic foot?

10. A dealer buys a half dozen saws at $30 a score, and sells them so as to gain 50 cents apiece. What is the selling price of each?

11. How many feet in 17 fathoms? 28 fathoms?

12. How many hands are equal to 6 ft.? To $7\frac{1}{2}$ ft.?

13. Which are the next 3 leap years? How do you tell the leap years? Will 1926 be a leap year?

14. What was the first day in the 18th century? The last day? Which century were you born in? In which century is Dec., 1900? How many leap years in the 20th century?

15. What is the greatest difference of longitude two places can have? What is the longitude of your nearest city? Its latitude?

16. Which is heavier, a pound of gold or a pound of meat? An ounce of which of these is the heavier?

17. How would you find the number of cu. in. in a barrel containing $31\frac{1}{2}$ gallons? In 10 bushels?

18. How would you find the number of gallons that a bin, containing 100 bushels, will hold?

19. Which is the greater quantity, 6 dozen dozen or half a dozen dozen?

EXERCISE 94.

GENERAL REVIEW.

1. From $\frac{3}{8}$ A. take 75 sq. rd. 27 sq. yd. 5 sq. ft. 75 sq. in.

2. Reduce 25 da. 16 hr. 50 min. to the decimal of a week.

3. Add .07 year, $\frac{17}{25}$ day, and $\frac{7}{8}$ hr.

4. What would 8 gal. 2 qt. 1 pt. of wine cost at $6 a gallon?

5. If a cubic foot of water weigh 62.5 lbs., how many cu. yds. in a ton of water? How many ounces does a cu. in. of water weigh?

6. How many steps, of 28 in. each, must a man take in walking 7 mi. 120 rds.?

7. If a man walk 64 mi. 256 rds. in 20 hr. 15 min., how long will he require to walk 31 mi. 64 rds.?

8. How far will the same gentleman walk in 6 hrs. 45 min. at the same rate?

9. A baker pays $4.90 for a barrel of flour. He bakes it into 2-lb. loaves of bread, which he sells at 7 cents each. What is his gain?

10. When it is 15 min. after 10 A. M. at a certain city, what is the time at a western city if the longitude is 48° 7′ 30″ greater?

11. At an observatory the sun is seen to have passed through 12° 51′ 45″ since noon. What time is it?

12. A dozen spoons, each weighing 1 oz. 8 pwt. 20 gr., were sold at $1.50 an oz. What was the total price?

13. If I buy a 5-gal. can of oil for 45 cents and spill 2 pints, what do I really pay for each gallon that I use?

14. A wholesale grocer bought 2 T. 12 cwt. 60 lbs. cheese for $313.50, and retailed it at $\frac{3}{4}$ ct. an oz. Find his gain.

15. From a farm containing 80 A. 60 sq. rd. was sold a portion containing 38 A. 156 sq. rd., at $62.80 an acre, and the balance at $74.40. What was the total selling price?

16. What part of 4 gal. 3 qt. is 2 qt. 1 pt. 2 gi.?

17. What decimal of a rod is 1 ft. 7$\frac{3}{5}$ in.?

18. Find $\frac{7}{8}$ of 5 bu. 2 pk. 6 qt. 1 pt.

19. How long is it between half-past nine P. M. of Jan. 17, 1834, and quarter before four A. M. of the following 4th of March? (Answer in days, hours, and minutes.)

20. If an express train travel 45 mi. an hour, how many feet does it move over each second?

21. If the diameter of a circle is 1 mile, the circumference is 3.141592 mi. Express this decimal in integers of lower units.

22. Which is the middle day of the year 1901?

23. If a family use 6 gas burners every evening of the winter months for 4 hours of each evening, and each burner consumes 18 ft. an hour, what will their gas bill be at $1.30 per thousand?

24. How many parcels, each weighing 3 lbs. 7 oz., can be made up from .924 T.?

25. If 6 horses eat 19 bu. 2$\frac{1}{4}$ pk. of oats in 11 days, how long will 25 bu. 2 pk. 3 qt. supply 13 horses?

26. Take from 180° the sum of 71° 4′ 46″ and 23° 55′ 39″.

27. Find the value of 12$\frac{1}{3}$ cwt. + 39$\frac{1}{8}$ lb. + 7$\frac{3}{8}$ oz.

28. If 3 lbs. of wheat make 2 lbs. of flour, how many barrels of flour can be made from 343 bu. of wheat?

29. Reduce £1.0735 and .3764 mi. to integral values.

30. Change 11 oz. 18 pwt. 15 gr. to the decimal of a pound.

31. How many square feet in the surface of a box a yard long, 8 inches wide, and 18 inches deep?

32. A stream 25 yards wide and 25 feet deep flows 3 miles an hour. Find the number of cubic feet of water which passes a certain point in a minute.

33. How many revolutions will a wheel 9 feet 4 inches in circumference make in passing a field 54 rods 4 yards 2 feet 4 inches long?

34. Change 51830.7125 hours to years, days, hours, minutes, and seconds, reckoning 365 days to a year.

35. If a laborer dig a certain trench in 39 days, 4 hours, 10 minutes, how long will it require 8 laborers to dig a similar trench three times as long?

36. A box 7 feet long, 4 feet 4 inches wide contains 3$\frac{1}{2}$ cubic yards. How deep must it be?

37. A bin 12 feet 4 inches long and 6 feet 6 inches deep is to contain 100 bushels of grain. How wide must it be made?

38. If a mile of a certain wire weigh a ton, what is the weight in ounces of one foot of it?

39. Reduce 75 A. 95 sq. rd, 25 sq. yd. to the decimal of a square mile.

40. Change 2.12345 years to units of lower denominations.

41. From the sum of 17° 31′ 28″ and 41° 19′ 22″ take the difference between 81° 18′ 43″ and 63° 31′ 52″.

42. A cellar 20 yards long and 30 feet wide is to be dug. What depth will make 520 cubic yards?

43. When a locomotive is traveling 55 miles an hour, how many feet is it running each second?

44. If a train is running 40 feet 4 inches each second, how long will it require to run 90 miles?

45. Change 185 rods, 3 yards, 1 foot, 10 inches to the decimal of a mile.

46. Some numbers occur several times in the different tables of compound numbers. Collect all the times the number 12 occurs. Same for 8. Same for 60. Same for 3. Same for 24. Same for 16.

47. Change 3297.147 yds. to mi. rd. yd. ft. and in.

48. How many cubic feet in a box 4 ft. 3 in. wide, 4 ft. 6 in. long, and 8 in. deep?

49. From 180° take one-half the sum of 46° 18′ 39″ and 57° 12′ 17″. Also take one-half their difference from 180°.

50. The distance from the earth to the sun is 93,000,000 miles. How long would it take a boy who can run 11 ft. a second to traverse that distance? How long would it take a locomotive running a mile in 50 seconds?

CHAPTER XII.

PRACTICAL MEASUREMENTS.

212. Illustrations.—The practical application of denominate numbers to a special kind of work is facilitated in many cases by the use of a special unit which is peculiar to that particular kind of work. For example, if it be required to determine how many shingles are necessary to cover a roof 40 feet long and 30 feet wide, the computation is greatly simplified by the knowledge of the fact that, on the average, a roof space of 100 square feet contains 1000 shingles. Thus, since the above roof contains 12 times 100 square feet, 12000 shingles will be required to cover it.

Similarly, if it be required to determine how many bricks will be needed to build a wall 50 feet long, 30 feet high, and 3 bricks thick, the reckoning is greatly facilitated by the knowledge that a piece of wall 1 foot square and 3 bricks thick contains 21 bricks. Hence, to build the above wall will require $50 \times 30 \times 21$ or 31500 bricks.

213. General Methods.—It will be observed that computations of this kind consist in

1st, the determination of the number of units of area or volume in a given object, from linear measurements (see Arts. 175, 179).

2d, the use of a special unit in each kind of work applicable to a unit of area or volume of the given material. In the numerical applications of these methods frequent opportunities occur to diminish the work by *cancellation*. All the operations to be performed should be grouped together, and all possible cancellations made, before the final reduction is made.

It should also be remembered that in computations based on measure-

ments, it is not necessary to carry the work beyond the fourth or fifth figure, since all ordinary measurements are not accurate beyond these figures.

APPLICATIONS RELATING TO AREAS.

214. Rectangular Areas of Land.—In computing the area of a piece of land, the ordinary unit is the *acre*. In such computations it is convenient to remember that

$$43560 \text{ sq. ft.} = 1 \text{ acre.}$$

Hence, an acre is a square, each side of which is 208 + feet, or 70 − paces.

Other units of area frequently used are:

$$160 \text{ sq. rds.} = 1 \text{ A.}$$
$$4840 \text{ sq. yds.} = 1 \text{ A.}$$
$$640 \text{ A.} = 1 \text{ sq. mi.}$$

Ex. 1. How many acres in a field 320 feet long and 213 feet wide?

$$\text{No. acres} = \frac{\overset{8}{\cancel{320}} \times \overset{71}{\cancel{213}}}{\underset{363}{\cancel{\underset{1089}{\cancel{43560}}}}} = \frac{568}{363} = 1\tfrac{205}{363} \textit{ Ans.}$$

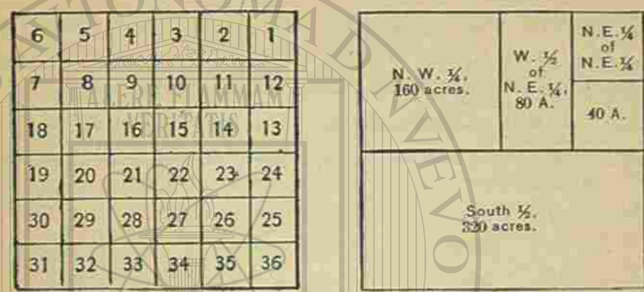
Ex. 2. How many acres in a meadow which averages $\frac{1}{4}$ mile in width and is $\frac{1}{2}$ mile long?

$$\text{Area} = \tfrac{1}{4} \times \tfrac{1}{2} \text{ sq. mi.} = \tfrac{1}{8} \text{ sq. mi.} = \tfrac{1}{8} \times 640 \text{ A.} = 80 \text{ acres.}$$

215. Townships and Sections.—In the eastern part of the United States, land, when settled, was divided according to the convenience or whim of the original settlers, and hence without any regular order or system. In the Western and Southern States most of the land was originally owned by the government, and has been divided according to a systematic plan, and disposed of in this form to settlers.

It is divided, first, into square **townships**, each side of which is 6 miles long; hence, each township contains 36 square miles. The sides of townships run east and west, and north and south.

Each township is subdivided into 36 **sections**, each containing 1 square mile, or 640 acres. The sections in a township are numbered according to a regular plan from 1 to 36.



TOWNSHIP. 1 SECTION = 640 ACRES.

A section is subdivided into *quarter-sections*, each containing 160 acres. Quarter-sections are subdivided into half-quarter-sections and quarter-quarter-sections or lots. A lot therefore contains 40 acres.

EXERCISE 95.

How many acres in a field*:

1. 3600 ft. long and 121 ft. wide.
2. 1815 yd. long and 256 ft. wide.
3. 495 yd. by 220 ft.
4. 84 ft. × 55 yd.*

Find the areas of the following rectangular surfaces:

5. 17 ft. long and 12 ft. wide in square yards.
6. 8 yd. long and $5\frac{1}{2}$ yd. wide in square rods.
7. 140 rd. by 72 rd. in acres.
8. 7 yd. 2 ft. by 3 yd. 1 ft. in square yards.
9. 45 rd. 3 yd. 2 ft. × 30 rd. 3 yd. in square rods.
10. 30 rd. 3 yd. 2 in. × 8 rd. 4 yd. in integral units.
11. A road 13 mi. long and 3 rds. wide in acres.
12. A ceiling 4 yd. 2 ft. × 3 yd. 1 ft. in square feet.

13. How many acres in three sections? In 5 sections? In $\frac{1}{2}$ section? In $\frac{3}{8}$ section? In $\frac{21}{40}$ section?

14. How many rods of fence are necessary to enclose a section? A half-section? A quarter-section?

* The statement of the dimensions of an object is often much abbreviated by the use of "by" or by the sign ×, which is then read "by."

15. How many square feet in the floor of a room 16 ft. 8 in. by 12 ft. 6 in.? How many sq. yds. in ceiling of same room?

16. How many sq. ft. on side of a barn 60 ft. 6 in. long by 22 ft. 4 in. high?

216. Circular Areas.—It is proved by geometrical methods that the area of a circle is determined with sufficient accuracy for all practical purposes by the formula (see Arts. 323, 325),

Area of a circle = 3.1416 × (*square of radius of the circle*).

Ex. What part of an acre is grazed over by a cow tied by a tether 100 feet long?

$$\text{Area} = 3.1416 \times 100 \times 100 \text{ sq. ft.} = 31416 \text{ sq. ft.}$$

$$= \tfrac{31416}{43560} \text{ A.} = 0.721 + \text{acre, } \textit{Area.}$$

EXERCISE 96.

Find the areas of the following circles:

1. Radius = 10 ft.
2. Diameter = 30 ft.
3. Radius = 8 yd. 2 ft.
4. Diameter = 78 rd. $3\frac{2}{3}$ yd.

How many acres in each of the following circles:

5. Radius = 180 rd.
6. Diameter = 125 rd.
7. Diameter = $63\frac{1}{2}$ rd.
8. Radius = 200 rd. $2\frac{1}{3}$ yd.

9. A pond in the shape of a circle has a radius of 25 rd. 5 yd. How many acres in its surface?

10. How many square inches on the face of a coin whose radius is 0.5 in.? Another, whose radius is 1.5 in.?

217. In paving, the unit of computation is the **square yard.** In **roofing, flooring**, etc., the unit is the **square**, which equals 100 sq. ft.

In roofing with shingles, the average shingle is taken to be 18 in. long, 4 in. wide, with 5 in. exposed to the weather. 1000 shingles, or a bundle, are allowed for shingling 1 square.

EXERCISE 97.

What will be the cost of:

1. Shingling a roof 48 ft. × 23 ft. 8 in. @ $8.45 per square?

2. Paving a walk 640 ft. long and 8 ft. 3 in. wide @ $2.10 per square yard?

3. Lining a surface 80 ft. 3 in. × 61 ft. 9 in. @ $1.35 per sq. yd.?

4. Tinning a roof 23 yd. 2 ft. 3 in. × 19 yd. 1 ft. 6 in. @ $1.10 a sq. yd.?

5. The shingles for a roof on a building 80 feet long, if the rafters are 36 ft., and shingles cost $7.25 per M.?

6. Paving a city street 3 miles long and 40 feet wide, at 80 cents per sq. yd.?

218. In **plastering, painting,** etc., the unit of computation is the **square yard.**

Custom varies as to the allowance to be made for openings in a wall, made by doors, windows, etc. One rule is that no deduction be made for openings in a room, aggregating less than 7 sq. yds., and that for openings aggregating more than 7 sq. yds. one-half their sum be deducted.

Ex. What will it cost to plaster the walls and ceiling of a room 22 ft. long, 14 ft. wide, and 10 ft. high, at $.33 a square yard, deducting half the area of 2 doors, each 7 × 3 ft., and 5 windows, each 6 × 3 ft.?

SOLUTION.

The walls and ceiling may be conveniently indicated as follows:

The walls (that is, the ends and sides combined) make a rectangle 2 × 22 + 2 × 14 feet, or 72 feet long.

Hence, area of walls = 72 × 10 sq. ft. = 720 sq. ft
area of ceiling = 22 × 14 sq. ft. = 308 sq. ft.
Total area = 1028 sq. ft.

Area of doors = 2 × 7 × 3 sq. ft. = 42 sq. ft.
Area of windows = 5 × 6 × 3 sq. ft. = 90 sq. ft.
132 sq. ft.

Area deducted = $\frac{1}{2}$ × 132 sq. ft. = 66 sq. ft.
Net area = 962 sq. ft. = $\frac{962}{9}$ sq. yds.

$$\text{Cost of plastering} = \frac{\overset{.11}{\$0.33} \times 962}{\underset{3}{9}} = \$35.27, \textit{ Cost.}$$

EXERCISE 98.

What will it cost to:

1. Paint the walls and ceiling of a room 30 ft. by 16 ft. by 9 ft. high at 8 ct. a sq. yd.?

2. Plaster a room 24 ft. × 20 ft. × 16 ft. at 42 ct. a sq. yd., allowing half of 3 doors each 7 ft. × $3\frac{1}{2}$ ft., and 5 windows each 5 ft. × $3\frac{1}{2}$ ft.?

3. Paint a room 40 ft. × 31 ft. 6 in. × 9 ft. 6 in. at 5 ct. a sq. yd. for the floor and 7 ct. for the rest, allowing a deduction of 10 sq. yd. for windows and doors?

4. Plaster the five rectangular faces of a church, whose inside dimensions are 60 yds. 2 ft. × 40 yds. 1 ft. 6 in. × 30 ft. at 9 ct. per sq. yd., if $\frac{1}{10}$ of the entire *wall* surface is deducted for openings?

219. Carpeting.—To determine the number of yards of carpet needed to carpet a room, it is necessary to *determine the number of strips which the room will require; and to multiply the number of strips by the length of each strip.*

The number of strips is determined by dividing the width of the room (if the carpet runs lengthwise), or the length of the room (if the carpet runs crosswise), by the width of a single strip (usually 1 yd. or $\frac{3}{4}$ yd.). A part of a strip is regarded as a whole strip, the part not needed being folded under. When the carpet is figured, in order to match the figures, strips of carpet must usually be taken a little longer than the length of the room (and the unused ends folded under).

Ex. How many yards of carpet, $\frac{3}{4}$ yd. wide, will be required to cover a floor 26 feet long and 17 feet wide, if the carpet runs lengthwise and $\frac{1}{6}$ of a yard is wasted in matching patterns?

Solution.

$$\text{No. of strips} = \tfrac{17}{3} \div \tfrac{3}{4} = 7\tfrac{5}{9}, \text{ or } 8.$$
$$\text{Length of a strip} = \tfrac{26}{3}\text{ yd.} + \tfrac{1}{6}\text{ yd.} = \tfrac{53}{6}\text{ yd.} = 8\tfrac{5}{6}\text{ yd.}$$
$$\text{No. yds.} = 8 \times 8\tfrac{5}{6} = 70\tfrac{2}{3}, \textit{ Result.}$$

EXERCISE 99.

1. A room $8\frac{1}{2} \times 7\frac{1}{3}$ yards is to be carpeted by unfigured carpet a yard wide, and strips are to run lengthwise. How many yards will be required?

2. If carpet is $\frac{3}{4}$ yd. wide, strips run lengthwise, and there is $\frac{1}{6}$ yd. wasted in matching patterns, how many yards must be bought for a room 9 yd. long and 5 yd. wide?

3. How many yards of carpet, $\frac{7}{8}$ yd. wide, will be required for a room 17 yd. $\times$ 17 ft., if strips run lengthwise? If strips run crosswise?

4. Find the cost of carpeting a room 19 ft. long and 14 ft. wide, with carpet $\frac{3}{4}$ yd. wide and costing \$1.50 a yard, when the strips run crosswise and there is a waste of $\frac{1}{4}$ yd. in matching.

5. A room 13 ft. $\times$ $10\frac{1}{2}$ ft. is to be carpeted with carpet $\frac{3}{4}$ yd. wide and worth \$2.25 a yard. There will be waste of $\frac{3}{16}$ yd. in matching. Will it be cheaper to run the strips lengthwise or across the room? How much cheaper?

220. Papering.—The unit used for wall paper is the *roll*, a roll being 8 yards long, 18 inches, or $\frac{1}{2}$ yard, wide. (The double roll, 16 yards long and 18 inches wide, is also used at times.)

To determine the number of rolls of wall paper needed to cover the walls of a given room,

1st. *Find the number of strips of paper by multiplying the number of yards in the distance around the room by 2;*

2d. *Find the number of rolls by dividing the number of strips required by the number of strips which can be cut from a single roll.*

A part of a strip of wall to be covered counts as a whole strip, and a part of a roll needed as a whole roll. (But in cutting paper, parts left over are rejected.)

Owing to waste in matching patterns, turning corners, etc., and gain due to windows, doors, etc., the estimate of the number of rolls required can only be approximate.

Borders used at the top of the wall are sold by the linear yard.

Ex. How many rolls of paper are required to cover the walls of a room 24 ft. long, 18 ft. 6 in. wide, and 10 ft. high?

Solution.

The distance around the room may be conveniently represented as follows:

$$\text{Hence, distance} = 2 \times 24\text{ ft.} + 2 \times 18\tfrac{1}{2}\text{ ft.} = 85\text{ ft.}$$
$$= 28\tfrac{1}{3}\text{ yds.} = 56\tfrac{2}{3}\text{ half-yds.}$$
$$\therefore \text{No. strips} = 57.$$
$$\text{No. strips cut from one roll} = \frac{24\text{ ft.}}{10\text{ ft.}} = 2.$$
$$\text{No. rolls} = \tfrac{57}{2} = 28\tfrac{1}{2}, \text{ that is } 29, \textit{ Result.}$$

EXERCISE 100.

1. How many rolls of paper are required to paper the walls of a room 15 ft. long, 11 ft. wide, and 9 ft. high?

2. What will be the cost of the paper for the walls of a room 40 ft. long, 32 ft. wide, and 11 ft. high, at 45 cents a roll?

3. What will it cost for paper in a room $21\frac{1}{2}$ ft. long, $16\frac{1}{2}$ ft. wide, and 15 ft. high, at 60 cents a roll, if the ceiling paper runs crosswise?

4. The walls of a room 28 ft. long, 25 ft. wide, and 15 ft. high are to be papered with paper selling at 75 cents a roll; there is a border of $2\frac{1}{2}$ ft. width at 4 cents a yd., and a baseboard of 6 in. What is total cost?

[Note.—Strips need be only 12 feet long.]

APPLICATIONS TO VOLUMES.

221. Board Measure.—In measuring boards and lumber, the unit is the **board foot,** which is a rectangular piece of wood 1 foot square and 1 inch thick.

Large quantities of lumber are sold in terms of the **hundred** or **thousand,** by which is meant 100 board feet or 1000 board feet.

Boards less than 1 inch in thickness are estimated as if they were 1 inch thick.

A cubic foot of lumber contains 12 board feet. Hence, the number of board feet in a piece of lumber is 12 times the number of cubic feet, and the number of cubic feet is $\frac{1}{12}$ the number of lumber feet.

All **square** lumber, as planks, joists, beams, etc., is estimated in board feet.

Round timber, as masts, etc., is estimated in cubic feet. To find the number of board feet in a given board or piece of lumber, *multiply two of the dimensions in feet by the other dimension in inches.*

Ex. 1. How many board feet in a plank 16 feet long, 10 inches wide, and 3 inches thick?

The width = 10 inches = $\frac{10}{12}$ feet.

$$\therefore \text{No. board feet} = \frac{\overset{4}{\cancel{16}} \times 10 \times \cancel{3}}{\underset{\cancel{4}}{\cancel{12}}} = 40, \textit{ Result.}$$

Ex. 2. How many feet in a board 14 feet long, 10 inches wide, and $\frac{7}{8}$ inch thick?

Since $\frac{7}{8}$ inch is taken as 1 inch,

$$\text{No. board feet} = \frac{\overset{7}{\cancel{14}} \times \overset{5}{\cancel{10}} \times 1}{\underset{\cancel{6}\;3}{\cancel{12}}} = 11\tfrac{2}{3}, \textit{ Result.}$$

EXERCISE 101.

How many board feet in:

1. A plank 12 ft. long, 8 in. wide, and 2 in. thick? $\frac{3}{4}$ in. thick?

2. 5 beams 9 ft. long, 10 in. wide, and 4 in. thick? $4\frac{1}{2}$ in. thick?

3. 20 rafters 24 ft. long, 6 in. wide, and 4 in. thick?

4. 45 joists 18 ft. × 8 in. × 6 in.? If $5\frac{1}{2}$ in. thick? If $\frac{1}{2}$ in. thick?

What is the cost of:

5. 75 boards 16 ft. × 9 in. × 1 in. @ $18 per M.?

6. 12 posts 8 ft. × 5 in. × 6 in. @ $2.20 a hundred?

7. 85 joists 12 ft. × 11 in. × 4 in. @ $23 per M.?

222. Capacity of Bins.—Instead of determining the number of bushels which a bin will contain by actually filling the bin and counting the number of bushels, it is much more convenient to compute the capacity of the bin in bushels from the linear dimensions of the bin.

Since a bushel contains 2150.42 cubic inches, to find the capacity of a bin in bushels, *divide the number of cubic inches in the volume of the bin by 2150.42.*

Ex. 1. How many bushels will a bin 20 feet long, 8 feet wide, and 4 feet deep contain?

$$\text{No. bushels} = \frac{20 \times 8 \times 4 \times 1728}{2150.42} = 514.28.$$

Since 2150.42 cubic inches = $1\frac{1}{4}$ cubic feet nearly, in many cases a sufficiently accurate computation of the number of bushels in a bin is obtained by dividing the number of cubic feet in the bin by $1\frac{1}{4}$, that is, by multiplying it by $\frac{4}{5}$. Hence, *the approximate number of bushels in a bin = $\frac{4}{5}$ the number of cubic feet in the bin.*

Ex. 2. Approximately how many bushels will a bin 15 × 8 × 6 feet hold?

$$\text{Approx. no. bush.} = \frac{\overset{3}{\cancel{15}} \times 8 \times 6 \times 4}{\cancel{5}} = 576.$$

Grain, seeds, and small fruits are sold by **stricken measure.** Coarser materials, as potatoes, corn in the ear, etc., are sold by **heaped measure.**

The number of bushels by heaped measure = $\frac{4}{5}$ *the number by stricken measure.*

Ex. 3. How many bushels of corn in the ear will a bin $15 \times 8 \times 6$ feet hold?

$$\text{No. bushels} = \frac{\overset{3}{\cancel{15}} \times 8 \times 6 \times 4 \times 4}{\cancel{5} \times 5} = 460\tfrac{4}{5}$$

EXERCISE 102.

1. How many cubic inches in a box 16 in. long, 10 in. wide, and 3 in. deep? What part of a cu. ft. is that?

2. Required the number of cu. yds. in a wall 2 yd. 2 ft. 8 in. long, 1 yd. 1 ft. 6 in. wide, and 2 yd. 9 in. high.

3. How many bushels of oats can be put in a bin $15 \times 7 \times 6$ ft.? (Accurate.)

4. How many bushels of wheat can be put into a bin $24 \times 16 \times 10$ ft.? (Approximate.)

5. A bin $22 \times 20 \times 8$ ft. is full of potatoes. About how many bushels in it?

6. A crib of corn is $60 \times 15 \times 6$ ft. About how many bushels of corn in the crib?

7. A bin $6 \times 7 \times 8$ ft. is $\frac{2}{3}$ full of wheat. It is bought by approximate measurement at 70 ct. a bushel, and sold under accurate measurement at 90 ct. a bushel. What was the gain?

223. Capacity of Cisterns.—It is convenient to be able to determine the capacity of a cistern or tank, in gallons, from the linear dimensions of the tank.

Since a gallon contains 231 cu. in., to find the capacity of a cistern in gallons, *divide the number of cubic inches which the cistern contains by* 231.

The capacity of a large cistern is also obtained sometimes

in terms of larger units, as the barrel ($31\frac{1}{2}$ gals.), or the hogshead (63 gals.).

Ex. How many gallons will a tank $22 \times 10 \times 6$ ft. contain?

$$\text{No. gals.} = \frac{\overset{2}{\cancel{22}} \times 10 \times \overset{2}{\cancel{6}} \times 1728}{\underset{\underset{7}{\cancel{21}}}{\cancel{231}}} = 9874\tfrac{2}{7}.$$

EXERCISE 103.

How many gallons in a cistern:

1. Containing 125 cu. ft.?

2. Containing 132 cu. yds.?

3. $12 \times 11 \times 8$ ft.?

4. $22 \times 21 \times 20$ ft.?

5. $8 \times 7 \times 6$ yds.?

6. $90 \times 14 \times 10$ ft.?

224. Excavations and Embankments.—In moving earth the unit is a cubic yard, or 27 cubic feet, called a *load.*

Ex. What is the cost of excavating a cellar $18 \times 24 \times 6$ ft., at 10 cents a load?

$$\text{No. loads} = \frac{\overset{2}{\cancel{18}} \times 24 \times \overset{2}{\cancel{6}}}{\underset{\cancel{9}}{\cancel{27}}} = 96.$$

$$\text{Cost} = 96 \times \$.10 = \$9.60.$$

225. Stone Work and Masonry.—In stone work and masonry the usual unit is the *perch,* or $24\frac{3}{4}$ cubic feet.

A perch of stone is a rectangular pile $16\frac{1}{2}$ feet long, $1\frac{1}{2}$ feet wide, and 1 foot deep, and containing, therefore, $24\frac{3}{4}$ cubic feet.

As used in a wall, the perch is regarded as consisting of 22 cubic feet of stone, with $2\frac{3}{4}$ cubic feet allowed for mortar and filling.

Sometimes, however, masonry is estimated by the cubic foot.

EXERCISE 104.

Find the number of perches of masonry in these walls:

1. $75 \times 3 \times 2$ ft.

2. $12 \times 11 \times 4\frac{1}{2}$ ft.

3. 55 yd. $\times$ 16 ft. $\times$ 4 ft.

4. 60 yd. $\times$ 15 ft. $\times$ 5 ft.

How many cubic yards of earth in the following excavations:

5. Cellar $30 \times 20 \times 7$ ft.?

6. Tunnel 90 rd. 4 yd. $\times$ 12 yd. $\times$ 20 ft.?

7. Find the cost of opening a railroad cut 90 yards long, averaging 40 ft. wide and 32 ft. deep, at $1.25 a cu. yd.

8. A wall 374 ft. long and 6 ft. square at the end is to be built at $3.75 a perch of 22 cu. ft. Find cost.

9. A foundation 70 ft. long, 9 ft. deep, and 8 ft. wide is to be laid. For necessary excavations the charge is 60 cts. a load, and for building the wall $4.50 a perch ($24\frac{3}{4}$ cu. ft.). What is the entire cost?

226. Brickwork.—In brickwork the unit is usually **one thousand bricks**, but sometimes the cubic foot is used.

The average size of bricks is $8 \times 4 \times 2$ inches.

The following practical units are also used:

1. A square foot of wall, 1 brick or 4 inches thick, contains 7 common bricks.

2. A square foot of wall, 2 bricks or 9 inches thick, contains 14 bricks.

3. A square foot of wall, 3 bricks or 13 inches thick, contains 21 bricks.

Hence, to find the number of common bricks required for a wall,

Multiply the number of square feet in the wall by 7, if the wall is 1 brick thick; by 14, if it is 2 bricks thick; by 21, if it is 3 bricks thick.

In a building, the corners, doors, and window-spaces are deducted in estimating the number of bricks, but not in estimating labor.

Ex. How many common bricks will be required to build a house 40 ft. long, 24 ft. wide, 18 ft. high, the walls being 13 in. thick, allowing 280 sq. ft. for doors and windows?

SOLUTION.

Entire distance around the building = 2(40 + 24) ft. = 128 ft.
Entire area of wall = 128×18 sq. ft. = 2304 sq. ft.
Deduction for 4 corners = $\frac{13}{12} \times 18 \times 4$ sq. ft. = 78 sq. ft.
Deduction for windows, etc. = 280 sq. ft.
Entire deduction = 358 sq. ft.
Net area of walls = 2304 sq. ft. − 358 sq. ft. = 1946 sq. ft.
No. bricks = 1946×21 = 40,866, *Result.*

Bricks are often of special shapes and sizes. Thus, Philadelphia and Baltimore bricks are $8\frac{1}{4} \times 4\frac{1}{8} \times 2\frac{3}{8}$ in.; Maine bricks are $7\frac{1}{2} \times 3\frac{3}{8} \times 2\frac{3}{8}$ in.; Milwaukee bricks are $8\frac{1}{2} \times 4\frac{1}{8} \times 2\frac{3}{8}$ in.; North River bricks are $8 \times 3\frac{1}{2} \times 2\frac{1}{4}$ in., etc. To determine the number of bricks of a special kind required to make a wall, *increase each of the three dimensions of the brick by $\frac{1}{4}$ in. to allow for mortar, and divide the contents of the wall by the contents of 1 brick.*

In computing the number of bricks in a pavement, no change is made in the dimensions of a brick, since no allowance is made for mortar.

EXERCISE 105.

1. How many common bricks are required to build a house 36×32 ft., and 26 ft. high, the wall to be 3 bricks thick, if an allowance of 150 sq. ft. is made for openings?

2. What will they cost at $16 a thousand?

3. How much will the bricks for a house 40×36 ft., 32 ft. high, and walls 13 in. thick, cost at $18 per M.? What is the cost of laying them at $2.25 a sq. yd.?

(No allowance for corners or openings.)

4. What will be the cost of the bricks and the laying of them, for a house 45 ft. sq., 28 ft. high, walls 3 bricks thick, after a deduction of 425 sq. ft. of surface for openings and corners, if bricks are worth $14 a thousand and laying them costs $2.75 per M.?

5. A pavement, 90 ft. long, $10\frac{1}{2}$ ft. wide, is laid with Milwaukee bricks on the edge, 60 to a square yard. What is the cost, at the rate of $10 per M. for the bricks and 2 cts. a sq. ft. for the labor?

227. Other Units of Weight or Volume.—It is sometimes useful to determine the number of tons in a heap or bin of coal, from the linear dimensions of the heap or bin.

From 36 to 40 cu. ft. of ordinary anthracite coal make 1 ton.
From 36 to 45 cu. ft. of bituminous coal make 1 ton.
About $34\frac{1}{2}$ cu. ft. of Lehigh white-ash coal (egg size) make 1 ton.
About 35 cu. ft. of Schuylkill white-ash coal (egg size) make 1 ton.
About 36 cu. ft. of Schuylkill gray or red-ash coal make 1 ton.

Similarly, to determine the number of tons of a quantity of hay from its linear dimensions:

About 500 cu. ft. of hay loose or in loads = 1 ton.
About 400 cu. ft. of hay in a mow = 1 ton.
About 270 cu. ft. of hay in a settled stack = 1 ton.

Coal is at times sold in small quantities by the *bushel*. A bushel weighs 72 lbs. and is about $\frac{1}{28}$ ton.

EXERCISE 106.

1. A box 6 × 5 × 4 ft. is full of ordinary coal. Counting a ton to 38 cu. ft., how many tons are there in the box? How many tons if it were full of Lehigh white-ash coal?

2. A mow of hay is 40 × 28 × 20 ft. How many tons of hay will it contain?

3. How many tons of bituminous coal in a train of 42 cars averaging 36 × 7 × 5 ft.?
(Allow 40 cu. ft. to the ton.)

4. A farmer hauled 60 loads of hay, and they averaged 18 × 9 × 10 ft. What was its value at $18 a ton?

5. What advantage is there in determining the number of tons of hay or of coal by measurements of the dimensions of the mows or bins? In what other ways might these weights be determined?

EXERCISE 107.

REVIEW.

1. Find the area of a circle whose radius is 3 inches. Another, whose radius is 7 inches. Radius is 8 feet. Radius is 20 rods. Radius is .15 inch. Radius is $\frac{3}{4}$ yard.

2. Find in acres the area of a farm 1 mile 250 rods × 85 rods.

3. Which is the larger volume, $111\frac{3}{4}$ gallons or 12 bushels?

4. How many square inches in the entire exterior surface of a box 10 × 9 × 8 inches? What part of a cubic yard will it contain?

5. About a square lawn 40 rods on a side is laid a drive 3 yards wide. How many square rods in the drive?

6. A box is made of 2-inch material, and the outer dimensions are 12 × 10 × 9 inches. How many cubic inches in the material of the box including the lid? How many cubic inches will the box contain?

7. How many board feet in a load of 40 rafters, each 18 feet long and 8 × 6 inches at the end?

8. A railroad passes through a farm, taking a strip $1\frac{1}{2}$ miles long and 66 feet wide. What is the value of this land at $80 an acre?

9. To dig a sewer 3 miles long, 8 feet deep, and 4 feet wide, the contract called for 80 cents per cubic yard. What did it cost?

10. How many bushels of grass seed will a bin 14 × 10 × 9 feet contain? How many bushels of potatoes?

11. What will it cost to lay a wall 200 rods long and 5 × 8 feet on the end, at $2.60 a perch (22 cubic feet)?

12. What will be the cost of carpeting a room 7 × 11 feet with carpet $\frac{3}{4}$ yard wide, worth $1.50 a yard, and put down crosswise?

13. The walls of a room 16 feet × 13 feet, and 9 feet high, are papered with paper worth 60 cents a roll, and the ceiling is painted at 30 cents a square yard. What will it cost?

A large parlor is 50 × 38 feet, and 14 feet high.

14. What will carpet cost, $\frac{7}{8}$ yard wide and worth $2.50 a yard, if put down lengthwise and every strip wastes $\frac{1}{4}$ yard in matching?

15. What will papering its walls cost at 90 cents a roll, allowing $\frac{1}{2}$ of one end for windows and doors?

16. What will painting the ceiling cost at 18 cents a square yard?

17. What is the area of the largest circle that can be drawn in the ceiling?

18. What will it cost to roof a barn whose rafters are 18 feet 6 inches long, and the ridge-pole 35 feet, at $8.40 per square?

19. What will it cost to floor a 3-story house, 60 × 42 feet, with 2-inch boards, at $34.60 per M.?

20. What will it cost to cement the floor and side walls of a cellar 25 × 24 feet, and 8 feet high, at 48 cents a square yard?

21. What is the cost of the carpet, running crosswise, $\frac{3}{4}$ yard wide, and worth $1.60 a yard, on a room 23 × 22 feet, if there must be a waste of $\frac{1}{4}$ yard for matching?

22. A bin occupying $\frac{2}{3}$ of a cellar which is $20 \times 18 \times 11$ feet is $\frac{3}{4}$ full of coal (ordinary anthracite). Find its approximate value at $5.25 a ton.

23. How many gallons will a cistern 7 feet cube contain?

24. Water weighs $62\frac{1}{2}$ pounds to the cubic foot. What will the water in a tank containing 1000 gallons weigh?

25. A county in one season built 17 miles of macadamized road 12 feet wide, at the rate of 75 cents per square yard. What was the total cost?

26. If a roll of paper a mile long will just cover $193\frac{3}{5}$ square yards, how wide is the paper? Its width is what decimal of its length?

27. What will be the cost of the palings necessary to inclose a lawn 36×41 yards, if they are 2 inches wide, placed 2 inches apart, and sell for $2.75 a hundred?

28. How many bushels will a bin hold whose dimensions are $15 \times 11\frac{1}{4} \times 8\frac{3}{4}$ feet? How many gallons?

29. A bin $15 \times 8\frac{1}{2} \times 6\frac{2}{3}$ feet was full of apples, from which enough cider was pressed to fill a tank $9\frac{1}{3} \times 4\frac{1}{2} \times 3$ feet, $\frac{3}{4}$ full. The apples were bought at 36 cents a bushel (approximately) and the cider was sold at 33 cents a gallon. Required the gain.

30. How many square yards in a circular flower-bed 56 feet across?

31. A certain tank contains 1000 gallons. How many bushels would it hold?

32. There are to be laid two cement walks, one on each side of a certain street 2 miles long. The walks are to be 60 in. wide and to cost 40 cts. a sq. yd. What is the total cost?

33. A circular pond 28 yd. across, is surrounded by a path 4 ft. wide. What is the area of the path?

34. My farm is 220 rd. long and 144 rd. wide. It is entirely surrounded with a four-wire fence. The wire cost me half a cent a foot and the posts, which are 11 ft. apart, cost 9 cents each. What did the material cost?

35. What will be the cost of plastering the walls and ceiling of a room 16 ft. 6 in. long × 11 ft. 4 in. wide × 10 ft. high @ 36 ¢ a sq. yd., allowing 15 sq. yd. for doors, etc.?

36. If a cubic foot of water weighs $62\frac{1}{2}$ lbs., what will a barrel of water weigh?

37. How many gallons of water required to weigh a ton?

38. If the diameter of a pail is 6 in., how many square inches in the bottom and lid together?

39. There is an iron cistern made of 3-in. metal plates, and without any top. If the inner dimensions are 8 ft. long, 5 ft. wide, and 10 ft. deep, how many cubic feet of iron in the material?

40. Find the weight of that iron if iron is 7 times as heavy as water.

CHAPTER XIII.

PERCENTAGE.

228. Illustration.—A man has two investments, one of $800 in real estate, which brings a return of $48 a year; and another of $500 in railroad bonds, which brings in $35 a year. Which is the better investment?

The comparison of the two investments is much facilitated by determining the proceeds of a single hundred dollars in each case, and comparing them.

Thus, if $800 in real estate brings in $48 a year.
$100 " " " $6 a year.
If $500 in bonds brings in $35 a year,
$100 " " $7 a year.

Since $7 per $100 invested is a better return than $6 per $100, the investment in bonds is relatively more profitable.

229. Value of 100 as a Basis of Comparison.—This use of a standard base, as 100, in making estimates, has two advantages: (1) it facilitates comparisons, as in the above example; (2) it leads in time to an instinctive grasp of the various rates used with reference to one hundred.

Thus, 6 per cent. (6 out of every hundred), 7 per cent., etc., come to have a sharp and definite meaning in the mind, which meaning rises instantly when such words are used.

230. Definitions and Symbols.—**Percentage** is the process of computing with reference to 100 as a base.

Per cent. (from *per*, by, and *centum*, one hundred) means by or on the hundred. Thus, when a merchant gains 15 per cent., he means that he gains $15 for every $100 invested

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Per cent. (from *per*, by, and *centum*, one hundred) means by or on the hundred. Thus, when a merchant gains 15 per cent., he means that he gains $15 for every $100 invested in goods. If a poultry-raiser lose 8 per cent. of his fowls, he means that he loses 8 fowls out of every hundred that he has.

Hence, so many per cent. means so many hundredths.

The symbol, %, is used for the words "per cent." Since *per cent.* means *hundredths*, per cent. may also be indicated by a common or *decimal fraction*, with 100 as a denominator. Thus,

$$6\% = \tfrac{6}{100} = .06;\quad 30\% = \tfrac{30}{100} = .30 = .3.$$

The **quantities considered** in computations in percentage are the *base, rate, percentage*, and the *amount* or *difference*.

The **base** is the number of which a certain number of hundredths is taken. It is denoted by the symbol, b.

The **rate** is the number of hundredths which is taken (or the number of units taken with reference to every hundred units in the base). When expressed decimally it is denoted by r.

The **percentage** is the number obtained by taking a certain per cent. of the base (or the number of units taken with reference to all the hundreds in the base). It is denoted by the symbol, p.

For example, if a farmer has 800 peach-trees and loses 5 per cent. of them, he loses 5 out of every hundred, or 8×5 (or 40) out of 8 hundred.

$$\therefore\ 800 = \text{base.}$$
$$5\% = \text{rate.}$$
$$40 = \text{percentage.}$$

Pupils are likely to confuse the terms *rate per cent.* and *percentage*, because of their similarity in sound. In order to distinguish them, it may be helpful to remember that

rate per cent. means the number by (on, or of) every *single* hundred;
percentage " " " out of *all* the hundreds in the base.

The **amount** is the sum of the base and percentage.

The **difference** is the difference of the base and percentage.

Proceeds is a general term for either amount or difference.

231. I. Given the base and rate, to find the percentage (or proceeds).

Ex. 1. A school of 250 pupils is 60% boys. How many boys are there in the school?

SOLUTION.

The number of boys in each 100 pupils = 60.
The number of hundreds of pupils = $\frac{250}{100}$.

Hence, the entire number of boys in the school

= (No. of hundreds of pupils) × (No. boys in every hundred)
$= \frac{250}{100} \times 60 = 250 \times \frac{60}{100} = 250 \times .60 = 150.$

Abbreviated form of computation.

250, *Base.*
.60, *Rate per cent.*
150.00, *Percentage.*

Hence, in general, to find the percentage,

Multiply the base by the rate expressed decimally (or in symbols, $p = b \times r$).

Also, to obtain the proceeds, *multiply the base by* 1 *plus the rate, or* 1 *minus the rate, expressed decimally.*

Ex. 2. The population of a town, which contained 18000 people, decreased 5% in a year; what was the population at the end of the year?

OPERATION.

18000, *Base.*
.95, *Final rate per cent.*
90000
162000
17100.00, *Final population.*

EXPLANATION.

If a number be diminished by 5% of itself, it becomes 95%, or .95 of itself. 95% of 18000 is 17100.

EXERCISE 108.

Find:

1. 3% of 800 boys.
2. 25% of 40 days.
3. 10% of 70 lbs.
4. 18% of $420.
5. 50% of 90 bu.
6. 12% of 530 tons.
7. 6% of $325.
8. 80% of $4500.
9. $4\frac{1}{2}$% of 600 pupils.
10. $7\frac{1}{4}$% of 80 days.

11. Thirty per cent. of a drove of 240 horses were sold in one day. How many were sold?

12. I invested \$675 for a year at 6%. What amount was due me at the end of the year?

13. Which is the greater, $9\frac{1}{2}\%$ of \$750, or $27\frac{1}{3}\%$ of \$240?

14. Smith had 230 lambs and bought 20% more. How many had he then?

15. Mr. Cox lost 15% of his 640 tons of hay. How many tons did he lose? How many remained?

16. A city of 78000 inhabitants gained 12% in three years. What was its population after this gain?

17. A gentleman's salary of \$3500 was increased 16%. What is his present salary?

18. A house formerly valued at \$8500 decreases 8%. What is the valuation now?

Find the percentage and the proceeds:

19. \$7000 gaining 11%.
20. 950 pupils losing 16%.
21. 875 men gaining 24%.
22. 15 ft. gaining 20%.
23. $12\frac{1}{2}$ bu. decreasing 5%.
24. 240 trees increasing $7\frac{1}{2}\%$.

EXERCISE 109.

ORAL.

1. 3% of 200.
2. 7% of 500.
3. 8% of 40.
4. 20% of 65.
5. 25% of 4.
6. 30% of 5.
7. 40% of 6.
8. 25% of 90.
9. 16% of 50.
10. 18% of 30.
11. 45% of 40.
12. $7\frac{1}{2}\%$ of \$800.

13. Of 75 examples, a boy did 40%. How many did he solve?

14. A man 60 years old has a son 35% of his age. How old is the son?

15. A certain salary of \$2500 is to be increased 12%. What will it then be?

16. My expenses last year were \$1400, and this year will be 5% more. What will they be this year?

17. The length of a field is 30% more than its width, which is 230 rds. What is its length?

18. The price of a carriage was 8% less than that of a horse which cost \$150. What was the price of the carriage?

19. Lost 5% of an investment of \$8000 and then gained 30% of the remainder in speculation. What had I then?

20. What is 25% of 10? Of $2\frac{2}{5}$? Of $5\frac{1}{3}$? Of 100?

232. Special Cases.—*When the rate per cent. is an aliquot part of 100* (see Art. 74), the process of computing the percentage may often be abbreviated.

Thus, $33\frac{1}{3}\% = \frac{1}{3}$, and if it be required to find $33\frac{1}{3}\%$ of a given number, we may substitute the simpler process of taking $\frac{1}{3}$ of the number. Similarly,

$6\frac{1}{4}\% = \frac{1}{16}$.	$20\% = \frac{1}{5}$.	$50\% = \frac{1}{2}$.
$8\frac{1}{3}\% = \frac{1}{12}$.	$25\% = \frac{1}{4}$.	$62\frac{1}{2}\% = \frac{5}{8}$.
$12\frac{1}{2}\% = \frac{1}{8}$.	$33\frac{1}{3}\% = \frac{1}{3}$.	$66\frac{2}{3}\% = \frac{2}{3}$.
$16\frac{2}{3}\% = \frac{1}{6}$.	$37\frac{1}{2}\% = \frac{3}{8}$.	$87\frac{1}{2}\% = \frac{7}{8}$.

Ex. 1. A man owns $12\frac{1}{2}\%$ of a vessel valued at \$24000. What is the value of his share?

$12\frac{1}{2}\%$ of \$24000 = \$24000 × $\frac{1}{8}$ = \$3000, *Result.*

Care should also be exercised in dealing with cases where *the rate % is very small or very large*, as $\frac{1}{8}\%$ or 225%.

Thus, $\frac{1}{8}\%$ is $\frac{1}{8}$ of 1% = $.00\frac{1}{8}$ = .00125; 225% = 2.25 when expressed decimally.

Ex. 2. A broker bought \$1200 worth of stocks and charged $\frac{1}{8}\%$ commission. What was his commission?

\$1200, *Base.*
$.00\frac{1}{8}$, *Rate.*
\$1.50, *Percentage or commission.*

EXERCISE 110.

Find:

1. 25% of 44 books.
2. $16\frac{2}{3}\%$ of 96 lambs.
3. $33\frac{1}{3}\%$ of 750 boys.
4. $37\frac{1}{2}\%$ of 640 pounds.
5. 20% of 36 miles.
6. $66\frac{2}{3}\%$ of 450 days.
7. 50% of 225 acres.
8. 100% of 325 years.
9. 120% of 290 tons.
10. 215% of 480 cities.
11. $\frac{1}{2}\%$ of \$800.
12. $\frac{3}{4}\%$ of \$68.
13. $\frac{2}{3}\%$ of \$720.
14. $\frac{7}{8}\%$ of \$3500.

15. What is the commission on a sale of \$42000 at $\frac{1}{8}\%$?

16. A city had a population of 8750 some years ago, but has now 240 per cent. of its numbers at that time. What is its present population?

17. A man weighs 270 per cent. of what he did when as a boy he weighed 60 pounds. What is his weight?

18. A gentleman spent $2\frac{1}{8}\%$ of an income of \$2800 in books. How many dollars was that?

19. Mr. Jones owned $\frac{2}{3}$ of a farm worth \$8000, but sold $37\frac{1}{2}\%$ of his share. What part does he still own, and what is its value?

20. Which is greater, $87\frac{1}{2}\%$ of \$250 or $\frac{5}{8}\%$ of \$35000?

EXERCISE 111.

ORAL.

Find:

1. 50% of 90.
2. $33\frac{1}{3}\%$ of 66.
3. 75% of 52.
4. $87\frac{1}{2}\%$ of 96.
5. $66\frac{2}{3}\%$ of 75.
6. $37\frac{1}{2}\%$ of $5\frac{1}{3}$.
7. $\frac{1}{2}\%$ of 600.
8. 125% of 60.

233. II. **To find what per cent. one number is of another (given the percentage and the base, to find the rate).**

Ex. 1. A house which cost \$6000 rents for \$540. What per cent. of the cost is the rent?

SOLUTION.

If \$6000, or 60 hundreds of dollars, bring in \$540,
1 hundred dollars " " $\$\frac{540}{60}$ or \$9.

Hence, \$540 is 9% of \$6000, or the income from the investment is 9%.

The same rate expressed decimally may be obtained by regarding \$540 as a fractional part of \$6000 and reducing till the denominator of the fraction is 100.

Thus, $\frac{540}{6000} = \frac{9}{100} = .09.$
Hence, \$540 is 9% of \$6000.

Hence, in general, to obtain the rate per cent., *divide the percentage by the base*—in symbols, $r = \frac{p}{b}.$

Sometimes the *proceeds* and *base* are given instead of the percentage. In this case the percentage may be obtained by subtracting the base from the proceeds.

Ex. 2. A dealer bought a horse for \$60 and sold it for \$75. What was his gain per cent.?

OPERATION.

\$75, *Proceeds.*
\$60, *Base.*
\$15, *Percentage.*
60)15.00(.25, *Rate.*
120
300
300

EXPLANATION.

Since the horse cost \$60 and was sold for \$75, the gain was the difference between \$75 and \$60, or \$15. Hence, the problem reduces to finding what per cent. 15 is of 60.

EXERCISE 112.

ORAL.

Express in per cent. with decimals:

1. $\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{4}{5}, \frac{2}{3}, \frac{3}{8}, \frac{1}{4}, \frac{3}{10}, \frac{1}{6}, \frac{1}{8}, \frac{5}{12}, \frac{5}{8}, \frac{9}{10}, \frac{2}{5}.$
2. $\frac{1}{7}, \frac{5}{6}, \frac{7}{8}, \frac{7}{10}, \frac{5}{7}, \frac{4}{9}, \frac{7}{100}, \frac{19}{40}, \frac{7}{11}, \frac{11}{10}, \frac{12}{5}, \frac{25}{6}.$
3. What per cent. of 8 is 4? Is 2? Is 1? Is 5?
4. What per cent. of 12 is 6? Is 4? Is 8? Is 2?
5. What per cent. of 24 is 3? Is 6? 8? 9? 12? 15? 16?
6. 12 is what per cent of 18? Of 24? Of 30? Of 48?
7. 15 is what per cent. of 25? 30? 35? 50? 75?
8. 18 is what per cent. of 27? 42? 54? 48? 180?

EXERCISE 113.

Find the rate per cent. when,

1. Base is \$45 and percentage is \$9.
2. Base is \$240 and percentage is \$84.
3. Base is 428 bushels and percentage is $160\frac{1}{2}$ bushels.
4. Base is 150 days and percentage is 12 days.
5. Percentage is $23\frac{1}{3}$ tons and base is 320 tons.
6. Percentage is 3250 boys and base is 1400 boys.
7. Percentage is \$73.50 and base is \$84.
8. Percentage is \$1.40 and base is \$56.

9. Cost is $236 and gain is $29.50.
10. Gain is $0.98 and cost is $1.40.
11. Gain and cost are each $75.
12. Gain is $\frac{1}{3}$ cost; $\frac{3}{4}$ cost; $\frac{5}{8}$ cost.
13. Cost is $2350 and gain is $173.90.
14. Cost is $85 and loss is $6.80.
15. Cost is $125 and loss is $22.50.
16. Loss is $2.76 and cost is $7.36.
17. Amount is $800 and base is $640.
18. Amount is $4397.61 and base is $4261.25.
19. Amount is $4420 and percentage is $420.
20. Amount is $470.61 and percentage is $209.16.
21. Selling price is $120 and gain is $30.
22. Selling price is $60 and cost is $48.
23. Selling price is $10.83 and cost is $9.50.
24. Selling price is $1462.50 and loss is $1037.50.
25. Selling price $642.60 and cost is $765.
26. Base is $900, percentage is $4.50.
27. Amount is $7110, percentage is $47.40.
28. Selling price is $136, cost is $134.98.
29. Cost is $30, gain is 25 cts.
30. Loss is 36 cents, selling price is $62.64.
31. Cost is $180, loss is $1.

What per cent. of

32. $\frac{5}{6}$ is $\frac{1}{2}$?	35. $3\frac{1}{3}$ is $1\frac{1}{5}$?	38. 15 is $12\frac{1}{2}$?
33. $\frac{3}{4}$ is $\frac{1}{2}$?	36. $5\frac{1}{4}$ is $3\frac{1}{2}$?	39. $10\frac{2}{3}$ is $9\frac{1}{3}$?
34. $\frac{5}{8}$ is $\frac{3}{16}$?	37. $8\frac{3}{4}$ is 7?	40. $21\frac{1}{4}$ is $1\frac{1}{16}$?

41. In a school of 320 pupils, 176 are boys. What per cent. of the school is boys? What per cent. is girls?

42. From a farm containing 146 acres, the owner sold 124.1 acres. What per cent. of the farm did he sell?

43. For threshing a crop of grain amounting to 872 bu. 2 pk., the thresher took 209 bu. $1\frac{3}{5}$ pk. as his pay. What per cent. of the crop did he take?

44. Of a regiment numbering 980 men, 147 were sick and the rest able-bodied. What per cent. of the regiment entered battle?

45. If 271 men were killed in battle and all of the sick recovered, what per cent. of the whole regiment (1000 men) would be able to enter another fight?

46. A house which cost $3500 was sold for $4060. What was the gain per cent.?

47. A pencil which cost 5 ct. sold for 6 ct. What was the gain per cent.?

48. Which investment returns the greater per cent., $40 in a bicycle which sold for $50, or 3 ct. in a newspaper which sold for 4 ct.?

49. A man has $4500 with which to speculate. Will he do better buying cattle at $10 and selling at $18 a head, or buying railroad stock at $75 and selling it at $132? What will be his gain per cent. in the better investment?

234. III. To find a number from a given per cent. of it (given the rate and percentage, to find the base).

Ex. 1. $30 is 12% of what number of dollars?

SOLUTION.

12% of the required number $= \$30$.
$\therefore$ 1% of the required number $= \$\frac{30}{12}$.
$\therefore$ 100% of the number, or the number itself $= \$\frac{30}{12} \times 100 = \$250$.

Or, more briefly, $\frac{\$30}{.12} = \$250$,

(since $\$\frac{30}{12} \times 100 = \$\frac{30}{12} \times \frac{1}{\frac{1}{100}} = \$\frac{30}{12} \times \frac{1}{.01} = \$\frac{30}{.12} = \$250$.)

Ex. 2. A house is sold for $4800 at a gain of 20%. What was the cost of the house?

SOLUTION.

Making the cost of the house the base,
120% of the cost of the house $= \$4800$.
$\therefore$ 1% of the cost of the house $= \frac{\$4800}{120} = \$40$.
$\therefore$ 100%, or the whole of the cost of the house $= \$40 \times 100 = \$4000$.

Or more briefly, cost of house $= \frac{\$4800}{1.20} = \$4000$.

Hence, in general, to find the base, either proceed by analysis, *that is, obtain* 1 *per cent. of the base by dividing the percentage by the per cent., and multiply the result by* 100, *in order to obtain* 100 *per cent. of the base;* or, *divide the percentage by the rate, expressed decimally* (in symbols $b = \frac{p}{r}$).

The amount or difference being given to find the base, *divide the amount by* 1 *plus the rate;* or, *divide the difference by* 1 *minus the rate, i. e.,* $base = \begin{cases} amount \div (1 + rate). \\ difference \div (1 - rate). \end{cases}$

EXERCISE 114.

Find the base if it is given that:

1. Percentage is $120 and rate is 6%.
2. Percentage is 760 bu. and rate is 5%.
3. Percentage is 115 gal. and rate is $2\frac{1}{2}$%.
4. Rate is 4% and percentage is 550 da.
5. Rate is $3\frac{1}{2}$% and percentage is $553.
6. Rate is 23% and percentage is $27.83.
7. Proceeds are $1086.40 and gain is 12%.
8. Proceeds are 23 tons and loss is 20%.
9. Rate of loss is 13% and proceeds are $453.27.
10. Gain is 18% and proceeds are $372.88.
11. 140 is 8% of what number?
12. 169 is $3\frac{1}{4}$% of what number?

Of what number is

13. 30, 5%? | *15.* 725, 25%? | *17.* 7, 4%?
14. 120, 6%? | *16.* 486, 30%? | *18.* 17, 5%?

Find the quantity of which:

19. 48 bu. is 20%. | *25.* 97 is $12\frac{1}{2}$%.
20. 75 da. is 25%. | *26.* 123 is $66\frac{2}{3}$%.
21. 145 tons is 5%. | *27.* 429 is $37\frac{1}{2}$%.
22. 67 mi. is 40%. | *28.* 45 is $3\frac{1}{2}$%.
23. $73.35 is 15%. | *29.* 19 is $\frac{1}{2}$%.
24. $1476 is 48%. | *30.* $31\frac{1}{4}$ is $\frac{3}{5}$%.

What quantity increased by:

31. 9% of itself is $54.50? | *34.* 15% of itself is $11.04?
32. 27% of itself is $439.42? | *35.* $33\frac{1}{3}$% of itself is $114.48?
33. 60% of itself is 12? | *36.* $8\frac{2}{5}$% of itself is 1.163403?

What quantity diminished by:

37. 25% of itself is 96? | *40.* $16\frac{2}{3}$% of itself is $76\frac{1}{2}$?
38. 30% of itself is 175 da.? | *41.* 19% of itself is $6\frac{3}{4}$ mi.?
39. $28\frac{1}{2}$% of itself is $167.31? | *42.* 88% of itself is 0.3?

Find cost if:

43. Selling price is $50 and gain is 25%.
44. Selling price is $308.14 and gain $8\frac{1}{2}$%.
45. Selling price is $7.14 and loss is $33\frac{1}{3}$%.
46. Selling price is $286.02 and loss is $9\frac{1}{5}$%.

47. A city whose population is now 74250, has gained 10% in a year. What was its population a year ago?

48. If during a year I spend 45% of my earnings, and have $1100 saved, what was my income?

49. A boy weighing 90 lbs. gained 20% during the last year. What was his weight a year ago? What will be his weight a year hence at the same rate per cent. of growth?

50. A pencil which sold for 8 cents brought 60% gain. What did it cost?

51. A grocer sold a score of eggs for 55 cents, thereby gaining 10%. What did a dozen eggs cost him?

52. A farmer lost 12% of his lambs by death, and 23% by theft, and there were 585 lambs remaining. How many had he at first?

53. A housekeeper wishes to use $124\frac{4}{5}$ yards of muslin and knows it will shrink $2\frac{1}{2}$%. How many yards should she buy?

54. I sold 2 horses for $60 each; on the one I gained 20%, and on the other I lost 20%. Did I gain or lose on them both together? How much?

55. The distance between two stops was $42\frac{3}{8}$ miles, or $7\frac{1}{2}\%$ of the entire journey. What was the length of the journey?

56. The gain was \$39.60, or 20% of the cost. What was the cost?

57. The loss was \$76.40, or 80% of the cost. What was the cost?

58. In a transaction a man gained 8%, but actually gained \$1256. How much did he invest?

EXERCISE 115.

ORAL.

1. If percentage is \$75 and rate is 3%, what is the base?
2. What is the base when rate is 8%, and percentage is 24 bu.?
3. \$24 is 20% more than what number of dollars?
4. 27 is 10% less than what number?
5. Gain is \$40, or 5%. What was the cost?
6. Loss was \$30, or 15%. What was the cost?
7. Selling price was \$84, and loss was 30%. What was cost?
8. Gained 20% when I sold for \$60. What was cost?
9. A knife selling at 70 cents yielded 40% gain. Find cost.
10. If percentage is 63 bu. and rate is 70%, find base.
11. If proceeds are 525 tons and rate is 5% gain, find base.
12. What was the base if rate was 16% and percentage \$16?

235. Algebraic Treatment of Percentage.—If the student is familiar with the first principles of algebra, the treatment of percentage may be simplified by their use. All three of its cases may be reduced to a single formula.

Thus, since $p = b\,r$, if any two of the three quantities, p, b, r, are known, the third may be found by substituting for the two known quantities and solving the resulting equation.

Ex. 1. What per cent. is 16 of 64?

Here, $p = 16$, $b = 64$.
Substituting for p and b in $p = br$
we obtain $16 = 64\,r$

$$\therefore\ r = \tfrac{16}{64} = .25,\ \textit{Rate.}$$

The above formula may also be made to cover cases where the proceeds occur instead of the percentage.

In such cases let p = proceeds,
r = final rate, *i. e.*, 1 + rate, or 1 − rate,
b = base.

Ex. 2. A property is sold for \$3360 at a gain of 12%. What was its cost?

Here, proceeds (or p) = \$3360,
final rate (r) = 1.12.
Substituting for p and r in $p = b\,r$, \$3360 = 1.12 r

$$r = \frac{\$3360}{1.12} = \$3000,\ \textit{Base.}$$

EXERCISE 116.

GENERAL REVIEW.

1. Find 7% of 1456 feet. Of \$351.70. Of $9\frac{1}{3}$.
2. 24 is what per cent. of 36? Of 64? Of 480? Of 18?
3. 30 is what per cent. of 50? Of 120? Of 24? Of 18?
4. 42 is 6% of what number? 8% of what? 21% of what?
5. 48 is 18% of what? $33\frac{1}{3}\%$ of what? 120% of what?
6. Find $101\frac{1}{2}\%$ of 64 acres. Of $7\frac{3}{5}$ miles.

In the following group of examples, two of these three quantities are given, and the problem in each case consists in ascertaining the missing one.

	Base.	Percentage.	Rate.
7.	\$236	\$34.22	?
8.	?	\$12.41	$21\frac{1}{4}\%$.
9.	736 pounds.	?	$2\frac{1}{3}\%$.
10.	$58\frac{3}{5}$ bushels.	8.76 bushels.	?
11.	?	\$355	$\frac{1}{2}\%$.
12.	\$460	?	$1\frac{1}{4}\%$.
13.	$427\frac{3}{5}$	$534\frac{1}{2}$	?
14.	?	750.527	$105\frac{1}{2}\%$.
15.	\$896	?	$96\frac{3}{4}\%$.

16. What per cent. of an hour is a minute? Of a peck is a quart? Of a week is a day? Of a gallon is a quart? Of an inch is a foot? Of a yard is a foot? Of a yard is an inch? Of a day is an hour?

17. Out of an examination of 9 problems, a boy solved $7\frac{1}{2}$ correctly. What should be his per cent. grade?

18. A man owning $\frac{3}{4}$ of a ship sold $33\frac{1}{3}\%$ of his share for \$9200. What was the whole vessel worth?

19. A field was 48 × 35 rods, and the owner increased each dimension 40% of itself. By how many acres and what per cent. did he increase the field?

20. A carpenter built a house at an expense of $5200, and sold it for $7098. What was the gain %?

21. A wholesale grocer buys coffee at 30 cents and sells it at 36 cents a pound. The local grocer buys at 36 and sells at 45 cents. What per cent. does each make? What per cent. would be the wholesale grocer's gain if he sold directly to the consumer for 45 cents?

22. Owning 30% of an office building, a man sold 25% of his share and then valued the balance at $9000. What was the entire building worth at this valuation?

23. What per cent. of 8 bushels, 3 pecks, 4 quarts is 4 bushels, 6.9 quarts?

24. Find the value of 80% of 60 acres, 125 square rods, 20 square yards at $60 an acre.

25. Find $\frac{1}{2}$ % of 90. Of 200. Of 8756. Of $\frac{4}{5}$.

26. Find $\frac{2}{3}$ % of 900. Of 4980. Of $\frac{9}{15}$. Of 1.6.

27. Write decimally, and as a common fraction; ten per cent.; four-fifths per cent.; one-half of one per cent.; one hundred twelve per cent.; eight per cent.; three-tenths per cent.; and eleven hundredths per cent.

28. Write *as* per cent.; $\frac{3}{10}$; $\frac{3}{4}$; $\frac{5}{8}$; $\frac{1}{2}$; $\frac{7}{40}$; $1\frac{1}{3}$; 6; $1\frac{9}{100}$; $\frac{2}{17}$; $\frac{3}{25}$; $\frac{4}{11}$; $\frac{22}{25}$.

29. On an examination a boy got 460 credits out of a possible 500. What should his grade be, expressed in per cent.?

30. A watch was sold for $190 at a loss of 24%. What should it have been sold for to obtain a gain of 5%?

31. A speculator bought 500 shares of railroad stock at $68 a share and sold it at $85 a share. What was his gain %?

32. If a teacher's salary is $2400 and he pays in a year, 15 % of it for board, 4% for room, 3% for clothes, 8% for incidentals, and gives his mother a quarter of his salary, how much is left for saving?

33. A man owned $\frac{4}{5}$ of a hotel and sold $12\frac{1}{2}$% of his share for $7540. At the same rate, what is the value of the hotel?

34. 707.84 is 12% more than what number?

35. If the cost was $\frac{4}{5}$ of the selling price, what is the gain %? Prove your answer in the case where the cost is $420.

36. What per cent. of the year 1904 are the Sundays? Of the year 1903?

CHAPTER XIV.

APPLICATIONS OF PERCENTAGE.

236. Applications of Percentage.—The method of reckoning with reference to 100 as a standard or base has so many advantages that it is widely used in many different departments of practical life.

The computations in these different applications are alike, in that, *first*, they all use 100 as a base, and, *second*, they are all concerned with the three quantities, *base*, *rate*, and *percentage*.

The various applications of percentage, however, differ from the general subject and from each other in that (1) different special names are assigned to one or more of the quantities used (thus the percentage is sometimes called *commission*, or *tax*, or *profit*, etc.); or (2), the base may be determined in some peculiar way; or (3) certain special standard rates are used.

In all cases, however, it will be found that the three quantities, base, rate, and percentage, appear in some form, and that two of them are given to find the remaining one.

PROFIT AND LOSS.

237. The subject of **Profit and Loss** differs very slightly from the general subject of percentage.

Profit or *loss* is the name given the percentage, **profit** being the excess of money received over that expended, and **loss** being the excess of money expended over that received.

The student should carefully note that the *base is the money paid out or invested* (not the money received).

Ex. A man sold his horse for $60, which was a loss of 20%. What did the horse cost him?

19. A field was 48 × 35 rods, and the owner increased each dimension 40% of itself. By how many acres and what per cent. did he increase the field?

20. A carpenter built a house at an expense of \$5200, and sold it for \$7098. What was the gain %?

21. A wholesale grocer buys coffee at 30 cents and sells it at 36 cents a pound. The local grocer buys at 36 and sells at 45 cents. What per cent. does each make? What per cent. would be the wholesale grocer's gain if he sold directly to the consumer for 45 cents?

22. Owning 30% of an office building, a man sold 25% of his share and then valued the balance at \$9000. What was the entire building worth at this valuation?

23. What per cent. of 8 bushels, 3 pecks, 4 quarts is 4 bushels, 6.9 quarts?

24. Find the value of 80% of 60 acres, 125 square rods, 20 square yards at \$60 an acre.

25. Find $\frac{1}{2}$ % of 90. Of 200. Of 8756. Of $\frac{4}{5}$.

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27. Write decimally, and as a common fraction; ten per cent.; four-fifths per cent.; one-half of one per cent.; one hundred twelve per cent.; eight per cent.; three-tenths per cent.; and eleven hundredths per cent.

28. Write *as* per cent.; $\frac{3}{10}$; $\frac{3}{4}$; $\frac{6}{5}$; $\frac{1}{2}$; $\frac{7}{40}$; $1\frac{1}{3}$; 6; $1\frac{9}{100}$; $\frac{2}{17}$; $\frac{3}{25}$; $\frac{4}{11}$; $\frac{22}{35}$.

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The student should carefully note that the *base is the money paid out or invested* (not the money received).

Ex. A man sold his horse for \$60, which was a loss of 20%. What did the horse cost him?

SOLUTION.

Making the cost of the horse the base,

80% of the cost of the horse = \$60, the selling price.

$\therefore$ cost of horse $= \frac{\$60}{.80} = \$75$, *Result.*

EXERCISE 117.

1. Bought for \$80 and sold for \$100. Find the gain %.

2. Sold for \$40 and lost 20%. Find cost.

3. Gained \$63 or 7%. Find cost and selling price.

4. Lost $32\frac{1}{2}$% on an investment of \$6200. Find the actual loss.

5. Did I gain or lose by buying eggs at 18 ct. a doz. and selling at 32 ct. a score? What per cent.?

6. Who gained more money and who gained a greater per cent.—John, who bought for \$60 and sold for \$65, or James, who sold for \$240 at a gain of 20%?

7. I sold a horse for \$404.40 and gained $12\frac{1}{3}$%. What would have been my per cent. of loss if I had sold him for \$252?

8. Bought two farms for \$4500 each, and sold the one for \$6300 and the other at a loss of 37%. Did I gain or lose on the transaction and how much?

9. Bought two books for \$4 each, and sold one at a gain of 30% and the other at a loss of 90 cents. Did I gain or lose on the transaction? How much? What per cent.?

10. By selling cloth for \$1.20 per yd., a salesman lost 20%. How should he have sold it to gain 20%?

11. I sold two watches for \$84 each. On one I gained 40% and on the other I lost 40%. Did I gain or lose on the whole? How much? What %?

12. Carpet is bought at 75 ct. a yd. Expenses amount to 20 cents additional on each yard. What must be the selling price that the dealer may realize an advance of 20%? How must he mark the carpet so that he can allow a 5% reduction and still gain 20%?

13. The farmer charges 10% profit on his wheat; the miller, 25% profit on his flour; the grocer, 20% gain. The consumer pays \$5.28 per bbl. What is the first cost to the farmer of the wheat in a barrel of flour?

14. The actual cost of a certain piano is \$200; the maker charges an advance of 60%; the agent realizes a profit of 25%, and the deliverer gains 5% for hauling. What is the cost to final owner?

EXERCISE 118.

ORAL.

1. Can a man gain 125%? Can he lose 125%?

2. How many per cent. is it possible to gain? To lose?

3. What is always the divisor in determining the gain or loss per cent.?

4. If I gain 200% on a purchase of \$5, what is the selling price?

5. What is the gain per centum if I buy at 40 ct. and sell at 60 ct.?

6. What is the loss per centum if I buy at 60 ct. and sell at 40 ct.?

7. If I lose a new 5-ct. pencil, what was my loss %?

8. If I find a dime, can you tell the gain per cent.? (We cannot, because to have gain % there must be *cost*, and here there was no cost.)

9. If the gain is $\frac{1}{5}$ the cost, what is the gain %?

10. If the selling price is $\frac{3}{4}$ the cost, what is the loss %?

11. If the cost is $\frac{5}{6}$ the selling price, what is the gain %?

12. If the selling price is twice the gain, what is the gain per cent.?

13. If the selling price is three times the cost, what is the gain per cent.?

14. If the cost is double the selling price, what is the loss %?

15. Sold for $\frac{2}{3}$ what cost $\frac{1}{2}$. What was the gain %?

16. Does a merchant gain or lose by buying coal by the long ton and selling it by the short ton, at the same price per ton? How would you find the per cent.?

17. A farmer planted a peck of corn and raised 250 bushels. How would you find the gain %?

TRADE DISCOUNTS.

238. The subject of **trade discounts** has the peculiarities that the percentage is termed **discount**, and that frequently several discounts are applied in succession.

239. Commercial Discounts.—It is the custom of manufacturers and various dealers in merchandise to have a fixed or catalogue price for goods, and to make deductions from this, called discounts. Thus, a manufacturer may allow a discount of 25%, owing to the fact that goods are produced more cheaply than when the catalogue was issued, and a further discount of 5% for payment of the bill within a certain time.

The catalogues of goods and prices issued by business houses, are frequently expensive, and when the prices of goods change, owing to cheapened processes of production, it is more economical to print off a brief list of discounts than to issue a new catalogue.

The catalogue price is called the **list price**; the price after the discount has been deducted is called the **net price.**

240. Successive Discounts.—In making several successive discounts *deduct the first discount from the list price, then compute the next discount on the remainder and deduct it from the remainder, and so proceed till all the discounts have been made.*

Ex. A bill of $250 for steam heating apparatus was subject to a discount of 60% and 20%, with 2% off for cash. What sum is needed to pay the bill?

SOLUTION.

Since after 60% is deducted 40% is left,
and " 20% " 80% of the remainder is left,
and " 2% " 98% of the second remainder is left,
the sum required = $250 × .40 × .80 × .98 = $78.40.

EXERCISE 119.

1. A librarian purchases a list of books amounting to $123.80, but is allowed 30% deduction and a 4% discount for cash. Find actual amount due.

2. From an assessment upon $8500 of value the owner obtained three successive reductions of 8%, 15%, and 10%. How large was the final valuation?

3. For damages on a large purchase of dry goods, amounting to $1800, a merchant discounted 5%, then on special sale 7%, and on cash payment 2%. What was final bill?

4. Which is the greater discount on a bill of $15600, and how much—a discount of 40% and then 8%, or one of 48%?

5. What single discount on a bill of $5000 is equivalent to the two discounts of 15% and 20%?

6. Find net cash amount of a bill of $675, subject to the three discounts, 20%, 16%, and 5%. Change the order of discounts in this example and ascertain whether or not there is change in the final amount of the same bill.

7. Prove that it is immaterial in what order several successive discounts on the same bill are made.

COMMISSION AND BROKERAGE.

241. Agents and Commissions.—Goods are frequently bought or sold through an agent, the advantages being that an agent may be in a more favorable place in which to buy or sell goods, and also that an agent by making a specialty of a certain line of goods may be able to buy or sell to greater advantage.

Thus, a farmer may receive a higher price for potatoes by selling them through an agent in a city, than by selling them himself in his own neighborhood.

An agent who buys or sells general merchandise is called a **commission merchant.** Goods sent to him to be sold are called a **consignment,** the person sending them being called a **consignor,** and the person receiving them being called a **consignee.**

An agent at a distance is sometimes called a *correspondent;* the person employing a correspondent is called a *principal.*

242. Commission is the percentage paid a person who buys or sells goods or collects money for another person.

The **base** of a commission is the amount of money paid out or received for goods by the agent.

Ex. 1. An agent sold \$3500 worth of goods for a commission of 5%. What was his commission?

$\$3500 \times .05 = \$175$, *Commission.*

Ex. 2. A gentleman sent an agent \$257.50 to expend in buying hay at \$20 a ton. The agent charged 3% commission. How many tons of hay did he buy?

OPERATION.

```
1.03)$257.50($250.
      206
      ---
       515
       515
       ---
         0
```

$\frac{250}{20} = 12\frac{1}{2}$, *No. of tons.*

EXPLANATION.

Since \$257.50 includes both the cost of the hay and 3% commission, \$257.50 is 103% of the cost of the hay. Hence, the cost of the hay = \$257.50 ÷ 1.03 = \$250.

If 1 ton of hay costs \$20, the number of tons which can be bought for \$250 is \$250 ÷ 20, or $12\frac{1}{2}$ tons.

243. Brokerage.—A broker is an agent who buys or sells stocks, bonds, bills of exchange, real estate, etc.

A commission merchant receives and sells goods in his own name, sending the net proceeds to the consignor. A broker does not handle the goods, and they are sent directly from the owner to the buyer. Since he is thus saved the labor of handling the goods, he is paid a less percentage for his work.

Brokerage is the commission charged by a broker. In the sale of stocks and bonds, brokerage is reckoned not on the selling price, but on the face or par value of the stocks. It is usually $\frac{1}{8}$%, or $12\frac{1}{2}$ cents on a share of \$100.

Ex. A broker sold 36 shares of N. Y. Central stock. What was his brokerage?

SOLUTION.

Since the par value of 1 share = \$100,
" " " " 36 " = \$3600.
$\$3600 \times .00\frac{1}{8} = \$4.50$, *Brokerage.*

EXERCISE 120.

1. A commission merchant sold a car of lime for \$80 and received 3% commission. What was his commission? How much did he remit to his employer?

2. I sold a lot of real estate for Mr. Jones for \$12500 on $3\frac{1}{2}$% commission. What amount should be sent him?

3. An agent sold 560 baskets of peaches at 90 cents a basket and charged \$25.20 for doing it. What was his rate of commission?

4. After selling a property worth \$8528, the agent sent to the former owner \$8229.52. What rate of commission did he charge?

5. A merchant charged 4% for selling a consignment of beef and received \$93.40 commission. What was the selling price of the beef?

6. After selling a load of grain, for doing which the agent retained $2\frac{1}{2}$%, he remitted \$2691 to his employer. Required the selling price of the grain.

7. A real estate agent sold a house on 3% commission and sent the owner \$7229.41. What commission did he retain?

8. If \$2388.33 includes commission at 2% and the amount invested in wool, how much was invested?

9. I sent my agent \$216.84 to invest in peaches, after deducting 4% commission. How many baskets at 75 cts. each will he purchase for me?

10. What is the brokerage on a sale of 75 shares of railroad stock (par value \$100) at $\frac{1}{8}$%? At $\frac{1}{4}$%?

11. Par value of P. R. R. stock is \$50. What will be the total cost of 48 shares at \$72, including $\frac{1}{8}$%?

12. Suppose in Ex. 11, shares were selling at \$63, what would be the total cost?

13. What is the total cost of 28 shares of D. L. & W. stock at \$124, counting brokerage at $\frac{1}{8}$%?

14. A speculator sold through his broker 90 shares of C. R. R. of N. J. at $\$112\frac{1}{2}$. What were the proceeds, brokerage $\frac{1}{4}$%?

15. I sent draft on Drexel & Co. for \$4631.25 to pay for stock at $92\frac{1}{4}$ (par \$100) and their commission at $\frac{1}{8}$%. How many shares could they buy?

TAXES.

244. Revenue.—Various governments, as the federal, state, county, city governments, need money in order to pay their general expenses, and also do the special work delegated to them, such as maintaining schools, building roads, caring for the insane, etc.

The different governments collect their revenues in different ways. State and local governments generally collect their revenue as *taxes.*

245. Taxes.—A **tax** is money assessed on property or persons by the government for public purposes.

A property tax is a tax assessed on property.

Property is of two kinds, *real estate* and *personal property*. Real estate is property not easily moved, as lands, buildings, etc.

Personal property is movable property, as money, stocks, bonds, household goods, cattle, etc.

A **poll tax** is a fixed sum, as \$1, or \$2, assessed on each voter in a community without regard to the amount of property he owns.

246. The **method of assessing taxes** in a state, for instance, is usually as follows: A representative body, the state legislature, determines by means of appropriation bills, the amount of money to be expended. The amount of taxable property in the state is determined by local officers called *assessors,* elected by the people in a township or borough or city, the reports of the assessors being summed up by a state official called the Auditor of Public Accounts. The auditor divides the total amount of money to be collected by the number of dollars of taxable property, and thus determines the amount of tax on one dollar. Hence, a *tax is a certain per cent. of the property assessed.*

Thus, if a state desires to expend \$1,200,000, and the amount of property in the state is \$800,000,000, the tax rate will be \$1,200,000 ÷ \$800,000,000, or .0015.

The same general method is followed in assessing city, borough, and county taxes.

The different rates of taxes of each government are sent to an official, who calculates the amount of each kind of tax to be paid by each person, corporation, or piece of property, and tabulates the results in a book. The book is given to a collector, who collects each tax and returns it to the proper (county, city, or state) treasurer.

It is the custom in many localities to assess property for not more than $\frac{1}{2}$ or $\frac{2}{3}$ of its real value. After property has been assessed, the owner may appear before the proper official and make claim for such reductions or corrections as he thinks he is entitled to.

If taxes are not paid when due, a certain per cent. is usually added to them as a fine.

Taxes are often stated as so many mills on a dollar.

Ex. What will be the county tax of Samuel Smith, the rate being $2\frac{1}{2}$ mills on a dollar, and his property being valued at \$3500.

The tax on \$1 = .0025.

Hence, tax on \$3500 = \$3500 × .0025 = \$8.75, *Tax.*

247. Computation of Taxes by Use of a Table.—The computation of the taxes of a community is greatly facilitated by the preparation and use of a table like the following, which gives the tax on various sums at the rate of 3 mills on a dollar.

Prop.	Tax.	Prop.	Tax.	Prop.	Tax.	Prop.	Tax.	Prop.	Tax.
1	.03	10	.30	100	\$3.00	1000	\$30	10,000	\$300
2	.06	20	.60	200	6.00	2000	60	20,000	600
3	.09	30	.90	300	9.00	3000	90	30,000	900
4	.12	40	1.20	400	12.00	4000	120	40,000	1200
5	.15	50	1.50	500	15.00	5000	150	50,000	1500
6	.18	60	1.80	600	18.00	6000	180	60,000	1800
7	.21	70	2.10	700	21.00	7000	210	70,000	2100
8	.24	80	2.40	800	24.00	8000	240	80,000	2400
9	.27	90	2.70	900	27.00	9000	270	90,000	2700

In this table, the columns headed "Prop." give the number of tens of dollars, not number of dollars of property tax

Ex. Compute by use of the table the tax on a property assessed at $5680.

SOLUTION.

Tax on $5000	=	$15.00
" " 600	=	1.80
" " 80	=	.24
		$17.04, *Tax.*

EXERCISE 121.

1. What will be R's tax on a farm valued at $4500 if the rate is .007? If it is .0102? If it is .012?

2. Mr. Smith owns a house assessed at $12000 and the tax rate is .021. What will his total tax be, including a poll at $1.50?

3. By the table find tax on the following amounts: $4175; $8925; $10328; $27030; $50409; $66666 at rate .003.

4. If my tax is $132.30 and my property is assessed at $7350, what is the rate?

5. I pay a poll tax of $1.25 and a total tax of $395.25; my property is assessed at $15760. What is the rate?

6. One year a gentleman paid $372.60 when the rate of taxation was $1\frac{2}{3}$%. What was the value of his property?

7. A tax of $30500 is to be assessed on a town; the real estate is valued at $3500000, and there are 500 polls taxed at $1.50 each. What will be the rate?

8. The real estate in a certain town is valued at $857400 and a tax of $13718.40 is to be assessed. What will A have to pay, his property being worth $14700?

9. Do you detect any similarity between the subject of taxes and of percentage? Any difference?

10. Find by use of the table the tax on amounts of Ex. 3 if the rate were 2 mills on the dollar instead of 3.

CUSTOMS OR DUTIES.

248. Revenues of the General Government.—The general government of the United States obtains its revenue from two

principal sources: (1) **Duties** (sometimes called tariff, or customs) which are a tax imposed on goods imported into the country from foreign countries;

(2) **Internal revenue,** that is, taxes charged on certain articles manufactured within the country, as spirituous liquors, articles made from tobacco, etc. Of all these, certain duties only are collected by the use of percentage.

249. Duties are of **two kinds,** *ad valorem* and *specific.*

An **ad valorem** duty is a certain percentage assessed on the value which imported goods have in the country from which they come. Thus, imported silk ribbons pay a duty of 40%; brushes, 40%; manufactured glass, 45%.

A **specific** duty is duty assessed on goods according to their weight or bulk without respect to their value. Thus, imported pig iron pays a duty of $4 a ton; iron ore 40 cents a ton.

Sometimes goods are subject to both an *ad valorem* and a specific duty. For example, preserved fruits, when imported, pay a duty of 1 cent a pound, and also 35% ad valorem.

Goods are said to be on the *free list* when no duty is charged on them.

Ad valorem duties are more just if honestly paid, but they present greater chance for fraud by undervaluation.

Hence, at present, most U. S. duties are specific.

Duties are collected at certain cities, called Ports of Entry, which are determined by law. Each port of entry has a building called a Custom House, where duties are collected under the oversight of a government official called the Collector of the Port.

A duty is computed on the cost of the goods at the port from which they are shipped. This cost includes both the cost price and all charges up to the final shipment. An invoice specifying the goods purchased, their cost, etc., is sent to the person or firm importing the goods, and is to be presented by them at the custom house where the goods are received.

Merchandise of certain kinds, imported but not intended for immediate consumption, may be placed in bonded warehouses provided by the United States, and remain there not longer than three years, the owner being at liberty to withdraw it at any time upon payment of the duties and charges.

Ex. What is the duty on 375 yards of cloth at $2.75 a yard, the duty being 20% *ad valorem?*

The duty $= \$\overset{.55}{\cancel{2.75}} \times 375 \times \frac{1}{\cancel{5}} = \$.55 \times 375 = \$206.25$, *Duty.*

EXERCISE 122.

1. What is the duty on 40 pairs lace curtains bought for $6.50 a pair, duty being 28%?

2. A jeweler receives from Switzerland a quantity of watch supplies, costing $2450, and charges amounting to $35. What will be the duty at 8% *ad valorem?*

3. Find the duty at 15% on 80 boxes of candles, each weighing 100 lbs. and costing $8\frac{3}{4}$ cts. per pound.

4. A liquor dealer imports 150 dozen bottles of wine at $2.50 a dozen, duty at 22%. What do the bottles cost him, provided charges *before* landing are $16 and those *after* landing are $9.50?

5. The duty on an invoice of lace goods at 24% was $211.50. What was the cost of the goods? What was the total cost? What must be the selling price to gain 20%?

6. A merchant imported dry goods valued at $7200, on which there was a duty of $1296. What was the custom rate?

7. In the above example, provided the goods cost $3 a yard, what must he ask per yard for them, to gain 10% above all given costs?

8. Compare and contrast this subject with the subject of percentage.

INSURANCE.

250. Insurance is a system of business whereby a certain sum is payable in case of loss of property in a specified way, or in case of injury or death of a person.

There are three principal kinds of insurance:

(1) Fire Insurance.
(2) Life Insurance.
(3) Accident Insurance.

Beside these, there are many special kinds of insurance, as marine, tornado, plate glass, employment insurance, etc.

251. The **insurer** or **underwriter** is the person or company taking a risk.

The **insured** or **assured** is the person protected.

The **policy** is the written contract between the insurer and the insured.

The **premium** is the amount paid for the insurance for a certain period of time, as one, three, or five years.

The **rate** of the insurance is either a certain per cent. to be charged on the face of the policy, or, what amounts to the same thing, a certain charge on every $100 or $1000 of the face of the policy. Thus, the rate of insurance on a building may be stated either as $1\frac{1}{2}$%, or as $1.50 on every $100 insured.

Business buildings are usually insured for a single year, the policy being renewed annually; dwellings and personal property for three years.

In case of loss the underwriter has the choice of replacing the insured property, or paying its value. Only the actual amount of the loss can be recovered.

Ex. A house is insured against fire for $4500 for 3 years. Find the premium, the rate of insurance being $3\frac{1}{2}$%.

OPERATION.

$$\begin{array}{r} \$4500 \\ .035 \\ \hline 22500 \\ 13500 \\ \hline \$157.500 \end{array}$$ *Premium.*

EXPLANATION.

Since the rate of insurance is $3\frac{1}{2}$%, the premium will be $4500 × .035, or $157.50.

EXERCISE 123.

1. What must be paid for insurance on a property worth $7530, at $1\frac{1}{4}$%? At $1\frac{3}{4}$%?

2. On a vessel worth $12000 the owner had paid insurance

3 years, at the rate of $1\frac{1}{2}\%$ annually; then the vessel was wrecked. What was the total loss? How much of it was not recovered by the insurance?

3. A merchant whose stock of goods is worth \$26000 gets them insured for $\frac{4}{5}$ of their value, at $\frac{3}{4}\%$. What premium does he pay?

4. A house cost me \$6000. I wish to insure it, so that in case of fire I lose nothing. For what must it be insured at 2%?

5. I pay \$11.90 premium on insurance of \$680. What is the rate?

•*6.* If it cost \$82.05 to insure $\frac{2}{3}$ of a store at $1\frac{1}{4}\%$, what is the whole value of the store?

7. I insure my house in one company for \$3500, at $\frac{3}{4}\%$, and my barn in another, for \$2500, at $\frac{1}{2}\%$. What rate do I pay on my entire insurance?

8. A bank building insured for \$75000, at $1\frac{4}{5}\%$, was destroyed by fire after the payment of 4 annual premiums. What was the loss to the company? To the bankers?

STOCKS AND BONDS.

252. Stock Companies and Stocks.—When a business enterprise, as the building and management of a railroad, is too large for the capital of a single person, it is customary for several persons to combine their resources and organize a stock company, for the purpose of carrying on the enterprise. The money thus invested is called the **stock**.

The stock is divided into a number of equal **shares**, each share being usually \$100, but sometimes \$50 or \$25.

The stock company, or corporation, has a **charter** secured by an act of a state legislature, or issued by a state officer in accordance with a general law. The charter specifies the name and purpose of the company, the amount of stock, the method in which the business is to be conducted, etc. A company is usually organized by electing a board of directors, each share of stock being allowed one vote. The board of directors elect officers, as president, secretary, treasurer, etc. Sometimes, however, the officers are elected directly by the stockholders.

Stock certificates are documents issued by a company, stating the number of shares of stock owned by each stockholder respectively.

253. Dividends, Assessments.—When the receipts of a company exceed its expenditures, it usually pays part or all of the net gains to the stockholders as a **dividend**. A dividend is paid out as a certain per cent. of the face or par value of the stock.

When a company is losing money, it often makes an **assessment** on its stockholders to cover a deficit or extraordinary expense. An assessment is also made as a certain per cent. of the par value of stock.

254. Par, Premium, and Discount.—According as a company is paying large or small dividends, and the public has or has not confidence in it, the stock of the company may sell for more or less than its face value.

The **market value** of stock is that for which it will sell. When stock sells for more than its face or par value, it is said to be **above par, or at a premium**; when for less, it is said to be **below par, or at a discount**.

255. Stocks, Bonds, etc.—Some companies issue stock of two kinds: (1) *preferred*, (2) *common stock*. In dividing the gains of a company, the preferred stock receives a dividend first, up to a certain amount, as 5%, after which the remainder of the net income of the company, if there be any, is apportioned to the common stock. About one-fifth of the railroad stock of the United States is preferred stock.

A **bond** is a note issued by a company to the person from whom the company or corporation borrows money, and specifying the amount, time to run, and rate of interest.

Bonds issued by a company are secured by a mortgage on the property of the company; those issued by a city, county, state, or national government are simply promises to pay, without any such security.

Bonds are either *coupon* or *registered* bonds.

Coupon bonds have small certificates of interest attached, which are cut off, as interest becomes due, and cashed at the proper place, as at a bank.

Registered bonds have the name and address of the owner; and interest, when due, is sent to the owner. Registered bonds must be indorsed in order to be sold.

Bonds receive special names indicating the year when they are due, or number of years which they have to run, the rate of interest, etc. Thus, 4's of 1907 are bonds which mature in the year 1907 and pay 4% interest.

Ex. 1. If I buy 12 shares of New York Central R. R. stock at 120 and receive a semi-annual dividend of 3½%, what is my annual income from the investment and what per cent. does the investment pay?

OPERATION.

```
      $12
       100
     $1200
      .035
      6000
     3600
   $42.000
        2
$1440)$84.00(.05⅚
     7200
     1200
     ---- = ⅚
     1440
```

EXPLANATION.

The par value of 12 shares at $100 a share is $1200. Since the dividend is paid on the par value, the semi-annual dividend on 12 shares will be 3½% of $1200 or $42. Hence, the annual income is $42 × 2, or $84. The cost of 12 shares at $120 per share is $1440.

In order to determine the per cent. paid by the investment, it is necessary to determine what per cent $84 is of $1440. This is 5⅚%.

EXERCISE 124.

1. I bought 25 shares of P. R. R. at $60 (par is $50) and received semi-annual dividends of 2½%. What is my annual income? What per cent. do I receive on the investment?

2. Which is the better investment, 9 shares of stock selling at 120 and yielding 7% on par value of 100, or 10 shares in stock selling at 108 and yielding 6%? (Find the rate per cent. of each investment.)

3. Which brings the greater income, $6438 invested in 6% bonds selling at $111 (par $100), or in 7% bonds selling at $58 (par $50)? What is the per cent. in each?

4. What sum must be invested in 5% bonds at 105 to yield an annual income of $1200?

5. What sum must be invested in 7% bonds at 121½ to yield an annual income of $3500?

EXERCISE 125.

GENERAL REVIEW.

1. A merchant sells an overcoat for $50 and gains 25% over total cost. If he had previously paid $9 duty on the goods, what was their first cost?

2. If a single fare to the city is 60 cents, what per cent. do I save by buying 100 tickets for $50?

3. What amount must be invested in Illinois 6's at 112 to realize an annual income of $2100?

4. At 1⅓% I insured my house by payment of $94.32 premium. What is the value of the house?

5. A school-house is to be built at a cost of $17484.25. Collector's commission is 3%; assessable property is valued at $721000. Find the rate of taxation.

6. A jockey sold two horses for $75 each. On one he gained 20% and on the other he lost 20%. Did he gain or lose on the whole transaction? How much? What per cent.?

7. On one occasion the price of kerosene fell from 12 cents per gallon to 8 cents. What per cent. is the decline? Again it rose from 8 cents to 12 cents. What per cent. was the advance?

8. An importer sold a line of goods for $15048, thereby gaining 20%. Previous to this, he had paid a duty of 10% on their cost. What was the cost?

9. A man buys a house for $12000; pays $230 tax and 1¼% insurance each of 5 years. He then sold it at a gain of 10% above all cost and expense. What was the selling price?

10. Mr. B. bought stock at $12 premium ($50 par) and sold it at a loss of 20%. At what rate was it sold?

11. The first cost of an importer's stock of goods was $13200; duty was levied at 12%; insurance was computed upon this total value at 2¼%. What would the goods have to sell for, to return the owner 15 per cent. above all cost?

12. By selling 3% bonds at 102¾ and investing in stock at 137, a man doubles his income. What is the annual dividend (%) of the stock?

13. If $7384.80 includes the price paid for a farm and the agent's commission at 2%, find his commission.

14. A coal dealer ordered through his agent 6000 tons coal at \$3.30 a ton, paid $3\frac{1}{2}\%$ commission, \$56.25 cartage, and \$210.75 freight. He sells it at \$3.98 a ton. What is his gain %?

15. I bought 60 shares Lehigh Valley at 32 and sold it at 60. My gain I invested in N. Y. Central at 120. What was the income from this, under their $5\frac{1}{2}\%$ dividends?

16. Bought a lot of goods at 20% below market price and sold them at 20% above market price. Find my gain per cent.

17. A stationer sold paper at 16 cents a quire, having paid \$2.50 a ream for it. Find his gain %.

18. I sold two houses for \$5400 each, having gained 20% on one and lost 20% on the other. Did I gain or lose in the double transaction, and what %?

19. A drover bought 75 cows at \$30 a head. Ten of them were killed by accident. He sells the rest so as to gain 10% on the transaction. At what price did he sell each cow?

20. A teacher spends 25% of her salary for board, 10% for clothes, 15% for books, $12\frac{1}{2}\%$ for traveling expenses, and saves the balance, which is \$450. What is her salary?

21. I lost 18%, or \$600, in the sale of a property. For what did I sell the property?

22. I sold a watch for \$90, losing 25%. What is my per cent. gain on a second watch which I sell for \$90, so as to profit as much on the one as I lose on the other?

23. What number increased by $33\frac{1}{3}\%$ of itself is 900?

24. Mr. X. raised 496 bushels of wheat, which is $33\frac{1}{3}\%$ more than $\frac{3}{4}$ of Mr. Y.'s crop. How many bushels did Mr. Y. raise?

25. A clerk spent 65% of his salary and saved \$385. How much did he spend?

26. If a man spends \$45.75, which is 60% of his money, how much remains?

27. Which gives the larger percentage return, a \$2400 investment yielding \$112, or an \$8400 investment yielding \$378?

28. Who makes the greater per cent., the lad who buys chestnuts at \$1.28 a bushel and sells them at 5 cents a quart, or the broker who buys bonds at \$144 and sells them at \$168?

29. What per cent. is gained by buying coal by the long ton (2240 lbs.) and selling it at the ordinary ton, \$5 a ton in each case?

30. What is the difference between the single discount of 35% on a bill of \$640 and three successive discounts of 20%, 10%, and 5%?

31. If I pay \$37.50 for insurance on my house at $1\frac{1}{4}$ per cent., what is the amount of insurance?

32. A grocer mixes two kinds of tea which cost him 36 cents and 60 cents per pound. He sells the mixture at 56 cents a pound. What is his per cent. profit?

33. Bought \$128 worth of apples at 80 cents a bushel. Part were damaged, and I sold the rest at an advance of 30%, receiving \$137.28. How many bushels were worthless? What was my per cent. profit withal?

34. I sold a horse for \$150 and with the money bought another. On the first of these horses I gained 20%, and when the second was sold I lost 20%. On the whole transaction, did I gain or lose? How much? What per cent.?

35. In a cubic foot of water there are 889 ounces of oxygen and 111 ounces of hydrogen. What per cent. of water is oxygen (by weight)? What per cent. is hydrogen? What per cent. of the oxygen is the hydrogen?

36. A merchant by selling a pound of butter gains the cost of an ounce. Find his gain %.

37. The volume of a gallon is what % of a cubic foot?

38. The population of a certain city is 84000, and that of the state in which the city is, 189000. What per cent. of the entire population of the state is that of the city?

39. A merchant bought goods for \$1200 and then sold $\frac{1}{3}$ of them at a loss of 25%. For what must he sell the remainder to gain 20% on the whole?

40. An agent received \$40.625 for selling a house worth \$1625. What can you find? Do so.

41. An agent sold 1470 bushels of oats at 60 cents a bushel and charged \$26.46 for doing it. What may be found? Find it.

42. For what sum is a house insured if the premium is \$17.50 and the rate $\frac{7}{8}\%$?

43. Out of a possible 72 points in an examination, A got 27 points and B got $58\frac{1}{2}$. Find their percentages of grade.

44. A merchant wishes to mark some goods which cost \$1.20 a yard, so that he may reduce them 20% from marked price, and still gain 10%. At what price must they be marked?

45. What single discount is equivalent to the successive discounts of 20% and 10%?

46. What is $16\frac{2}{3}\%$ of $116\frac{2}{3}$?

47. $212\frac{1}{2}$ is what per cent. of $916\frac{2}{3}$?

48. A desk was sold for \$60, at a gain of 20%. What would have been the loss % had it been sold for \$40?

49. My house, worth \$6400, was insured for $\frac{3}{4}$ of its value at $\frac{2}{3}\%$. It was destroyed after the second premium had been paid. What was my actual loss? The company's?

50. Write as a rate per cent. $\frac{1}{8}$; $\frac{7}{10}$; $\frac{7}{16}$; $\frac{9}{40}$; $\frac{1}{300}$; $\frac{13}{40}$; $\frac{5}{3}$; $\frac{27}{8}$.

51. Write as a common fraction and as a decimal: $\frac{3}{4}$%; $\frac{3}{10}$%; $7\frac{1}{2}$%; 340%; 40%; 4%; $14\frac{2}{7}$%; 625%.

52. A horse worth \$150 was bought for \$25 less and sold for \$25 more than his real value. What % was gained?

53. At \$5 a ton, how many tons of coal can be bought with \$8526, after paying commission of $1\frac{1}{2}$%?

54. Which is a greater per cent. change, when sugar drops from 6 to 5 cents a pound, or when it advances from 5 to 6 cents? Why?

55. If the cost is $\frac{3}{5}$ of the selling price, find the gain %.

56. If the selling price is $\frac{4}{5}$ of the cost, find the loss %.

57. Bought 320 shares of a certain stock (par \$10) when $3\frac{1}{2}$% below par, and sold them when $1\frac{1}{4}$% above par. Find my gain in dollars and in per cent.

58. Bought 500 shares of railroad stock at $28\frac{1}{2}$ and sold it at $45\frac{3}{4}$. Allowing $\frac{1}{8}$% commission on both purchase and sale, what was my gain?

59. Sold 400 shares of United States Steel preferred, at $98\frac{1}{4}$. With the proceeds I bought 1000 shares of Southern Railway at $31\frac{5}{8}$. Allowing $\frac{1}{8}$% commission in each transaction, how much money remained unemployed?

60. Mr. A. sold 800 Reading bonds (4%) at $96\frac{3}{8}$ and bought Erie at 51. How many shares did he buy, allowing commission on bonds at $\frac{1}{4}$% and on stock at $\frac{1}{8}$%?

61. I sent my broker \$10000, asking him to buy some Southern Pacific R. R. stock under 45 and sell it over 60. He bought all he could with my remittance at $42\frac{3}{4}$ and sold it at $60\frac{1}{2}$. Compute the number of shares purchased; his total commission at $\frac{1}{8}$%; and my profit.

62. In a certain school there are 288 boys and 162 girls. What per cent. of the school is boys? Girls?

63. Into 80 gallons of alcohol are mixed 40 gallons of water. What per cent. of the mixture is water?

64. To bake a certain kind of bread 9 measures of rye meal are mixed with 13 measures of corn meal. What per cent. of the bread is rye?

65. After a reduction of 8% a man's wages were \$22.54. What were they before the reduction?

66. What amount of insurance may I procure on my house by paying \$35 if the rate is $\frac{5}{8}$%?

67. Twenty per cent. of the selling price of my horse was \$36. What was my profit if I gained 20%?

68. In a certain city there are 73000 white, and 25000 black citizens, What per cent. of the entire population is colored?

69. Sold two horses for \$210 each. On one I gained 20% and on the other I gained \$20. Find my total gain. Also gain per centum.

CHAPTER XV.

INTEREST.

256. Interest.—If a business man does not have money enough of his own to carry on a certain enterprise, he may be enabled to proceed, by borrowing money from another person. In such cases it is customary to pay a certain sum (per annum) in return for the use of the borrowed money.

As money paid for the use of a house is called *rent*, and money paid for the use of a horse is called *hire*, so money paid for the use of money is called **interest**.

Interest is usually reckoned as a certain annual per cent. of the money borrowed.

257. Interest and Time.—The length of time for which borrowed money is used varies greatly according to the needs of the borrower. The time may be only a few days, or it may be a number of years, or of years, months, and days. Hence, interest differs from percentage in general, in that the element of *time* is to be carefully considered in connection with every problem.

258. The **quantities considered** in interest are the *principal, rate, interest, time, amount.*

The **principal** is the sum of money on which interest is paid.

The **rate** is the per cent. of the principal paid for the use of the principal for one year.

The **interest** is the sum of money paid for the use of the principal for the entire time.

The **amount** is the sum obtained by adding the interest to the principal.

259. Legal interest is interest determined according to a

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The **interest** is the sum of money paid for the use of the principal for the entire time.

The **amount** is the sum obtained by adding the interest to the principal.

259. Legal interest is interest determined according to a rate fixed by law. In business transactions, when no rate is specified, the legal rate is understood.

Usury is interest at a rate higher than that fixed by law. The exaction of usury is punishable by law.

Different states and nations have different legal rates of interest, according to their financial needs.

In 1900 the legal rate was 6% in all the states, with the following exceptions: 5% in La. and Ill.; 7% in Ariz., Cal., Ga., Idaho, Minn., Neb., Nev., N. Dak., Okla., S. C., S. Dak., and Wash.; 8% in Ala., Colo., Fla. Ore., Utah, Wyo.; 10% in Mont.

In many of the states the law allows a higher rate than the legal one to be used by mutual agreement.

260. Methods of Computing Interest.—Persons who have many problems in interest to compute (as, for instance, bankers) do so by the use of tables. Other methods, however, are needed for persons to whom tables are not accessible, and to give mastery of other problems related to interest. Of the many methods by which interest can be computed, three principal ones (beside the use of tables) will be considered here, another being presented later in connection with the subject of discount.

I. When the Time is an Exact Number of Years or Months.

261. Method.—When the period of time for which interest is to be computed is an exact number of years, or years and months, *find the interest for one year and multiply this by the number of years.*

Ex. What is the interest on $350 for 2 years 3 months, at 5%?

OPERATION.

2 yr. 3 mo. = $2\frac{1}{4}$ yr.

$350	
.05	
$17.50,	*Interest for 1 year.*
$2\frac{1}{4}$	
3500	
$437\frac{1}{2}$	
$39.38,	*Interest for 2 yr. 3 mo.*

EXPLANATION.

Since 3 mo. = $\frac{1}{4}$ yr., the entire time is $2\frac{1}{4}$ yr. Since 5% of $350 is $17.50, the interest for 1 yr. is $17.50. Hence, the interest for $2\frac{1}{4}$ yr. is $2\frac{1}{4}$ times $17.50, or $39.38.

EXERCISE 126.

Find the interest of

1. $340 for 4 yr. 6 mo. at 5%.
2. $275 for 3 yr. 8 mo. at 6%.
3. $515 for 5 yr. 4 mo. at $4\frac{1}{2}$%.
4. $108.50 for 6 yr. at 3%.
5. $214.61 for 8 yr. 6 mo. at 5%.
6. $2075 for 10 yr. 2 mo. at 6%.
7. $489.30 for 2 yr. 5 mo. at 7%.
8. $307.15 for 9 mo. at $2\frac{1}{2}$%.
9. $2560.60 for 7 yr. 11 mo. at $3\frac{1}{2}$%.
10. $1971.40 for 10 mo. at $5\frac{1}{2}$%.
11. $7327.50 for 3 yr. 8 mo. at 6%.
12. $956.70 for 5 yr. 6 mo. at $4\frac{1}{2}$%.

EXERCISE 127.

ORAL.

What is the interest of

1. $3 for 4 yr. at 6%? At 2%? At 5%?
2. $8 for 1 yr. 6 mo. at 5%? At 3%?
3. $9 for 2 yr. 4 mo. at 3%? At 4%?
4. $10 for 3 yr. 3 mo. at 4%? At 6%?
5. $15 for 5 yr. 8 mo. at 2%? At 5%?
6. $100 for 6 yr. 9 mo. at 5%? At 6%?

II. Six Per Cent. Method.

262. The **six per cent. method** of finding interest consists essentially in finding

(1) the interest on $1 for the given time at 6%,

(2) the interest on the entire principal for the given time at 6%,

(3) the interest on the entire principal at any other desired rate.

The interest on $1 for the required time at 6% is readily obtained by the use of the following:

Interest on $1 for 1 year = $0.06.
" " $1 " 1 mo. = .005.
" " $1 " 6 da. = .001.
" " $1 " 1 da. = $.000\frac{1}{6}$.

Hence, in computing interest according to this method, each month is considered as containing 30 days, and 1 year as containing 360 days.

Ex. 1. Find the interest on $312 for 2 yr. 7 mo. 15 da. at 6%.

SOLUTION.

Interest on $1 for 2 yr. = $0.06 × 2 = $0.12.
" " $1 " 7 mo. = $0.005 × 7 = .035.
" " $1 " 15 da. = $\$0.000\frac{1}{6}$ × 15 = .0025.
" " $1 " 2 yr. 7 mo. 15 da. = .1575.
Interest on $312 " " " = $312 × .1575.

```
   $312
  .1575
  -----
   1560
  2184
 1560
 312
 --------
 $49.1400, Interest.
```

Ex. 2. Find interest on $312 for 2 yr. 7 mo. 15 da. at 5 %.

SOLUTION.

By Ex. 1, the interest on $312 for the given time at 6% = $49.14.
Hence, " " " " " " " 1% = 8.19.
" " " " " " " 5% = $40.95.

Ex. 3. Find the interest on $317.25 from Apr. 1, 1892, to Nov. 19, 1896, at $4\frac{1}{2}$%.

SOLUTION.

From Apr. 1, '92, to Nov. 19, '96, is 4 yr. 7 mo. 18 da.
Interest on $1 for 4 yr. 7 mo. 18 da. at 6% = $0.278.
Interest on $317.25 for 4 yr. 7 mo. 18 da. at 6% = $0.278 × 317.25 = $88.20.
" " $317.25 " " " " " $1\frac{1}{2}$% = $22.05.
" " $317.25 " " " " " $4\frac{1}{2}$% = $66.15, *Result.*

Hence, in general, *obtain the interest on $1 for the given time, by multiplying the number of years by .06, the number of months by .005, and the number of days by $.000\frac{1}{6}$, and taking the sum of the results; multiply the number of dollars in the principal by the interest on $1; this will give the interest at 6%; to obtain the interest at any other rate, add or subtract such a fractional part of this interest, as the rate exceeds or falls below 6%, or find the interest at 1% and multiply it by the required rate.*

The first part of this method, sometimes stated in other ways, as *reduce the years to months, take the number of months as cents, one-third the number of days as mills, and multiply their sum by half the principal;* or *multiply dollars by days and divide by 6000.*

EXERCISE 128.

Find the interest, by the 6% method, of:

1. $260 for 2 yr. 8 mo. 24 da. at 6%.
2. $450 for 4 yr. 6 mo. 15 da. at 4%.
3. $846 for 7 yr. 10 mo. 12 da. at 5%.
4. $2350 for 5 yr. 9 mo. 6 da. at 3%.
5. $246.70 for 6 yr. 1 mo. 20 da. at $4\frac{1}{2}$%.
6. $93.45 for 3 yr. 11 mo. 3 da. at 7%.
7. $928.50 for 8 yr. 3 mo. 25 da. at 8%.
8. $1250.40 for 10 yr. 10 da. at $3\frac{1}{2}$%.
9. $760 from July 4, 1875, to Feb. 22, 1890, at 4%.
10. $45.50 from Oct. 19, 1890, to Mar. 11, 1899, at 5%.
11. $325.60 from Nov. 8, 1888, to June 23, 1897, at 3%.
12. $50.60 from Dec. 25, 1789, to May 1, 1801, at 6%.
13. $2500 from Aug. 17, 1903, to Nov. 8, 1930, at 5%.
14. $3280 from July 4, 1776, to Jan. 3, 1850, at $4\frac{1}{2}$%.
15. $7650 from Sept. 10, 1888, to Oct. 1, 1900, at 7%.
16. $372 from Oct. 12, 1492, to to-day at 5%.
17. $9800 for 8 yr. 3 mo. 26 da. at $3\frac{1}{2}$%.
18. $7384.80 for 9 yr. 2 mo. 10 da. at $5\frac{1}{2}$%.
19. $12.75 for 7 yr. 18 da. at $6\frac{1}{2}$%.
20. $5.64 for 3 yr. 8 mo. 20 da. at $3\frac{1}{2}$%.

III. Exact Interest.

263. Exact interest is interest obtained by taking one year as equal to 365 days.

The U. S. Government computes interest by the exact method, as do also an increasing number of business men.

Ex. Find the exact interest on \$652 from Apr. 1, 1895, to Sept. 13, 1897, at 7%.

Solution.

From Apr. 1, '95, to Apr. 1, '97, is 2 years.
From Apr. 1, '97, to Sep. 13, '97, is 165 days.
165 days = $\frac{165}{365}$ yr. = $\frac{33}{73}$ of a year.
Interest on \$652 at 7% for 1 year = \$45.64.
Interest on \$652 at 7% for $2\frac{33}{73}$ years = \$45.64 × $2\frac{33}{73}$ = \$111.91, *Interest.*

EXERCISE 129.

Find the exact interest of:

1. \$400 for 146 days at 7%.
2. \$325 for 3 yr. 219 da. at 4%.
3. \$75.50 for 2 yr. 100 da. at 5%.
4. \$136.40 for 4 yr. 150 da. at $4\frac{1}{2}$%.
5. \$350 from Jan. 10, 1890, to Dec. 1, 1894, at 5%.
6. \$425 from May 16, 1887, to Jan. 4, 1895, at 6%.
7. \$170 from Feb. 20, 1900, to Oct. 16, 1906, at 6%.
8. \$90.50 from July 7, 1891, to Mar. 3, 1899, at 4%.

9. Find the difference between the exact interest and the interest determined by the 6% method, on \$7000 from Mar. 11 to Sept. 10, at 6%.

10. Find the difference between the exact interest and the interest found by the 6% method, on \$4500 from Nov. 16, 1889, to June 23, 1897, at 5%.

11. Find the difference between the two interests of \$5678000 from Jan. 10 to July 10 of same yr., at 7%.

12. When the time is less than a year and in days, what will always be the ratio between these two interests?

IV. Interest Tables.

264. If **interest tables** are available, the most convenient way of computing interest is by their use. In some cases the tables are formed regarding a year as 360 days; other tables give exact interest. In either case the exact number of days is reckoned between two dates in computing any problem in interest.

A part of a page from an interest table is inserted below. It gives the exact interest on various sums from \$10 to \$150 at 1, 5, 6, 7% for 60 and 61 days.

	60 days				61 days			
Dolls.	7	6	5	1	7	6	5	1
10	.1151	.0986	.0822	.0164	.1170	.1003	.0836	.0167
20	.2301	.1973	.1644	.0329	.2340	.2005	.1671	.0334
30	.3452	.2959	.2466	.0493	.3510	.3008	.2507	.0501
40	.4603	.3945	.3288	.0658	.4679	.4011	.3342	.0668
50	.5753	.4931	.4110	.0822	.5849	.5014	.4178	.0836
60	.6904	.5918	.4931	.0986	.7019	.6016	.5014	.1003
70	.8055	.6904	.5753	.1151	.8189	.7019	.5849	.1170
80	.9205	.7890	.6575	.1315	.9359	.8022	.6685	.1337
90	1.0356	.8877	.7397	.1479	1.0529	.9025	.7520	.1504
100	1.1507	.9863	.8219	.1644	1.1699	1.0027	.8356	.1671
110	1.2657	1.0849	.9041	.1808	1.2868	1.1030	.9192	.1838
120	1.3808	1.1836	.9863	.1973	1.4038	1.2033	1.0027	.2005
130	1.4959	1.2822	1.0685	.2137	1.5208	1.3035	1.0863	.2173
140	1.6109	1.3808	1.1507	.2301	1.6378	1.4038	1.1699	.2340
150	1.7260	1.4794	1.2329	.2466	1.7548	1.5041	1.2534	.2507
Etc.	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. .

Ex. Find exact interest on \$146.50 at 5% for 60 days by the use of tables.

SOLUTION.

Interest on $140 for 60 days at 5%	=	$1.1507
" " 6 (= $\frac{1}{10}$ of $60) for 60 days at 5%	=	.0493
" " .50 (= $\frac{1}{100}$ of $50) for 60 days at 5%	=	.0041
		$1.2041,
		or $1.20, *Interest.*

PROBLEMS IN INTEREST.

Any three of the five quantities considered in interest, the *principal, rate, interest, time, amount*, being given, the other two may be found. The case when the principal, rate, and time are given, to find the interest (and amount) has already been considered.

265. I. *Given the* **principal, interest** (*or amount*), *and* **time**, *to find the* **rate per cent.** The method is best shown by an example.

Ex. 1. At what rate per cent. will $360 produce $66 interest in 3 yr. 8 mo?

SOLUTION.

Interest on $360 for 1 yr. at 1% = $3.60.
" " $360 " 3 yr. 8 mo. at 1% = $3.60 × $3\frac{2}{3}$ = $13.20.

If the given principal produces $13.20 interest in the given time at 1%, it will take as many per cent. to produce $66 as $13.20 is contained times in $66.

$$\text{Rate required} = \frac{\$66}{\$13.20} = 5, \textit{ Rate.}$$

Ex. 2. At what rate per cent. will $120 amount to $144 in 5 years?

The interest = amount − principal = $144 − $120 = $24. Proceeding as in Ex. 1.

$$\begin{array}{r} \$120 \\ \underline{.01} \\ \$1.20 \\ \underline{5} \\ \$6.00)\$24.00(4, \textit{ Rate.} \\ \underline{24.00} \end{array}$$

Hence, in general, *divide the given interest by the interest on the principal for the given time at one per cent.*

It is useful also to state this rule briefly, thus:

rate = *interest* ÷ (*principal* × .01 × *time*),

or using symbols, $r = \dfrac{i}{p \times t}$,

where r denotes the rate per cent. expressed decimally.

EXERCISE 130.

Find the rate if:

1. Interest on $420 for 3 yr. 6 mo. is $73.50.
2. Interest on $56 for 4 yr. 3 mo. is $9.52.
3. Interest on $760 for 2 yr. 8 mo. is $121.60.
4. Interest on $840 for 5 yr. 3 mo. 15 da. is $311.15.
5. Interest on $900 for 2 yr. 4 mo. 20 da. is $96.75.
6. Interest on $45 for 3 yr. 9 mo. 24 da. is $10.305.
7. Interest on $370 for 6 yr. 6 mo. 6 da. is $72.335.
8. Interest on $49.50 for 1 yr. 4 mo. 10 da. is $2.695.
9. Interest on $1780 for 6 mo. 25 da. is $55.748.
10. Interest on $100 for 16 yr. 8 mo. is $100.
11. Amount of $360 in 3 yr. 10 mo. is $415.20.
12. Amount of $3700 in 2 yr. 7 mo. is $4225.71.
13. Amount of $75 in 6 yr. 11 mo. 10 da. is $93.23.

266. II. *Given the* **principal, interest** (*or amount*), *and* **rate**, *to find the* **time.**

Ex. In what time will the interest on $424 be $37.10 at $2\frac{1}{2}$%?

SOLUTION.

Interest on $424 at $2\frac{1}{2}$% for 1 year = $10.60.

Hence, it will take as many years to produce $37.10 interest as $10.60 is contained times in $37.10.

$$\therefore \text{ Number of years} = \frac{\$37.10}{\$10.60} = 3\tfrac{1}{2}. \quad \therefore \textit{Time} = 3 \text{ yr. } 6 \text{ mo.}$$

Hence, in general, *divide the given interest by the interest of the principal at the given rate for one year*;

or *time* = *interest* ÷ (*principal* × *rate expressed decimally*),

or using symbols, $t = \dfrac{i}{p \times r}$

EXERCISE 131.

Find the time if:

1. Interest on \$75 at 6% is \$15.75.
2. Interest on \$240 at 5% is \$63.
3. Interest on \$475 at 4% is \$33.25.
4. Interest on \$76.80 at $4\frac{1}{2}$% is \$9.792.
5. Interest on \$570 at 7% is \$332.50.
6. Interest on \$65 at 6% is $\$12.51\frac{1}{4}$.
7. Interest on \$820 at 3% is \$140.77.
8. Interest on \$980 at $3\frac{1}{2}$% is \$259.54.
9. Amount of \$420.75 at 4% is \$475.915.
10. Amount of \$31250 at 5% is \$35625.
11. Amount of \$8.25 at 6% is \$13.134.
12. Amount of \$2460 at $5\frac{1}{2}$% is \$3224.82.

267. III. *Given the* **interest** (*or amount*), **time,** *and* **rate,** *to find the* **principal.**

Ex. 1. What principal will produce \$33.75 interest in 2 yr. 3 mo. at 6%?

SOLUTION.

Interest on \$1 for $2\frac{1}{4}$ yr. at 6% = \$0.135.

If one dollar produces \$0.135 interest in the given time, it will take as many dollars to produce \$33.75 interest as \$0.135 is contained times in \$33.75, or

$$\frac{\$33.75}{\$0.135} = 250. \quad \therefore \textit{Principal} = \$250.$$

Hence, in general, *divide the given interest (or amount) by the interest on (or amount of) \$1 for the given time and rate,*

or *principal* = *interest* ÷ (*time* × *rate expressed decimally*), or $p = \dfrac{i}{r \times t}$.

It is to be noted that if the amount be given instead of the interest, it is necessary to *divide the given amount by the amount of \$1 for the given time and rate.*

Ex. 2. What principal will amount to \$263.50 in 4 years at 6%?

SOLUTION.

Amount of \$1 at 6% in 4 years = \$1.24.

It will take as many dollars to amount to \$263.50 as \$1.24 is contained times in \$263.50, or

$$\frac{\$263.50}{\$1.24} = 212.50. \quad \therefore \textit{Principal} = \$212.50.$$

EXERCISE 132.

What principal will:

1. Produce \$156.40 interest in 5 yr. 8 mo. at 6%?
2. Produce \$30.38 interest in 7 yr. 9 mo. at 5%?
3. Produce \$7.15 interest in 3 yr. 3 mo. at 4%?
4. Produce \$36.72 interest in 2 yr. 10 mo. at $4\frac{1}{2}$%?
5. Produce \$83.72 interest in 6 yr. 8 mo. 15 da. at 3%?
6. Produce $\$5.49\frac{1}{2}$ interest in 4 yr. 4 mo. 10 da. at 5%?
7. Amount to \$99.22 in 3 yr. 6 mo. at 6%?
8. Amount to \$195.09 in 5 yr. 9 mo. 20 da. at 5%?
9. Amount to \$121.98 in 4 yr. 8 mo. 24 da. at 6%?
10. Amount to \$327.384 in 7 yr. 4 mo. 12 da. at 7%?

268. The **present worth** of a sum payable at a future time (without interest) is such a sum, as being put at interest, will amount to the given sum in the given time.

The **true discount** is the difference between the sum payable at a future time and its present worth.

Hence, in determining the present worth of a sum due at a certain future date, we have given the *amount, time,* and *rate,* to find the *principal* (see Art. 267).

EXERCISE 133.

Find the present value and true discount upon:

1. A debt of \$378.75 due in 3 mo. without interest, but money being worth 4%.

2. A debt of \$4377.80 due in 6 mo. 15 da. without interest, but the usual rate being 6%.

3. A debt of \$8255 due in 1 yr. 8 mo. without interest, when the regular interest is at $3\frac{1}{2}$%.

4. I owe you \$1492 payable in 8 mo. 20 da., not bearing interest. What sum ought you accept now if interest is reckoned at 5%?

269. General Algebraic Method.—For students who are familiar with the elements of algebra, all the problems in interest may be combined as the treatment of a single simple equation.

Thus, let p = principal, r = rate per cent. expressed decimally.
i = interest, t = time expressed in years.

Then by the definition of interest, Art. 256 and by Art. 261,

$$i = p\,r\,t \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

Any three of the four quantities, i, p, r, t, being given, the remaining quantity is found by solving equation I.

$$\text{Thus, divide I by } p \times t,\ r = \frac{i}{p\,t} \quad . \quad . \quad . \quad . \ \text{II.}$$

$$\text{“} \quad \text{I “} \ p \times r,\ t = \frac{i}{p\,r} \quad . \quad . \quad . \ \text{III.}$$

$$\text{“} \quad \text{I “} \ r \times t,\ p = \frac{i}{r\,t} \quad . \quad . \quad . \ \text{IV.}$$

Also letting a = amount,

$$a = p + p\,r\,t = p\,(1 + r\,t)$$

$$\therefore p = \frac{a}{1 + r\,t}.$$

EXERCISE 134.

GENERAL REVIEW.

1. At what rate will \$705.60 yield \$170.52 interest in 4 yr. 10 mo.?
2. What principal will produce \$14.30 interest in 3 yr. 8 mo. at 4%?
3. In what time will \$72.50 amount to \$113.64⅜ at 6%?
4. If the rate is 4½%, the time 4 yr. 3 mo., and the amount \$123.89, find the principal.
5. If the principal is \$318, the rate 5%, and the amount \$422.675, find the time.
6. If the interest of \$441 is \$95.67¼ in 5 yr. 11 mo., find the rate.
7. What is the present worth of \$800, payable in 9 mo., money being worth 6%? 7%? 5%?
8. In 4 yr. 8 mo. a note I hold will be worth \$385.60. What ought I accept now, if money is worth 5%?

9. In what time will a sum of money double itself at simple interest, the rate being 4%? 5%? 6%? 7%? 8%?
10. In what time will \$126.50 yield \$6.66 interest at 4%?
11. At what rate will \$562 yield \$160.86 in 3 yr. 6 mo. 28 da.?
12. What principal will give \$207.71 interest in 1 yr. 6 mo. 17 da. at 5%?
13. What is the interest on \$963.45 from April 10, 1883, to July 4, 1895, at 2½%?
14. At what rate will \$250 gain \$35 in 2 yr. 9 mo. 18 da.?
15. Find the amount of \$392.10 for 6 yr. 9 mo. 15 da. at 3¾%.
16. On an investment of \$5620 I receive \$1803 in 2 yr. 3 mo. 15 da. What is the rate?
17. In what time will \$2275 amount to \$2673.12½ at 5%?
18. What principal will in 5 yr. 8 mo. 15 da., at 5%, give \$287.70?
19. At 6%, what sum of money will amount to \$666 in 6 yr. 6 mo. 6 da.?
20. In what time will \$500 produce \$50 interest at 3½%?

CHAPTER XVI.

APPLICATIONS OF INTEREST.

PROMISSORY NOTES.

270. Promissory Notes.—If a person borrows money of another person, or buys goods on credit (that is, without paying for them immediately), he usually gives a *note*, or *written promise* to pay the given sum of money at a given date in the future (or on demand), with or without interest, as the case may be.

This written promise is called a **promissory note.** A promissory note given by one person to another is usually of the following form:

$350. PHILADELPHIA, PA., *Oct. 12, 1900.*

Thirty days after date, I promise to pay James Scudder, or order, three hundred and fifty dollars, value received, with interest at six per cent.

WILLIAM HEYWOOD.

From the general form as here given, notes may vary by the omission of the words "or order," or by the omission of the time clause ("thirty days after date"), or by the substitution of a definite date, as "on Nov. 12," for the time clause; or by the omission of the interest clause ("with interest at six per cent."); or by the insertion of clauses not given in the above form, as "without defalcation," or of a clause specifying the bank where the note is payable; or by being signed by several persons.

The essential parts of the note are the **date,** the **promise to pay, person to whom, amount,** the words **"value received,"** and **signature.**

If the words "value received" are omitted, the holder of the note may be required to prove that the value of the note had been received by him.

271. Definitions.—The **maker** of a note is the person who signs the note, as William Heywood in the note of Art. 270.

The **payee** is the person to whom the note is made payable, as James Scudder in the above note.

The **face** (or principal) is the sum promised to be paid.

The face of the note is written in the body of the note in words, not figures, to avoid fraud or error. The number of cents, however, is usually expressed in figures, as the hundredths of a dollar.

272. Maturity of Notes.—In some states, as New York, New Jersey, Pennsylvania, and Illinois, a note matures, that is, is legally due, at the end of the time specified in the note. Thus, the above note (Art. 270) is due 30 days after Oct. 12, 1900, that is, on Nov. 11, 1900.

In other states, as California, Tennessee, etc., a note is due three days after the time specified in the note. These three days are called **days of grace.** Thus, if the above note had been given in Knoxville instead of Philadelphia, it would have been due Nov. 14, instead of Nov. 11.

Days of grace were formerly allowed in all states, but their use is gradually being abolished by law.

When the time of a note is specified in months, calendar months are used. Thus, if a three-months note is given on June 5, it falls due on Sept. 5 (or Sept. 8, if days of grace are allowed). If, however, a note for 90 days is given on June 5, it falls due on Sept. 3 (or Sept. 6).

If a note falls due on a Sunday or a legal holiday, it matures on the nearest business day preceding, except in Pennsylvania, where it falls due on the first business day following.

It is becoming customary to specify in the note the day on which the note becomes due, instead of stating the number of days or months which the note is to run.

If no time for which the note is to run is specified in the note, the note is payable at any time on which the holder of the note may choose to call for its payment, that is, it is due *on demand.*

273. Interest on Notes.—If a note contain the words "with interest," interest is computed on the note from the date at which the note is given. Thus, interest on the above note (Art. 270) is computed beginning with Oct. 12.

If, however, the words "with interest" are omitted, interest is computed from the day on which the note becomes due; thus, if, in the above note, the words "with interest at 6%" were omitted, interest would be computed beginning with Nov. 11.

Interest on notes is usually computed for the exact number of days (even when the note runs for a certain number of months), allowing either 360 or 365 days to the year.

Since notes are usually given for 30, 60, or 90 days, it is generally convenient to find the interest at 6% for 60 days (which interest equals 1% of the principal), and for such parts of 60 days as are needed, to take their sum and then obtain the interest at any other rate if such rate is used. This, in effect, constitutes still another method of computing interest than those given in Chapter XV., and is called the **Two Months Method.**

Ex. Find the amount due on the following note at maturity.

$860.42. TRENTON, N. J., *May 12, 1898.*
Three months after date, I promise to pay Stephen Blake, Eight Hundred Sixty $\frac{42}{100}$ Dollars, value received, with interest at 5%.
JAMES EVANS.

SOLUTION.

The note is due 3 mo. after May 12, that is, on Aug. 12.
From May 12 to Aug. 12, the number of days is as follows:

May, 19 days.	Int. on $860.42 for 60 da. at 6%	= $8.604
June, 30 "	" " " 30 " "	= 4.302
July, 31 "	" " " 2 " "	= .2868
Aug., 12 "	" " " 92 " "	= 13.193
Total, 92 "	" " " " " 1%	= 2.199
	" " " " " 5%	= 10.99
		860.42

Amount due Aug. 12 = $871.41, *Result.*

EXERCISE 135.

Find the date of maturity and the amount due then, on each of the following notes:

1. *$360.* ALBANY, N. Y., *April 7, 1901.*
Four months after date, I promise to pay Bradley Goold, Three Hundred Sixty Dollars, with interest, at 5%, for value received.
JONATHAN SCUDDER.

2. *$935.* PRINCETON, N. J., *Dec. 5, 1892.*
On demand, we, or either of us, promise to pay, with interest at $4\frac{1}{2}$%, to Charles R. Watson, Nine Hundred Thirty-five Dollars, without defalcation. HORACE DAY,
(Paid, 8 mo. from date.) EARNEST F. KEIGWIN.

3. *$700.* PHILADELPHIA, *Oct. 8, 1897.*
Ninety days after date, I promise to pay William Black, Seven Hundred Dollars, with interest at $3\frac{1}{2}$%, for value received.
ELLERY FRANKLIN.

4. *$520.60.* CHICAGO, ILL., *Dec. 14, 1899.*
Six months after date, I promise to pay Sherman Thatcher, Five Hundred Twenty Dollars and $\frac{60}{100}$, with interest, at 6%, for value received. BENJAMIN CARROLL.

5. *$1400.* TRENTON, N. J., *March 8, 1873.*
On July 1, 1893, I promise to pay Robert Dolliver, or order, Fourteen Hundred Dollars, with interest at six per centum, for value received. ROGER SOMERVILLE.

6. *$250.* ALBANY, N. Y., *Feb. 1, 1875.*
On the 21st of January, 1899, I promise to pay Dwight Ogden, Two Hundred Fifty Dollars, with interest at five per centum.
F. I. SHELL.

7. A 90-day note for $2800, with interest at $5\frac{1}{2}$%, was paid at maturity. What amount was due?

8. A 3-month note for \$6750 at $4\frac{3}{4}\%$ was paid when due. What amount was due?

9. Write a 60-day note with your teacher as payee and yourself as maker, for \$100, interest at 5%. Then compute its value when due.

10. Write a 6-month promissory note for \$1670, bearing interest at $4\frac{1}{2}\%$, with Horace Mansfield as maker and John Douglass payee. Then compute its value at maturity.

11. Write a demand note for \$3250, at $5\frac{1}{2}\%$, with yourself as payee and Richard Smith as maker. Compute its value 6 yr. 7 mo. 8 da. from date.

12. What is due on a promissory note for \$6840.90 at 4%, 23 yr. 11 mo. 23 da. after date? Write such a note, giving its date and its date when due.

Some other facts relating to the use of promissory notes are of value, though they do not directly affect arithmetical computations. Among them are the following:

274. Transfer of Notes.—If a note be written in a certain form, it may be sold or transferred by the payee or holder to another person. Such a note is said to be **negotiable.**

In order to be negotiable, a note must be made payable to the "order of" the payee, or to the "bearer." (Details in the form of notes, and in methods of computing interest on notes, vary in different states and localities. The pupil should question a local banker and discover the forms and methods necessary in his state, or customary in his locality.)

In order to transfer a note, the payee or holder must write his name across the back of the note. If only the signature be written, the indorsement is said to be "general indorsement" or "indorsement in blank"; if the words "please pay to the order of" a specified person, are written with the signature underneath, the indorsement is said to be "special."

If a note is made payable to "bearer" merely, it may be transferred without indorsement.

275. Failure to Pay a Note.—If the maker of a note fails to pay the note when it becomes due (or, if the note be payable on demand, when it is presented for payment), every indorser of the note becomes liable for the payment of the entire note. (If an indorser has written the words "without recourse" above his name, he is not liable. Several indorsers may have an agreement to share the amount of the note between them.)

In case of the failure of the maker of a note to pay the note, the usual method of the holder is to collect the amount of the note from the last indorser, who collects it in like manner from the preceding indorser, etc., up to the first liable indorser. The holder, however, may, if he chooses, collect the amount of the note from any liable indorser, such collection, however, releasing all indorsers *subsequent* to the one from whom collection is made, though prior indorsers are still liable.

A **protest** is a written notice sent by a notary public at the request of the holder of a note to the indorsers of the note, that the note has not been paid when due.

If a protest is not sent to an indorser on the day on which the note matures, the indorser is released by law, unless the words "waiving demand and notice" have been written above the indorser's signature. If the notice has been properly made out and mailed by the notary, the law assumes that a protest has been made.

276. Kinds of Notes.—In what has been said, reference has been made to different kinds of notes, which may be termed **time note, demand note, negotiable note,** and **non-negotiable note.** The pupil may give formal definitions of these. Beside these are other kinds of notes, as follows:

A **joint note** is a note signed by two or more persons, each of whom is liable for his share of the amount of the note.

A **joint and several note** is a note signed by several persons, each of whom is liable, not only for his share of the note, but also for the entire note if the other signers fail to pay. Instead of "I promise," the note reads "we, or either of us, promise" to pay, etc.

Bank bills or notes are promissory notes issued by banks and payable on demand.

PARTIAL PAYMENTS.

277. Partial Payments.—It is frequently convenient for the maker of a note, instead of paying the note all at one time, to make **partial payments** on it, as money comes into his possession, till the entire note is paid.

278. Indorsements.—The partial payments made are written on the back of the note and called *indorsements.* Each indorsement specifies the amount paid and the date of payment, with the signature of the holder of the note, and is thus, in effect, a receipt for each amount paid on the note.

Several kinds of indorsements occur in business, but all have reference to a writing of some sort on the back of a business paper, the word indorsement being obtained from the Latin word "*dorsum*," meaning "back."

279. United States Rule *for the settlement of a note on which partial payments have been made.* The Supreme Court of the United States, reasoning that interest shall not be reckoned on interest nor on any payment, has fixed the following rule for allowing for partial payments which have been made in the settlement of a note. The same rule has been adopted by most of the State Governments, the chief exceptions being New Hampshire, Vermont, and Connecticut.

Find the amount of the principal until the time of the first payment; if the payment exceeds the interest due, subtract the payment from the amount and use the remainder as a new principal; if, however, the payment is less than the accrued interest, find the amount of the principal to the date of that payment at which the sum of the payments exceeds the interest to date; subtract the sum of the payments from the amount and use the remainder as a new principal; proceed in like manner till the amount due on the day of settlement is determined.

It is seldom that a payment less than the accrued interest is made; when such a payment occurs, it is usually easy to determine that it is less than the accrued interest by a simple mental trial.

The time between dates in partial payments is usually found by subtracting months and days, and not by determining the exact number of days between dates.

Ex. Find the amount due on the following note on Jan. 1, 1898.

$800. NEW YORK, *Jan. 1, '96.*

Two years from date, I promise to pay James White, or order, Eight Hundred Dollars, for value received, with interest.

SAMUEL HILLMAN.

Indorsed with the following payments, Apr. 1, '96, $10; Jan. 1, '97, $100; Apr. 10, '97, $200.

SOLUTION.

As the interest on $800 for 3 months is $12, the first payment is less than the interest to date. Hence, we have

Interest on $800 from Jan. 1, '96, to Jan. 1, '97,	=	$48
Principal,	=	800
Amount,	=	848
Sum of payments to Jan. 1, '97,	=	110
Remainder for new principal,	=	738
Interest on $738 from Jan. 1, '97, to Apr. 10, '97,	=	12.17
Amount,	=	750.17
Payment Apr. 10, '97,	=	200
New principal,	=	550.17
Interest from April 10, '97, to Jan. 1, '98,	=	23.93
Amount due on Jan. 1, '98,	=	$574.10, *Result.*

EXERCISE 136.

1. $600. HARRISBURG, PA., *November 1, 1888.*

Three years after date, I promise to pay George Morris, Six Hundred Dollars, with interest at 5%.

THEODORE JOHNSON.

The following endorsements were made: Jan. 1, 1889, $60; July 1, 1890, $100; Dec. 10, 1890, $220.

What remained due at maturity?

2. What is due July 1, 1899, on a note for $2100, dated Jan. 1, 1897, with interest at 4%, on which are the following payments endorsed:

Sept. 1, 1897, $450; Aug. 1, 1898, $620; Feb. 1, 1899, $500?

3. A note for $2400 dated July 15, 1880, drawing interest at 5%, bears following endorsements:

Dec. 20, 1880, $740; Mar. 30, 1881, $250; June 18, 1882, $600. What is due Dec. 1, 1882?

4. How much is due Jan. 1, 1900, on a 6% note for $1500, dated Oct. 20, 1897, and endorsed as follows:

Dec. 30, 1897, $450; July 9, 1898, $25; Nov. 7, 1899, $600?

5. How much remains due Oct. 15, 1888, on a 5% note for $3600, dated June 23, 1885, and endorsed as follows:

Oct. 9, 1885, $100; Jan. 7, 1886, $200; Aug. 20, 1887, $300; Jan. 31, 1888, $1500?

280. Merchants' Rule.—When a note is settled within a year of the date at which it is given, merchants frequently use the following rule to determine the amount due when partial payments have been made.

Find the amount of the principal to the date of settlement; find the amount of each payment from the time it was made to the date of settlement; take the sum of the amounts of the payments, and subtract it from the amount of the principal; the remainder will be the balance due.

Since the periods of time are short in these notes, interest is usually calculated for the exact number of days. The method of exact interest is used in the following problems:

Ex. Find the amount due on Dec. 31, 1897, on a note for $300, interest at 7%, dated Feb. 15, 1897, on which the following payments have been made: Mar. 25, $75; June 1, $37.50; Oct. 10, $50.

SOLUTION.

Amount of $300	from Feb. 15	to Dec. 31	(319 da.) =		$318.35
" 75	" Mar. 25	"	(281 da.) =	$79.04	
" 37.50	" June 1	"	(213 da.) =	39.03	
" 50	" Oct. 10	"	(82 da.) =	50.79	
					168.86
			Balance due Dec. 31, '97,		$149.49

EXERCISE 137.

1. A note for $790 at 6%, dated Jan. 15, 1891, bears two indorsements: July 1, $400; Nov. 19, $60. What is due Dec. 31, 1891?

2. Find the amount due Dec. 15, 1898, on a 5% note for $2500, dated Mar. 12, 1898, and indorsed as follows: May 17, $800; Aug. 10, $350; Nov. 1, $90.

3. Find the amount due Nov. 30, 1894, on a 4% note of $1500, dated Feb. 6, 1894, and indorsed as follows: Apr. 25, $350; June 6, $240; Aug. 7, $120; and Nov. 1, $400.

BANK DISCOUNT.

281. Bank Discount.—The holder of a promissory note may need money immediately. If either the maker of the note or an indorser of it be of good financial standing, he can usually obtain money on the note by taking it to a bank and selling it ("putting it in bank"). He indorses the note by writing his name on the back, deposits the note in the bank, and in return the bank pays him the sum due at the maturity of the note, less the interest on this sum from the date at which the note is deposited at the bank to the date of maturity.

The sum paid out by the bank is called the *proceeds* or *avails.*

The sum deducted from the amount due at maturity is called the **bank discount.** Hence, the bank discount on a note is *the simple interest from the date at which the note is discounted to the date of maturity computed on the amount of the note at the date of maturity.*

282. Kinds of Notes Discounted.—(1) A merchant who sells goods on time frequently receives a note promising to pay for the goods at the end of one, two, or three months, or of a certain number of days. Such a note may or may not bear interest. He obtains money by indorsing it and depositing it in a bank. In this case the money is paid to the *indorser.*

Such notes are frequently discounted on a date later than the date on which the note was given, called the date of discount.

(2) A person desiring money may make out a note payable to a friend, sign it himself, and get his friend to indorse. He takes it to the bank himself and secures the money on it. In this case the money is paid to the *maker* of the note; the note

does not bear interest, and the note is discounted on the date on which it is made.

(3) A person desiring money may make out a note payable to a bank, sign it himself, take it to the bank, and obtain the money, depositing property of value, as stocks, bonds, etc., called "collateral," to secure the payment of the note. Such a note needs no indorsement.

Banks reckon time by months or days, according as one or the other is specified in the note, and compute discount by the use of tables, which usually count 360 days in a year, 30 days in a month.

In Pennsylvania, Delaware, Maryland, Mississippi, and the District of Columbia, discount is reckoned on the *day of discount*, as well as the day of payment. Thus, a 60 days note in Pennsylvania is discounted for 61 days; in Mississippi (where days of grace are allowed) for 64 days.

Ex. 1. What are the proceeds on a note for $500 for 60 days (with grace) discounted at a bank at 6%?

SOLUTION.

The note has 63 days to run.

$$\text{Interest on } \$1 \text{ for 63 da.} = \frac{63}{6 \times 1000} = \$.0105$$

Interest on $500 for 63 da. = $500 × .0105 = $5.25

$500 − $5.25 = $494.75, *Proceeds.*

Ex. 2. Find the proceeds of the following note:

$650. PHILADELPHIA, PA., *Jan. 6, '98.*

Ninety days after date, I promise to pay to the order of Anthony Fisher, Six Hundred and Fifty Dollars, value received, with interest.

Discounted at 5%, Jan. 26, '98. ROBERT ALLEN.

SOLUTION.

The note is an interest-bearing note, hence it is necessary first to find its amount at maturity. It is to be observed that the note is given in Pennsylvania, and that therefore there are no days of grace, and that discount is computed on the day of discount as well as the day of maturity.

Interest on $650 for 90 da. at 5%,	=	$8.13
Amount of note when due (Apr. 6),	=	658.13
From Jan. 26 to Apr. 6 (inclusive), = 71 da.		
Discoun. on $658.13 at 5% for 71 da.,	=	6.49

$658.13 − $6.49 = $651.64, *Proceeds.*

In the following examples it is important that the pupil, before working an example, observe carefully:

1. Whether the time to run is *days* or *months*;
2. Whether the note bears *interest* or not;
3. The *date* when the note is discounted;
4. The *State* in which the note is given, and therefore whether *days of grace* are allowed, and whether the *date of discount* is included in the time.

EXERCISE 138.

1. Find the proceeds of a note for $1200 on 90 days, discounted at 7%. At 8%.

2. Find the bank discount on a note for $870, due in 90 days, at 6%. Find same with 3 days of grace.

3. A note for $800, due in 60 days, was discounted at the bank at 7%. Find proceeds.

4. A 6-mo. note for $650, dated June 1, was discounted July 15, at 6%. Find discount and proceeds.

5. A 3-mo. note for $1400, dated July 15, was discounted Aug. 10, at $5\frac{1}{2}$%. Find the proceeds.

6. A 90-day note for $5000, *with interest* at 6%, dated May 4, was discounted June 8 at 8%. Find the proceeds.

In each of the following, determine the day of maturity, time to run, discount, and proceeds:

7. *$1600.* TRENTON, N. J., *June 1, 1895.*

Three months after date, I promise to pay George Williams, Sixteen Hundred Dollars, for value received.

Discounted at 6%, July 15. WASHINGTON NORRIS.

8. *$2000.* BALTIMORE, MD., *Feb. 20, 1897.*

Ninety days after date, I promise to pay Jacob Warren, Two Thousand Dollars, with interest at 6%.

Discounted at 7%, April 1. ANDREW FLEMING.

9. *$870.* PHILADELPHIA, *July 10, 1896.*

On the 10th of January next, I promise to pay John Wanamaker, Eight Hundred Seventy Dollars, with interest at 5%.

Discounted Oct. 1, at $6\frac{1}{2}$%. FREDERIC TOWNSEND.

10. A note for \$1200, bearing interest at $4\frac{1}{2}\%$, dated Oct. 25, 1896, due Feb. 28, 1897, and discounted Dec. 1, at 7%.

11. A 90-day note given Dec. 16, 1895, for \$1800, bearing interest at 3%, was discounted Dec. 30 at 6% (with 3 days of grace).

12. What is the difference between the true discount and the bank discount on \$500 for 3 yr. 6 mo. at 5%?

13. Find the proceeds of a note dated Oct. 15, 1896, for \$460.30, payable in 9 months, bearing interest at 6%, and discounted February 25, 1897, at 7%.

14. A note for \$2300, dated July 30, 1899, and payable in 90 days, with interest at 5%, was discounted September 23, at 7%. Find proceeds.

283. Proceeds to be a Certain Sum.—If a person wishes to obtain a certain sum, as \$200 on a 3 mos. note at 6%, from a bank, it will be necessary to determine the face of the note that will yield that sum.

Thus, on a note for \$1 the bank would pay in the above case \$1 − .015, or \$0.985. Hence, to obtain \$200, the face of the note must be as many dollars as \$0.985 is contained times in \$200, or \$203.05.

The student may verify this result by obtaining the proceeds of \$203.05 for 3 mo. at 6%.

Hence, in general,

Divide the given proceeds by the proceeds of \$1 for the given time and rate.

EXERCISE 139.

Find the face of the note which,

1. Is to run 60 days, and when discounted at 6% to realize \$891.

2. Is to run 6 months, and when discounted at 7% realizes \$3493.30.

3. When discounted at the bank for 4 mo. 9 da. at 8% will give \$72850.

4. I owe a man \$800 and wish to pay him by a 6-mo. note. What sum must the note demand so that when discounted at the bank at 8%, the debt is exactly paid?

5. Let the student verify all of these results by employing the principles of the few preceding exercises.

COMPOUND INTEREST.

284. Compound Interest.—If interest is not paid when it becomes due, under some circumstances interest on the unpaid interest, as well as on the principal, is computed for the next period of time, and so on. Such interest is called **compound interest.**

Ex. Compute the compound interest on \$400 for 3 years at 5%.

SOLUTION.

Principal,	=	\$400,	(1st principal)
Interest for the first year,	=	20	
Amount at end of first year,	=	420,	(2d principal)
		.05	
Interest for second year,	=	21.00	
		420	
Amount at end of 2d year,	=	441.00,	(3d principal)
		.05	
Interest for 3d year,	=	22.05	
		441	
Amount at end of 3d year,	=	463.05	
		400	
		\$63.05,	*Compound Interest.*

If the above example had called for the compound interest on \$400 for 3 yr. 2 mo. 15 da., instead of 3 years merely, we would find the compound amount for 3 years as above, or \$463.05, then find the simple interest on this sum for 2 mo. and 15 da., or \$4.82, and add this to the compound interest, giving \$67.87 as the compound interest for 3 yr. 2 mo. 15 da.

Savings banks usually compute interest on deposits in this way, adding the interest to the principal at the end of a stated period, as six months. Compound interest is not, however, allowed by law on ordinary debts.

When the annual rate of interest is 5%, for instance, and the interest is compounded semi-annually, the compound interest is obtained by computing the interest for twice as many periods of time as there are years, at half the annual rate, $2\frac{1}{2}$%.

Thus, to find the compound interest on $500 for 4 years at 7%, find the compound interest on $500 at $3\frac{1}{2}$% for 8 periods of time.

In computing compound interest, it is often useful to use tables giving the amount of $1 for various rates for different periods of time.

EXERCISE 140.

Find the compound interest of:

1. $250 for 4 yr. at 5%.

2. $1800 for 6 yr. at 4%. Also find amount.

3. $900 for 3 yr. at 6% compounded semi-annually.

4. $680 for 4 yr. at 5% compounded semi-annually.

5. $1600 for 5 yr. at $4\frac{1}{2}$%.

6. $600 for 3 yr. 4 mo. at 5%. Also find amount.

7. $3000 for 4 yr. 6 mo. at 4%.

8. $2500 for 5 yr. 3 mo. at 6%. Find amount too.

9. What is the difference between simple and compound interest on $1500 for 6 yr. at 5%?

10. What sum at 4% compound interest will amount to $1000 in 4 yr.? In 5 yr.? In 3 yr. 9 mo.?

ANNUAL INTEREST.

285. Annual interest is simple interest on the principal, together with simple interest on each unpaid installment of interest.

In some States, if a note or bond contains the words "with interest payable annually," simple interest can be collected on each unpaid year's interest from the date at which it becomes due to the date of settlement, *i. e.*, annual interest can be collected.

Ex. Find the annual interest on a note for $500 for 4 years at 6%.

SOLUTION.

Interest on $500 for 1 year at 6% = $30
" $500 " 4 " 6% = $120
" $30 " 3 + 2 + 1 (or 6) years = $10.80
$120 + $10.80 = $130.80, *Annual Interest.*

EXERCISE 141.

Find the annual interest and amount of:

1. $700 at 5% for 3 yr.

2. $1200 at 4% for 5 yr.

3. $90 at $3\frac{1}{2}$% for 4 yr.

4. $125 at 6% for 3 yr. 6 mo.

5. $750 at 4% for 4 yr. 4 mo.

6. Find the difference between annual interest and compound interest of $360 at 5% for 5 yr.

7. Find the simple, the exact, the compound, and the annual interest on $4800 at 5% for 6 yr. 6 mo.

EXCHANGE.

286. Exchange is a system of business whereby payments are made at a distance by means of drafts or bills of exchange, which largely cancel each other, and hence call for little actual transmission of money.

Illustration.—James Smith, of Chattanooga, owes Daniel Compton, of New York City, $250. To send the money in the mail would involve risk of loss; to send it by express would be expensive. A direct check on a Chattanooga bank might be expensive for Compton to collect. Hence, Smith goes to a bank in Chattanooga (the Farmers'), which has money on deposit in a New York bank (the Chemical) and buys a draft, or order, in which the Farmers' Bank directs the Chemical Bank to pay the required sum to Daniel Compton, thus:

FARMERS' BANK,
$250. CHATTANOOGA, TENN., *Jan. 25, 1900.*

At sight, pay to the order of Daniel Compton, two hundred and fifty dollars, value received.

THOMAS FORSYTH,
Cashier.

To the CHEMICAL NATIONAL BANK,
NEW YORK, N. Y.

287. A **draft** is a written order in which one party directs another to pay a specified sum to a third party.

The **maker** or **drawer** is the person who signs the draft.

The **drawee** is the person directed to pay the sum.

The **payee** is the person to whom the money is to be paid.

In the above draft, the maker is Thomas Forsyth; the drawee is the Chemical National Bank; and the payee is Daniel Compton.

288. Par, Premium, Discount.—If, in buying a draft, a person has to pay the exact face of the draft, exchange is said to be **at par**; if more than the face, exchange is said to be **at a premium**; if less than the face, exchange is said to be **at a discount.**

Premium and **discount** of exchange (apart from paying for the labor involved in the exchange) arise from the fact that the banks of one large money-center, as San Francisco, may owe the banks of another large center, as Chicago, a considerable sum, as $1,000,000. Hence, the San Francisco banks must either be at a considerable expense in sending this money to Chicago, or must pay interest on it. In this case, a person in San Francisco, buying a draft on Chicago, would have to pay a considerable premium, since his draft would increase the balance at Chicago against San Francisco. On the other hand, a person at Chicago, buying a draft on San Francisco for a large amount, might get the draft at a discount, since it would diminish the balance against San Francisco and the cost of transmitting the same.

289. Sight and Time Drafts.—A sight draft is one which is to be paid immediately on presentation.

A **time draft** is one payable after a specified time.

A time draft is to be presented immediately to the drawee, and, if he agrees to pay it, he writes the word "accepted" across the face, with the date and his signature. This is called an acceptance, is equivalent to a promise to pay, and, in effect, makes the draft a promissory note.

It is evident that, since a time draft is not payable until its maturity, its value, apart from the cost of exchange, is the face value, less the bank discount on it up to the time of maturity.

Ex. 1. Find the cost of a draft on New York for $500 at ½% premium.

SOLUTION.

Premium = ½% of $500 = $500 × .005 =	$2.50
Face of draft	= $500
Cost of draft	= $502.50, ***Result.***

Ex. 2. What must be paid in Boston for a draft on St. Louis, at 30 days, for $1200, exchange being at ¼% premium?

SOLUTION.

Discount on $1200 for 33 days	= $6.60
Proceeds of $1200 = $1200 − $6.60	= 1193.40
Premium on $1200 at ¼%	= 3.00
Cost of draft = $1193.40 + 3.00	= $1196.40, ***Result.***

EXERCISE 142.

Find the cost of a draft for:

1. $900 at ¼% premium.

2. $2500 at ⅛% discount.

3. $7600 at ¼% premium.

4. $100000 at ⅛% discount.

*5.** $570, payable in 30 days, exchange being at ¼% premium, and interest at 6%.

6. $3000, payable in 60 days, exchange being at ⅛% discount, and interest at 5%.

7. $2400, given by a Boston merchant to a Chicago manufacturer, payable in 60 days, exchange at ¾% premium.

8. What is the difference between a check and a draft? Between a negotiable note and a draft?

* Allow no days of grace in solution of these examples.

9. How large a sight draft can be bought for \$2500, exchange being at $\frac{3}{8}\%$ premium?

10. What will be the cost of a draft for \$1680, payable in 60 days after sight, exchange being $\frac{1}{2}\%$ premium, and interest being at 6%?

11. Find face of a draft on New York, at 90 days' sight, bought for \$450, exchange being $1\frac{1}{2}\%$ premium, and interest at 5%.

290. Foreign exchange is exchange between different countries.

Exchange between two places in the same country is called **domestic exchange.**

In foreign exchange, a draft is usually called a **bill of exchange.** Usually three bills, forming a **set of exchange,** are drawn. To prevent loss or delay, each is sent by a different route. Each specifies that when it is paid the other two of the set become void.

Foreign exchange is based on the par of exchange, or the legal value of the currency of one country in terms of that of another. For instance, the par value of a pound sterling is \$4.8665.

Ex. What is the cost in Philadelphia of a bill of exchange on London for £250, when exchange is at \$4.875 for the pound sterling?

OPERATION.

$$\begin{array}{r} \$4.875 \\ 250 \\ \hline 243750 \\ 9750 \\ \hline \$1218.75, \textit{ Cost.} \end{array}$$

EXERCISE 143.

1. What is the cost of a bill of exchange on London for £2250 at \$4.87 a pound sterling?

2. What is the cost in New York of a bill of exchange on London for £6300 at \$4.865 to a pound sterling?

3. What is the face of a bill of exchange on London that was purchased for \$13406.25, exchange being quoted at \$4.875 to a pound sterling?

4. When exchange is quoted at $5.18\frac{3}{8}$ francs to \$1, what will be the face of a bill of exchange on Paris that is bought for \$2300?

5. When exchange on London is quoted at 4.85, what will be the face of a draft that can be bought for \$7779.40?

6. What will a bill of exchange on Liverpool for £135 15s. 6d. cost, exchange being at 4.86?

7. Find the cost of a bill of exchange on Antwerp for 14176.75 francs, exchange at $5.17\frac{1}{4}$.

8. Find cost of bill of exchange on Berlin for 7648 marks, exchange at 23¢.

9. How much must be paid in Boston for a bill of exchange on Hamburg for 1330 marks, exchange at $23\frac{3}{4}$¢?

10. How large a bill of exchange can be bought on Paris for \$8000, exchange being at 5.21?

11. How large a bill of exchange can be bought on Berlin for \$8000, exchange at $24\frac{1}{8}$¢?

EQUATION OF PAYMENTS.

291. Example.—William Smith owes Stephen Day the following sums:

\$500	payable	in	4 mo.
300	"	"	8 mo.
400	"	"	9 mo.

It will be a useful exercise for the pupil to determine the date when in equity the entire debt of \$1200 can be paid as a single payment.

EXERCISE 144.

1. A man owes three accounts to the same person, \$750 due in 8 mos., \$560 due in 5 mos., and \$600 due in 6 mos. When can the entire amount be paid in one sum?

2. On Jan. 1st, X. gives Y. four notes as follows: 1st for \$700 due in 9 mos., 2d for \$850 due in 6 mos., 3d for \$400 due in 4 mos., and 4th for \$600 due in 7 mos. On what date will a single payment equitably cancel all notes?

3. What is the average date for paying three notes, due, 1st, March 20, \$500; 2d, April 25, \$600; 3d, June 3, \$400?

4. Four notes for \$750 each are due respectively Aug. 1, Oct. 8, Nov. 20, and Dec. 5. What is the average date of maturity?

5. Find the equated time of paying \$430 due in 8 mos.; \$350 due in 9 mos.; \$1000 due in 6 mos.

6. Find the equated time of paying following bills: \$60 due in 30 days; \$100 due in 60 days; \$360 due in 90 days; and \$250 due in 30 days. They all bear date Oct. 17.

7. Of a debt, $\frac{1}{3}$ is due in 7 mos., $\frac{1}{4}$ in 6 mos., and the rest in a year. Find the equated time that one payment ought to pay it all.

8. Three bills are due as follows: Aug. 5, \$365, Oct. 10, \$470, Dec. 14, \$930. Find the average time of payment.

9. Three notes are due as follows: 1st for \$320, June 1st; 2d for \$480, Aug. 20; 3d for \$520, Oct. 30. I wish to substitute one note for \$1320. What should be its day of maturity?

10. A man bought a house for \$6400 on 8 months' credit. He paid \$2000 at time of purchase; when should the balance be due?

11. A man owes \$500 due in 8 mos., \$900 due in 6 mos., and \$1200 due in a year. After 5 months he pays \$1000. When in equity should the remainder be due?

12. A man owes \$12000 due in 9 months. If he pays \$6800 in 5 months and \$2700 in 2 months more, when ought the balance be paid?

13. A certain debt is to be paid $\frac{1}{4}$ down, $\frac{1}{3}$ in 8 months, and $\frac{1}{4}$ in 9 months, and the balance in a year. If the payments are all made in one, when is it equitably due?

CHAPTER XVII.

RATIO AND PROPORTION. PARTNERSHIP.

RATIO.

292. Ratio.—If the quotient of one number divided by another occurs in a problem, it is often of advantage not to perform the division immediately, but to indicate the division for the time being. Thus, in Ex. 2 of Art. 95, the quotient of 200 divided by 9 being indicated for the time being, it was not found necessary to perform the division at all, since 9 ultimately was canceled by a factor of the multiplier, 54.

A ratio is the indicated quotient of one number divided by another number of the same kind.

293. The **terms** of a ratio are the quantities whose quotient is indicated. The first of these (the indicated dividend) is called the **antecedent**; the second (the indicated divisor) is called the **consequent.**

Thus, in the ratio 12 to 8, 12 is the antecedent, 8 the consequent.

294. Symbols.—A ratio is usually indicated by the sign, :, between the numbers compared. This sign is probably an abbreviation of ÷, the sign of division.

Thus, the ratio of 12 to 8 is denoted by 12 : 8; it may also be indicated in the fractional form, $\frac{12}{8}$.

295. A compound ratio is the product of two simple ratios.

Thus, $\frac{2}{3} \times \frac{5}{7}$, or $\frac{2 \times 5}{3 \times 7}$, is a compound ratio.

It may also be expressed thus $\begin{cases} 2:3. \\ 5:7. \end{cases}$

2. On Jan. 1st, X. gives Y. four notes as follows: 1st for \$700 due in 9 mos., 2d for \$850 due in 6 mos., 3d for \$400 due in 4 mos., and 4th for \$600 due in 7 mos. On what date will a single payment equitably cancel all notes?

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Thus, $\frac{2}{3} \times \frac{5}{7}$, or $\frac{2 \times 5}{3 \times 7}$, is a compound ratio.

It may also be expressed thus $\left\{\begin{matrix} 2:3. \\ 5:7. \end{matrix}\right.$

296. Properties of Ratios.—From the nature of quotients and fractions it is evident that—

(1) *if both antecedent and consequent of a ratio be multiplied or divided by the same number, the ratio is not changed in value;*

(2) *the antecedent equals the product of the ratio by the consequent;*

(3) *the consequent equals the antecedent divided by the ratio.*

EXERCISE 145.

What is the ratio of:

1. 12 to 42?	*4.* $11\frac{2}{3}$ to $12\frac{5}{6}$?	7. 5.2 to 7.28?
2. 81 : 135?	*5.* $3\frac{4}{7} : 8\frac{13}{14}$?	*8.* 12.75 : 16.15?
3. $9\frac{3}{4}$ to $14\frac{2}{3}$?	*6.* $5\frac{4}{9} : 6\frac{20}{27}$?	*9.* 3.422 : 3.76?

Find the value of the parenthesis in each:

10. 12 : 30 = (?).	*13.* 28 to (?) = $\frac{4}{11}$.	*16.* $7\frac{3}{5}$: (?) = $\frac{4}{9}$.
11. 18 : (?) = $\frac{2}{5}$.	*14.* (?) to 81 = $\frac{4}{9}$.	*17.* (?) : $10\frac{1}{12}$ = $\frac{12}{11}$.
12. (?) : 18 = $\frac{1}{6}$.	*15.* 42 : (?) = $\frac{3}{7}$.	*18.* 4.45 : (?) = $\frac{9}{13}$.

19. If the consequent is 45 and the antecedent is 35, find the ratio.

20. If the antecedent is $14\frac{2}{5}$ and the ratio is $\frac{15}{23}$, find the consequent.

PROPORTION.

297. A **proportion** is an equality between ratios.

By the use of proportion a problem is often solved more readily than by analysis (see Arts. 94, 95), but sometimes the reason for the steps used is not so evident.

A proportion is indicated by placing the symbol, =, or, : :, between the two equal ratios.

Thus, 2 : 3 = 8 : 12, is a proportion.

It may be read in several ways, as "2 is to 3 as 8 is to 12;" or, "2 over 3 equals 8 over 12," or, "the ratio of 2 to 3 equals the ratio of 8 to 12," etc.

298. Terms.—Hence, a proportion contains four terms. The first and last terms are called the **extremes**; the second and third terms are called the **means**.

If the two means are alike, each is called a **mean proportional**, and the last term is called a **third proportional**.

Thus, in 2 : 4 = 4 : 8, 4 is a mean proportional, and 8 is a third proportional.

299. Properties of a Proportion.—The fundamental property of a proportion is that "*the product of the means is equal to the product of the extremes.*"

Thus, in the proportion in Art. 297, 3 × 8 = 2 × 12.

This property is seen to be true for any proportion, since, if $\frac{a}{b}$ and $\frac{c}{d}$ are two equal ratios, and each be multiplied by $b\,d$, we have $a \times d$ and $b \times c$ equal.

It follows at once from the above property that *either extreme equals the product of the means divided by the other extreme.*

What does either mean equal?

To obtain a method for solving problems by proportion, let us consider the following problem.

Ex. 1. If 12 books cost $20, what will 15 books cost?

Solving by analysis, we have

Cost of 1 book = $\$\frac{20}{12}$.

Cost of 15 books = $\$\frac{20}{12} \times 15$.

To show how this relation may be converted into a proportion we divide each of these equals by $20, and obtain

$$\frac{\text{cost of 15 books}}{\$20 \text{ (or cost of 12 books)}} = \frac{15}{12}.$$

Taking the last ratio first, and writing the denominator first

12 books : 15 books = $20 (cost of 12 books) : cost of 15 books

or, 12 : 15 = $20 : ( ).

Hence, in general, *write the required quantity as the last term of the proportion; use the quantity with which it is compared as the third term; write the ratio which is equal to the ratio of the 3d and 4th terms as the first two terms; solve by using the properties of a proportion.*

Ex. 2. If 20 acres of land produce 320 bushels of wheat, how many acres are needed to produce 400 bushels?

SOLUTION.

20 acres is compared with the unknown number of acres. Hence, we have

$$320 : 400 = 20 : (\quad).$$

$$\therefore \text{No. acres required} = \frac{\overset{25}{\cancel{400}} \times \cancel{20}}{\underset{\cancel{16}}{\cancel{320}}} = 25.$$

EXERCISE 146.

Supply the missing term in each proportion :

1. 3 : 8 : : 6 : (?).	*6.* $3\frac{1}{2} : 4\frac{2}{3} :: 9\frac{3}{7}$: (?).
2. 12 : 5 : : (?) : 40.	*7.* $41\frac{1}{3}$: (?) : : $196\frac{1}{3} : 232\frac{1}{2}$.
3. 8 : (?) : : 5 : 30.	*8.* $12\frac{4}{11} : 10\frac{1}{5}$: : (?) : $9\frac{5}{8}$.
4. 16 : 24 : : (?) : 15.	*9.* (?) : 9.75 : : 13.25 : 10.4.
5. (?) : 11 : : 14 : 33.	*10.* 2.76 : 3.45 : : 2.28 : (?).

11. If 75 acres of land cost \$5090.40, what will 175 acres cost? 411 acres?

12. If \$5890.50 buys 85 acres, how many acres will \$34650 buy? \$25410?

13. If 7 horses require a pasture of 18 acres, how large a pasture will 300 horses require?

14. If a pole 12 feet high casts a shadow $5\frac{1}{4}$ feet long, how long will be the shadow from a steeple 144 feet high at the same time?

15. If the shadow from a chimney $46\frac{1}{2}$ feet high is 38 ft. 9 in., what is the height of the tree whose shadow is 111 ft. 3 in.?

16. If $3\frac{3}{8}$ bu. of grain are used to sow $13\frac{3}{4}$ acres, how many bushels will be required to sow 100 acres?

17. If \$75 yields \$35 interest, how much must be invested to yield \$63 interest?

18. If I pay \$75.65 for the use of \$425, what should be paid for the use of \$545 for same time?

19. If 12 men accomplish a certain task in 30 days, how many days will 45 men require?

NOTE.—The pupil should observe that 45 men will not require as long as 12 men do. Hence, we invert the unknown ratio in the proportion. For this reason such proportions are called *inverse proportions*. The solution is arranged thus: 12 men : 45 men : : (? days) : 30 days.

20. If 18 men dig a ditch in 35 days, how long would it take 21 men to do the same? 14 men?

21. If 10 men lay a wall in 48 days, how many men will be needed to lay a similar wall in 30 days?

22. A man borrows \$72 for 5 years, and lends \$240 in return. How long ought he lend the latter sum to pay for the former loan?

23. If \$4500 is borrowed for a certain time at 5%, what sum must be loaned at $4\frac{1}{2}$% the same time, to compensate?

24. If a man can do in 8 days as much work as his son does in 15 days, and the son's wages are \$1.60 a day, what pay should the father receive per day?

25. Find a fourth proportional to 7, 17, and 21.

26. Find the number which has to $6\frac{3}{5}$ the same ratio which $11\frac{2}{3}$ has to $3\frac{1}{7}$.

27. Find a third proportional to $3\frac{1}{2}$ and $4\frac{2}{3}$.

28. Find the fourth proportional to 3.81, .056, and 1.67.

29. By a certain pipe a certain cistern can be emptied in $5\frac{3}{8}$ hours. In what time can another cistern $3\frac{1}{3}$ times as large be emptied by a pipe carrying only $\frac{3}{4}$ as much water?

COMPOUND PROPORTION.

300. A **compound proportion** is an equality between a simple ratio and a compound ratio, or between two compound ratios.

$$\text{Exs.} \quad \left\{\begin{matrix} 3 : 6 \\ 10 : 30 \end{matrix}\right\} = 2 : 12, \text{ or } \left\{\begin{matrix} 2 : 6 \\ 14 : 28 \end{matrix}\right\} = \left\{\begin{matrix} 1 : 2 \\ 2 : 6 \end{matrix}\right\}.$$

Ex. 1. If 12 men can earn \$180 in 5 days, how much can 16 men earn in 9 days?

SOLUTION.

\$180 is compared with the required number of dollars. The ratio which is equal to the ratio, \$180 : required No. \$, is 12 × 5 days' work : 16 × 9 days' work. Hence,

$$\left\{\begin{matrix}12:16\\5:9\end{matrix}\right\} = \$180 : (\quad)$$

$$\text{or } (\quad) = \frac{180 \times 16 \times 9}{12 \times 5} = \$432, \textit{ Result.}$$

Sometimes the terms of ratio vary *inversely*, and must be used accordingly (for instance, the number of days required to do a given piece of work varies inversely as the number of workmen, that is, the *greater* the number of workmen the *fewer* the days).

Ex. 2. If 15 men can dig a ditch 180 rods long in 8 days, how many days will it take 20 men to dig a ditch 300 rods long?

SOLUTION.

The final ratio is, 8 days: required No. days.

The longer the ditch, the greater the number of days required, hence, 180 : 300 is a part of the first ratio equal to the above ratio; but the greater the number of men the fewer the number of days, ∴ 20 : 15 is the other part of the first ratio. Hence,

$$\left\{\begin{matrix}20:15\\180:300\end{matrix}\right\} = 8 : (\quad).$$

$$\text{Hence, } (\quad) = \frac{8 \times 15 \times 300}{20 \times 180} = 10, \textit{ No. of days.}$$

If the first set of men had worked 8 hours a day, and the second set 12 hours a day, how would this have affected the solution?

EXERCISE 147.

1. If 15 men can earn \$360 in 8 days, how much can 7 men earn in 40 days?

2. Five clerks use 50 quires of paper in 16 days. At the same rate, how much paper will 9 clerks use in 15 days?

3. If 8 persons spend \$470 in 5 days, how much will 15 persons spend in 16 days at same rate?

4. If a block of stone 2 ft. × 3 ft. × 4 ft. weigh 1740 lbs., what will a block of like stone 3 × 5 × 7 ft. weigh?

5. If 7 men working 8 hours a day can accomplish a task in 15 days, how many days of 6 hours will 10 men require for the same task?

6. If a cistern $17\frac{1}{2}$ ft. long, $10\frac{1}{2}$ ft. wide, and 13 ft. deep, hold 546 bbl., how many barrels will a cistern hold that is 16 ft. long, 7 ft. wide, and 15 ft. deep?

7. If 22 men can cut 294 cords of wood in 7 days when they work 14 hours a day, how many days will it take 5 men to cut 375 cords, working 10 hours a day?

8. If 25 men dig a ditch 396 feet long in 36 days of 7 hours each, in how many days will 30 men dig a similar ditch 990 feet long, if they work 9 hours a day?

9. If 90 men build a wall 2304 ft. long, 8 ft. wide, and $2\frac{1}{2}$ ft. high in 45 days of $7\frac{1}{2}$ hrs. each, how long a wall 7 ft. wide and 4 ft. high can 125 men build in 35 days of 9 hrs. each?

10. If a slab of marble 9 ft. long, 3 ft. wide, and 4 in. thick weighs 1200 lbs., how much will another similar slab weigh which is 6 ft. long, 2 ft. wide, and 3 in. thick?

11. A certain bin 7 ft. × $2\frac{1}{2}$ ft. and 2 ft. deep contains 28 bushels of grain; what is the depth of a second bin 18 ft. × 1 ft. $10\frac{1}{2}$ in. which contains 120 bu.?

12. If 496 men, in 5 da. of 12 hr. 6 min. each, dig a trench of 5 degrees of hardness, 465 ft. long, 3 ft. 8 in. wide, and 4 ft. 8 in. deep, how many men will be required to dig a trench of 8 degrees of hardness, $168\frac{3}{4}$ ft. long, 7 ft. 6 in. wide, and $2\frac{4}{5}$ ft. deep, in 22 da. of 9 hr. each?

PROPORTIONAL PARTS.

301. Proportional Parts.—It may be required to divide a given number into parts which shall be proportional to a series of given numbers. We may do this either by the use of proportion, or by the use of fractions and fractional units.

Ex. Three men working a mine agree to divide the profits in the proportion of 2, 3, and 4. They make \$2700. What is the share of each?

SOLUTION.

Since $2 + 3 + 4 = 9$, we may regard the profits as forming 9 shares, of which the miners get 2, 3, and 4 shares respectively.

$9 : 2 = \$2700 :$ share of 1st, \$600.
$9 : 3 = \$2700 :$ " 2d, \$900.
$9 : 4 = \$2700 :$ " 3d, \$1200.

Or, since \$2700 is to be divided into 9 shares,

1 share $= \frac{1}{9}$ of \$2700 = \$300.
2 shares = \$600.
Etc.

EXERCISE 148.

1. Divide 20000 into four parts proportional to 5, 7, 8, 12.

2. Divide 6300 into four parts proportional to 3, 5, 11, 17.

3. Divide 31800 into parts in the relation of $1 : 2 : 3$.

4. Divide 3864 into parts in the relation of $2 : 3 : 4 : 5$.

5. The weights of three casks aggregate a ton, but their individual weights are as $11 : 13 : 16$. Find the weight of each.

6. A father divided his property of \$46500 among three sons, in parts proportional to their ages, 17 yr., 20 yr., and 25 yr. How much did each receive?

7. Divide 210 into three parts, which shall be proportional to $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$.

PARTNERSHIP.

302. Partnership.—When two or more persons can carry on a business to better advantage together than singly, they often unite and form a partnership.

Partnership is the combination of two or more persons as a single firm to carry on business.

The **capital** is the money invested by the different partners in the business.

303. In **simple partnership** all the partners invest their capital for the same length of time.

In **compound partnership** the partners invest their capital for different lengths of time.

Ex. 1. Three men, A, B, and C, are in business together, and gain \$3600. A's capital is \$2000; B's is \$4000; and C's is \$6000. What is each one's share of the profits?

SOLUTION.

Entire capital = \$2000 + \$4000 + \$6000 = \$12000.

A's share of the capital $= \frac{2000}{12000} = \frac{1}{6}$.
B's " " " $= \frac{4000}{12000} = \frac{1}{3}$.
C's " " " $= \frac{6000}{12000} = \frac{1}{2}$.

Hence, A's share of the gain $= \frac{1}{6}$ of \$3600 = \$600.
B's " " " $= \frac{1}{3}$ of \$3600 = \$1200.
C's " " " $= \frac{1}{2}$ of \$3600 = \$1800.

Ex. 2. A, B, and C form a partnership. A puts in \$500 for 8 months, B \$600 for 9 months, and C \$800 for 12 months. They gain \$3040. What share of this belongs to each partner?

SOLUTION.

\$500 invested for 8 mo. is the same as \$4000 for 1 mo.
\$600 " " 9 " " " " " \$5400 " 1 "
\$800 " " 12 " " " " " \$9600 " 1 "
Entire capital = \$19000 " 1 "

Hence, A's share of the gain $= \frac{4000}{19000}$ or $\frac{4}{19}$ of \$3040 = \$640.
B's " " " $= \frac{5400}{19000}$ or $\frac{27}{95}$ " " = \$864.
C's " " " $= \frac{9600}{19000}$ or $\frac{48}{95}$ " " = \$1536.

EXERCISE 149.

1. Three men contract for a pasture, into which A puts 7 horses, B puts 9, and C 13. Of the rent, which was \$17.40, how much ought each pay?

2. Into a partnership A places \$3200, B \$4100, and C \$1700. How should a gain of \$4500 be divided?

3. Three men gain \$500 after investing \$560, \$640, and \$800 respectively. How do they share the profit?

4. By a will a man leaves to his widow \$12000, to a son

$9000, and to a daughter $4000. But upon investigation the estate produced only $20000. How should it be divided equitably?

5. A firm lost in a year $3300. A's stock was $3200, B's was $7100, and C's was $6200. How is the loss to be distributed?

6. A, B, and C go into business with a capital of $12000. From the gain of one year A's share is $1250, B's is $1000, and C's is $750. What was each man's capital?

7. Three persons enter partnership. A puts into it $1600 for 3 months; B $800 for 5 months; and C $900 for 3 months. How should they justly share the profits of $575? The losses of $1035?

8. A pasture is rented by 3 persons for $760. A puts in 7 cows for 5 mos.; B 8 cows for 3 mos.; and C 9 cows for 4 mos. What rent should each pay?

9. Three laborers contracted to dig a trench for $49.50. The first worked 8 days of 7 hours each; the second 10 days of 8 hours each; and the third 14 days of 6 hours each. What should each receive?

10. A entered business with $5000, and in 3 mos. took in B with $4000. After 2 mos. more C entered the firm with $12000. At the end of the year they had gained $8100. How should it be divided equitably?

CHAPTER XVIII.

INVOLUTION AND EVOLUTION.

INVOLUTION.

304. Definitions.—The second power, or **square**, of a number is the number obtained by multiplying a given number by itself.

Thus, $23^2 = 23 \times 23 = 529$.

The **third power**, or **cube**, of a number is the number obtained by using the given number as a factor three times.

Thus, $8^3 = 8 \times 8 \times 8 = 512$.

Let the pupil define **fourth power**, **fifth power**, etc., of a number and give examples.

Involution is the process of computing any required power of a given number.

305. Memorizing Powers of Small Numbers.—It is important that the pupil calculate and commit to memory the following powers:

Squares of 1, 2, 3, 4 . . . . to 25.
Cubes of 1, 2, 3, 4 . . . . to 12.
Fourth powers of 1, 2, 3, 4, 5, 6.
Fifth powers of 1, 2, 3, 4, 5.
Sixth powers of 1, 2, 3, 4, 5.
Seventh powers of 1, 2, 3.
Eighth, ninth, and **tenth powers** of 1, 2.

306. Methods of Involution.—The powers of numbers may be obtained either by (1) **actual multiplication**, or (2) by the **use of tables**, or (3) by use of **logarithms** (see Art. 86).

$9000, and to a daughter $4000. But upon investigation the estate produced only $20000. How should it be divided equitably?

5. A firm lost in a year $3300. A's stock was $3200, B's was $7100, and C's was $6200. How is the loss to be distributed?

6. A, B, and C go into business with a capital of $12000. From the gain of one year A's share is $1250, B's is $1000, and C's is $750. What was each man's capital?

7. Three persons enter partnership. A puts into it $1600 for 3 months; B $800 for 5 months; and C $900 for 3 months. How should they justly share the profits of $575? The losses of $1035?

8. A pasture is rented by 3 persons for $760. A puts in 7 cows for 5 mos.; B 8 cows for 3 mos.; and C 9 cows for 4 mos. What rent should each pay?

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10. A entered business with $5000, and in 3 mos. took in B with $4000. After 2 mos. more C entered the firm with $12000. At the end of the year they had gained $8100. How should it be divided equitably?

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Squares of 1, 2, 3, 4 . . . . to 25.
Cubes of 1, 2, 3, 4 . . . . to 12.
Fourth powers of 1, 2, 3, 4, 5, 6.
Fifth powers of 1, 2, 3, 4, 5.
Sixth powers of 1, 2, 3, 4, 5.
Seventh powers of 1, 2, 3.
Eighth, ninth, and **tenth powers** of 1, 2.

306. Methods of Involution.—The powers of numbers may be obtained either by (1) **actual multiplication,** or (2) by the **use of tables,** or (3) by use of **logarithms** (see Art. 86).

It is also useful to be able to separate a number into parts (as tens + units), and form the product by multiplication of these parts. By this means properties of a power are discovered, which can be used in the inverse process of finding the root of a number. See Arts. 309 and 314.

EXERCISE 150.

ORAL.

1. State rapidly the squares of all the numbers up to 20.
2. What is the square of 18? 14? 17? 21? 19? 15? etc.
3. What is the square of $\frac{1}{2}$? $\frac{2}{3}$? $\frac{3}{4}$? $\frac{2}{7}$? $\frac{5}{8}$? $\frac{14}{23}$? $\frac{17}{15}$?
4. What is the square of .2? .3? .5? .8? .13? .18? 1.6?
5. What is the square of 1.8? 2.1? 2.5? 30? 50? 70? 600?
6. What is the cube of 9? 8? 3? 7? 6? 5? 11?
7. What is the cube of $\frac{2}{5}$? $\frac{5}{3}$? $\frac{4}{9}$? $\frac{6}{7}$? $\frac{8}{11}$? $2\frac{1}{2}$? $3\frac{1}{4}$?
8. What is the cube of .2? .3? .4? 20? 30? 1.2?
9. Tell the value of 8^3, 12^3, 2^4, 3^4, 6^3, 3^5, 2^7, 5^6, 4^3, 4^4, 5^4, 2^8, 3^6, 6^4, 5^5, $(1\frac{2}{3})^3$, $(5\frac{2}{3})^2$, $(2.2)^2$, $(.07)^3$, $(.012)^3$.

EXERCISE 151.

Find the value of:

1. 26^2.	6. 1.75^2.	11. $(1\frac{1}{3})^3$.	16. 15^4.
2. 28^3.	7. 23.1^2.	12. $(3\frac{1}{2})^2$.	17. 23^4.
3. 113^2.	8. 31.4^2.	13. $(5\frac{2}{3})^3$.	18. $(12\frac{1}{2})^4$.
4. 512^2.	9. $.038^3$.	14. $(4\frac{1}{2})^3$.	19. $(3.1\frac{1}{3})^3$.
5. 205^3.	10. $.66^3$.	15. $(7\frac{1}{2})^3$.	20. $(1.02\frac{1}{2})^3$.

21. $3^3 \times 4^2$.	24. $2^5 \times 3^4 \div 6^3$.	27. $15^3 \times 45^2$.
22. $4^3 \times 2^4$.	25. $6^3 \times 7^2 \div 14^3$.	28. $45^3 \div 15^3$.
23. $(\frac{1}{2})^2 \times 5^3$.	26. $(2.5)^2 \times (3.5)^2$.	29. $7.2^2 \times 7.5^2$.

EVOLUTION.

307. Definitions.—The **square root** of a number is that number which, multiplied by itself, will produce the given number. Thus, 13 is the square root of 169, since $13 \times 13 = 169$.

The **cube root** of a number is that number which, used as a factor three times, will produce the given number. Thus, 8 is the cube root of 512, since $8 \times 8 \times 8 = 512$.

Let the pupil define **fourth root, cube root,** etc.

The student should commit to memory the roots corresponding to the powers mentioned in Art. 305.

Evolution is the process of determining any required root of a given number.

308. The **methods of determining the roots** of numbers are (1) the use of **tables**, when a number has an exact root, or (2) the use of **logarithms**, or (3) the **direct methods** given in the remainder of this chapter, which are independent of tables and logarithms.

As stated in Art. 306, these methods are based on observing how the power of a number is formed when the number is dissected into parts (units and tens) and the product formed by the use of these parts.

SQUARE ROOT.

309. Squaring a Number by Parts.—Since, for example, $47 = 40 + 7$, the square of 47 may be formed thus,

$$\begin{array}{l} 40 + 7 \\ \underline{40 + 7} \\ 40^2 + 40 \times 7 \\ \underline{\quad\quad + 40 \times 7 + 7^2} \\ 40^2 + 2 \times 40 \times 7 + 7^2 = 1600 + 560 + 49 = 2209. \end{array}$$

Hence, if any number be separated into a number of tens + a number of units, its square will equal (*the square of the tens*) + (*twice the tens × the units*) + (*square of the units*), or, denoting the tens by t, and the units by u, $(t + u)^2 = t^2 + 2\,tu + u^2$.

NOTE.—This method of squaring may also be applied to numbers containing three or more figures, and the observed properties employed in extracting the roots of correspondingly large powers.

Thus, $346 = 300 + 46$,

$$346^2 = 300^2 + 2 \times 300 \times 46 + 46^2.$$

Having found the first and second figures (3 and 4) of a square root by the use of the square in this form, we may then proceed to find the third figure of the root by the use of the square, as if it were in the form,

$$346^2 = (340 + 6)^2$$
$$= 340^2 + 2 \times 340 \times 6 + 6^2.$$

310. Periods.—Since in any given number, as 2209, whose square root is to be extracted, the square of the tens (1600) is not given explicitly, it must be determined indirectly, and its root then extracted. This is done by marking off the figures of the number whose root is to be extracted, into periods of two figures each, beginning at the decimal point, and then determining the largest square number represented in the first period of figures to the left.

For the *square of a number contains twice as many figures as the number itself, or twice as many less one.*

For since

$$1^2 = 1$$
$$10^2 = 100$$
$$100^2 = 10000$$
$$1000^2 = 1000000$$
etc.

it follows that if a number contains one figure, its square is either 1, or lies between 1 and 100, and hence contains one or two figures; if a number contains two digits, its square is either 100, or lies between 100 and 10000, and hence contains three or four digits; similarly, if a number contains three digits, its square contains five or six digits, etc.

Hence, if any number be separated into periods of two figures each, beginning at the decimal point, the *number of periods thus formed will be the same as the number of figures in the square root*, and the square root of the largest square number represented in the left-hand period gives the first figure of the root.

311. Extraction of Square Root.—Ex. 1. Extract the square root of 2209.

OPERATION.

$$\begin{array}{lr|l} & \overset{\frown}{22}\overset{\frown}{09}. & 40 + 7 \\ \textit{Tens squared } (t^2) = 40^2 = & 1600 & \\ 2 \times \textit{tens} = 2 \times 40 = 80 & 609 & \\ (2 \times \textit{tens} + \textit{units}) \times \textit{units} = 87 \times 7 & 609 & \end{array}$$

ABBREVIATED FORM.

$$\begin{array}{r|l} & \overset{\frown}{22}\overset{\frown}{09} \,|\, 47 \\ & 16 \\ 87 & 609 \\ & 609 \end{array}$$

EXPLANATION.

Since 2209 contains two periods of two figures each, the root must contain two figures, a tens figure and a units figure. Since the largest square in 2200 is 1600, and the square root of 1600 is 40, the number of tens is 4. Subtracting the square of the tens, 1600, from 2209, the remainder, 609, must be $2 \times$ *tens* $\times$ *units* + (*units*)^2. Since (*units*)^2 is much less than $2 \times$ *tens* $\times$ *units*, much the largest part of 609 must be $2 \times$ *tens* $\times$ *units*, and if 609 be divided by $2 \times$ *tens*, or 80, it will give the units figure or a slightly larger number. We obtain 7 as the approximate quotient, and, by trial, determine that it is the exact number of units, since $87 \times 7 = 609$.

Ex. 2. Extract the square root of 119716.

We have

$$\begin{array}{r|l} & \overset{\frown}{11}\overset{\frown}{97}\overset{\frown}{16}. \,|\, 346, \textit{ Root.} \\ & 9 \\ 64 & 297 \\ & 256 \\ 686 & 4116 \\ & 4116 \end{array}$$

By use of the Note to Art. 309, we determine that 3 is the number of hundreds in the root, and that the number of units is 40 +. Having found the first and second figures by this means, we may then find the third figure by separating the given number, as in the latter part of the note, into tens plus units, thus 340 + units.

Let the student write out a detailed explanation of the entire process of extracting the square root of 119716.

312. Square Root of Decimal Numbers.—If it be required to extract the square root of a decimal number, we may proceed thus, for example:

$$\sqrt{.0225} = \sqrt{\tfrac{225}{10000}} = \tfrac{15}{100} = .15, \textit{ Root.}$$

It is better, however, to put the work in a different form, by marking off the given number into periods of two figures each, beginning at the decimal point. Thus, we have,

$$\begin{array}{r|l} & .\overset{\frown}{02}\overset{\frown}{25} \,|\, .15, \textit{ Root.} \\ & 1 \\ 25 & 125 \\ & 125 \end{array}$$

If necessary, annex a zero to complete the last period of figures to the right. In such cases, however, the root cannot be extracted exactly.

Ex. Extract the square root of 0.369 to 4 decimal places.

```
        .36900000|.6074 +, Root.
         36
   1207 |9000
        |8449
  12144 |55100
        |48576
```

313. Square Root of Common Fractions.—If the denominator of the fraction, whose square root is to be extracted, is a perfect square, extract the root of the numerator and of the denominator separately, and divide the one result by the other.

Ex. $\sqrt{\frac{289}{324}} = \frac{\sqrt{289}}{\sqrt{324}} = \frac{17}{18}.$

If the denominator is not a perfect square, reduce the fraction to a decimal and extract the root of the decimal.

Ex. $\sqrt{\frac{2}{3}} = \sqrt{0.66666666+}$

```
        0.66666666 + |0.8164 +, Root.
         64
    161 | 266
        | 161
   1626 | 10566
        |  9756
  16324 | 81066
        | 65296
```

Hence, in general, to extract the square root of a number,

Separate the number into periods of two figures each, beginning at the decimal point;

Find the greatest square in the left-hand period, and set down its root as the first figure of the required root;

Square this figure, subtract the result from the left-hand period, and to the remainder bring down the next period;

Double the root already found for a trial divisor, divide it into the remainder (omitting last figure of the remainder), and annex the quotient obtained, to the root and to the trial divisor.

Multiply the complete divisor by the figure of the root last found, and subtract the result from the remainder;

Proceed in like manner till all the periods of figures have been used.

EXERCISE 152.

Find the square root of:

1. 676.	*4.* 1764.	*7.* 6889.	*10.* 710649.
2. 841.	*5.* 3364.	*8.* 18496.	*11.* 879844.
3. 961.	*6.* 4489.	*9.* 173889.	*12.* 54804409.

13. 64272289.	*15.* 96177249.	*17.* 2181637264.
14. 82646281.	*16.* 1228292209.	*18.* 5416076836.

19. 61.7796.	*22.* 1.752976.	*25.* 11.67657241.
20. 6955.56.	*23.* 1419.7824.	*26.* 175351.5625.
21. 0.822649.	*24.* 0.50665924.	*27.* 25.81554481.

Find the square root of each of the following to three decimal places:

28. 40.	*32.* $13\frac{1}{2}$.	*36.* $5\frac{1}{4}$.	*40.* $7\frac{5}{12}$.
29. 8.	*33.* 0.9.	*37.* $3\frac{1}{3}$.	*41.* $26\frac{1}{11}$.
30. 31.	*34.* $\frac{3}{4}$.	*38.* 246.01.	*42.* $36\frac{1}{6}$.
31. 17.2	*35.* $\frac{7}{10}$.	*39.* $30\frac{3}{100}$.	*43.* $100\frac{1}{9}$.

EXERCISE 153.

ORAL.

State the square root of each of the following:

1. 256.	*6.* 121.	*11.* $\frac{64}{225}$.	*16.* .09.
2. 361.	*7.* 324.	*12.* $\frac{49}{100}$.	*17.* 1.21.
3. 400.	*8.* 169.	*13.* $\frac{81}{16}$.	*18.* .0036.
4. 144.	*9.* 529.	*14.* $1\frac{7}{9}$.	*19.* 2500.
5. 289.	*10.* 196.	*15.* $6\frac{1}{4}$.	*20.* $.062\frac{1}{2}$.

CUBE ROOT.

314. Cubing a Number by Parts.—In order to discover a method of extracting the cube root of a number, we separate a number, as 54, into its tens and units, $50 + 4$, and form its cube as follows:

$$
\begin{array}{l}
50 + 4 \\
\underline{50 + 4} \\
50^2 + 50 \times 4 \\
\underline{\qquad\quad 50 \times 4 + 4^2} \\
50^2 + 2 \times 50 \times 4 + 4^2 \\
\underline{50 + 4} \\
50^3 + 2 \times 50^2 \times 4 \qquad + 50 \times 4^2 \\
\underline{\qquad\quad 50^2 \times 4 + 2 \times 50 \times 4^2 + 4^3} \\
50^3 + 3 \times 50^2 \times 4 + 3 \times 50 \times 4^2 + 4^3 \\
= 125000 + 30000 + 2400 + 64 = 157464.
\end{array}
$$

Hence, if any number be separated into tens + units, its cube will be equal to

(*cube of the tens*) + (*three times the square of the tens times the units*) + (*three times the tens times the square of the units*) + (*cube of the units*), or in symbols

$$(t + u)^3 = t^3 + 3\,t^2\,u + 3\,t\,u^2 + u^3.$$

NOTE.—This method may also be applied in cubing a number which contains three or more figures, and the observed properties employed in extracting the cube root of a correspondingly large number. See Art. 309, Note.

315. Periods.—Since in any given number, as 157464, whose cube root is sought, the cube of the tens is not given explicitly, it must be determined indirectly, and its root then extracted. It is determined by marking off the figures of the number whose root is sought into periods of three figures each, beginning at the decimal point, and then determining the largest cube number represented in the first period of figures to the left.

For the *cube of a number contains three times as many digits* (*less one or two*) *as the number itself.*

For since,

$$
\begin{aligned}
1^3 &= 1 \\
10^3 &= 1000 \\
100^3 &= 1000000 \\
1000^3 &= 1000000000
\end{aligned}
$$

it follows that if a number contains one figure, its cube is either 1, or lies between 1 and 1000, and hence contains one, two, or three digits; if a number contains two digits, its cube is either 1000, or lies between 1000 and 1000000, and hence contains four, five, or six digits; similarly, if a number contains three digits, its cube contains seven, eight, or nine digits, etc.

Hence, if we begin at the decimal point and mark off the digits of any number in periods of 3 figures each, the *number of periods thus formed will be the same as the number of figures in the root.*

316. Extraction of cube root.

Ex. 1. Extract the cube root of 157464.

OPERATION.

$$
\begin{array}{rrr|l}
 & & 157464 & \underline{50 + 4} \\
t^3 = 50^3 = & & 125000 & \\
3\,t^2 = 3 \times 50^2 = & 7500 & 32464 & \\
3\,t\,u = 3 \times 50 \times 4 = & 600 & & \\
u^2 = 4^2 = & \underline{16} & & \\
u \times (3\,t^2 + 3\,t\,u + u^2) = & 4 \times 8116 = & 32464 &
\end{array}
$$

ABBREVIATED FORM OF OPERATION.

$$
\begin{array}{rr|l}
 & 157464 & \underline{54}, \textit{ Root.} \\
 & 125 & \\
3 \times 50^2 = 7500 & 32464 & \\
3 \times 50 \times 4 = 600 & & \\
4^2 = \underline{16} & & \\
8116 & 32464 &
\end{array}
$$

EXPLANATION.

Since 157464 contains two periods of three figures each, the cube root must contain two figures, a tens figure and a units figure. Since the largest cube in 157,000 is 125,000, and the cube root of 125,000 is 50, the number of tens is 5.

Subtracting the cube of the tens 125,000 from 157464, the remainder, 32464, must be $3 \times tens^2 \times units + 3 \times tens \times units^2 + units^3$, and since $3 \times tens^2 \times units$ is much the largest part of the remainder, if the remainder be divided by $3 \times tens^2$, or 7500, it will give the *units* figure or a slightly larger number as the quotient. Dividing we obtain 4 as the approximate quotient, and on trial find that it is the exact number of units, since,

$$(3 \times 50^2 + 3 \times 50 \times 4 + 4^2) \times 4 = 32464.$$

By use of the Note of Art. 314, the same method may be used in extracting the cube root of a number of more than two periods.

Ex. 2. Extract the cube root of 8,627,738.651.

OPERATION.

		$\overparen{8}\overparen{627}\ \overparen{738}.\overparen{651}$ | 205.1, *Root.*
		8
$3 \times (200)^2 =$	120000	627 738
$3 \times (200 \times 5) =$	3000	
$5^2 =$	25	
	123025	615125
$3 \times (2050)^2 =$	12607500	12613651
$3 \times (2050 \times 1) =$	6150	
$1^2 =$	1	
	12613651	12613651

Hence, in general, to extract the cube root of a number,

Separate the number into periods of three figures each, beginning at the decimal point;

Find the greatest cube in the left-hand period, and set down its cube root as the first figure of the required root;

Cube this figure, and subtract the result from the left-hand period, and annex the next period of figures to the remainder;

Take three times the square of the root already found as a trial divisor; divide the remainder by it, and set down the quotient as the next figure of the root;

Complete the trial divisor by adding to it three times the product of the first figure of the root with zero annexed, multiplied by the last figure, and the square of the last figure;

Multiply this complete divisor by the figure of the root last found, and subtract the result from the remainder;

Proceed in like manner till all the periods have been used.

EXERCISE 154.

Find the cube root of:

1. 19683.	*5.* 592704.	*9.* 119823157.
2. 97336.	*6.* 1906624.	*10.* 317214568.
3. 195112.	*7.* 31855013.	*11.* 371694959.
4. 250047.	*8.* 155720872.	*12.* 794022776.

13. 114.084125.	*18.* 5900304.943.
14. .270840023.	*19.* 14.154926059.
15. 487443.403.	*20.* 28877.930432.
16. 529.475129.	*21.* 185.485563927.
17. .773620632.	*22.* .494538357312.

Find the cube root to three decimal places:

23. 6.	*27.* 100.	*31.* $64\frac{3}{8}$.
24. 12.	*28.* $19\frac{1}{8}$.	*32.* 512.9.
25. 29.	*29.* $80\frac{3}{5}$.	*33.* 51.29.
26. 4.5.	*30.* 28.	*34.* $10\frac{4}{11}$.

EXERCISE 155.

ORAL.

State the cube root of:

1. 125; 64; 216; 729; 1000; 512; 1728.

2. 343; 27; 1331; $\frac{8}{27}$; $\frac{64}{125}$; $3\frac{3}{8}$; $\frac{1}{64}$; .008.

3. .001; .064; .001728; 1.728; 8000; 27000.

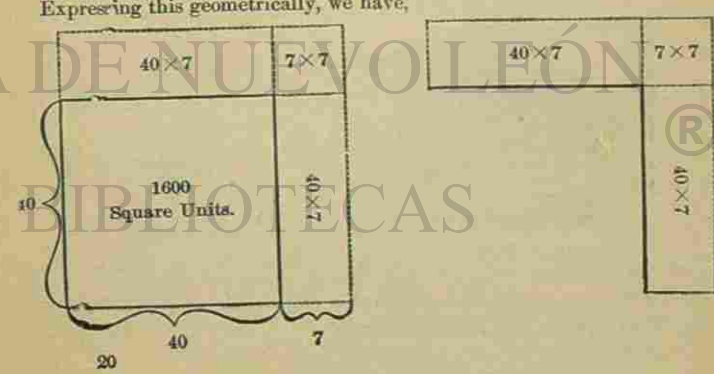
Perhaps no single mental aid to further mathematical study—excepting only the multiplication table—is of more constant benefit than a thorough familiarity with the perfect squares, cubes, and fourth powers of small numbers and the corresponding roots of these powers. Therefore the pupil should pause here until they are most carefully fastened in memory.

OTHER METHODS.

317. Geometrical illustration of square root.

By Art. 309, $47^2 = (40 + 7)^2 = 40^2 + 2 \times (40 \times 7) + 7^2 = 2209$.

Expressing this geometrically, we have,



or 40^2 is represented by a square 40 units of length on a side;

$2 \times (40 \times 7)$ by two rectangular strips, each 40 units long and 7 wide;

7^2 by a small square 7 units on a side.

In extracting the square root of 2209, the square of the tens, 1600 square units, is first removed, leaving a surface of 609 square units.

Much the largest part of this remaining surface is the two equal rectangles. Hence, dividing the area 609 by the combined length of these rectangles, 2×40 or 80, gives the width approximately, or 7.

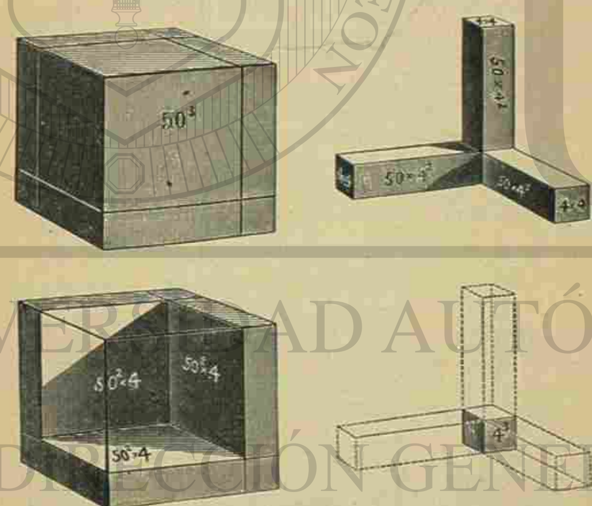
If this width is correct, the entire length of the three figures remaining after the large square (1600) is removed is $40 + 40 + 7$, or 87.

$87 \times 7 = 609$, the remaining area, hence, 7 is the correct width, or the second figure of the root.

318. Geometrical illustration of cube root.

$$54^3 = (50 + 4)^3 = 50^3 + 3 \times (50^2 \times 4) + 3 \times (50 \times 4^2) + 4^3.$$

Expressing this geometrically, we have,



or, 50^3 is represented by a cube, each edge of which contains 50 linear units;

$3 \times (50^2 \times 4)$ is represented by three rectangular solids, each 50 units long, 50 units wide, and 4 units thick;

$3 \times (50 \times 4^2)$ by 3 other solids, each 50 units long, 4 units wide, and 4 units thick;

3^3 by a small cube, each edge of which is 3 units.

Hence, in extracting the cube root of 157,464, the cube of the tens, 125,000 is first removed, leaving a volume of 32464 cubic units. Much the largest part of this is the 3 solids whose bases may be taken as 50×50 each, or $3 \times 50 \times 50$, or 7500 in all.

Hence, dividing the remaining volume, 32464, by 7500, gives the thickness of them approximately as 4.

If this thickness is correct, the sum of the bases of all the remaining solids (after the large cube, 125000, is removed) is

$$3 \times 50^2 + 3 \times 50 \times 4 + 4^2, \text{ or } 8116.$$

But $8116 \times 3 = 23877$, the remaining volume.

Hence, 4 is the correct thickness, or the second figure of the root.

319. Factorial Method of Extracting Roots.—If a number be separated into its prime factors, and each of these factors occurs an even number of times, the *square* root of the number may be obtained by multiplying together all the factors half the number of times they each occur; if each factor occurs three or a multiple of three times, the *cube* root may be obtained by multiplying together all the factors one-third of the number of times which each occurs, etc.

Ex. Extract the square root of 324.

$$\text{Since } 324 = 2 \times 2 \times 3 \times 3 \times 3 \times 3$$
$$= 2^2 \times 3^4$$
$$\sqrt{324} = 2 \times 3^2 = 18, \textit{ Root.}$$

EXERCISE 156.

Find the cube root of the following:

1. 13824.	*4.* 110592.	*7.* 884736.
2. 46656.	*5.* 250047.	*8.* 2460375.
3. 74088.	*6.* 421875.	*9.* 4251528.

320. Higher Roots Obtained by Successive Extractions.—From the meaning of an exponent it follows that the square of the square of a number gives the fourth power of the number. Hence, reversing the process, the *fourth root* of a number is the *square root* of the *square root* of the number. Similarly, the *sixth root* of a number is the *square root* of the *cube root* of the number. The *eighth*, *ninth*, *tenth* . . . . . . roots of a number may be found by similar methods.

Ex. Obtain sixth root of 7,529,536.

Extracting the cube root, we obtain 196.

Extracting the square root of 196, we obtain 14 as the sixth root of the original number.

EXERCISE 157.

Find the fourth root of the following:

1. 331776.	*3.* 47458321.	*5.* 1196883216.
2. 4879681.	*4.* 81450625.	*6.* 11574317056.

Find the sixth root of:

7. 148035889.	*8.* 2176782336.

Find the sixth root of the following to 2 places of decimals:

9. 30.	*10.* 55.	*11.* 78.	*12.* 101.

Compute to 2 decimals the values of:

13. $\sqrt[3]{7+\sqrt{5}}$.	*16.* $\sqrt[3]{\sqrt{11}-\sqrt[3]{11}}$.	*19.* $\sqrt{1+\sqrt{2}+\sqrt[3]{3}}$.
14. $\sqrt{10-\sqrt{7}}$.	*17.* $\sqrt[4]{40+\sqrt[3]{40}}$.	*20.* $\sqrt[3]{\sqrt{5}+\sqrt[3]{6}+\sqrt[4]{7}}$.
15. $\sqrt{19-\sqrt[3]{21}}$.	*18.* $\sqrt[6]{\sqrt{50}-\sqrt[3]{17}}$.	*21.* $\sqrt{3\sqrt{10}+7\sqrt{31}}$.

Extract to three decimal places:

22. The square root of 7.0763; of .70763; and of 4.0763.

23. The square root of .387; of .0387; and of .00765.

24. The square root of .938; of .0938; and of .000765.

Extract to two decimal places:

25. The cube root of 6.318; of .6318; of .075.

26. The cube root of .07165; of .007165; of 19.0019.

27. The sixth root of 2.175; of .2175; of .025.

CHAPTER XIX.

MENSURATION.

321. Mensuration is that branch of mathematics which treats of the measurement of lines, surfaces, and volumes.

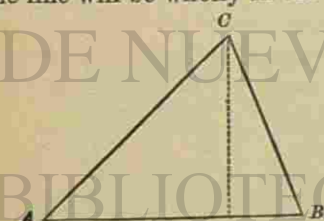
Since *lines* are measured more readily than any other kind of geometrical magnitude, it will be found that, in problems of mensuration, certain lines are usually measured first, and, from the results obtained, the lengths of *other lines*, or required *areas*, or *volumes*, are computed by principles determined by geometry.

It will not be possible to demonstrate fully these principles in the present brief treatment of the subject; but, wherever possible, they will be so presented and illustrated, as to make their truth clear to the pupil and enable him to recall them readily. He should constantly remember, however, that the complete demonstration of the rules and formulas used in this chapter belongs to another branch of mathematics, the subject of Geometry.

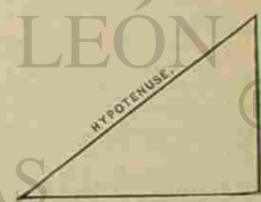
The limitations in the degree of accuracy with which a line can be measured are discussed in Art. 78, which should be reviewed.

I. Mensuration of Lines.

322. Definitions.—A **plane surface** is a surface such that if any two points in it be taken and joined by a straight line, the line will be wholly in the surface.



TRIANGLE.



RIGHT TRIANGLE.

A **triangle** is a portion of a plane surface bounded by three straight lines, as the figure $A\,B\,C$.

number is the *square root* of the *square root* of the number. Similarly, the *sixth root* of a number is the *square root* of the *cube root* of the number. The *eighth*, *ninth*, *tenth* . . . . . . roots of a number may be found by similar methods.

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Extract to three decimal places:

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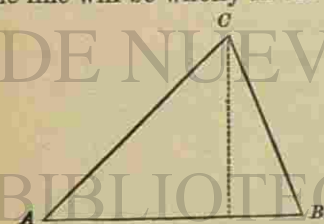
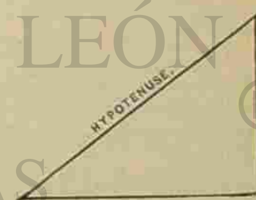


TRIANGLE.



RIGHT TRIANGLE.

A **triangle** is a portion of a plane surface bounded by three straight lines, as the figure $A\,B\,C$.

The **base** of a triangle is the side upon which it is regarded as standing.

The **altitude** of a triangle is the perpendicular distance to the base from the vertex, or point where the other two sides meet.

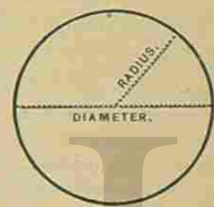
A **right triangle** is a triangle, one of whose angles is a right angle.

The **hypotenuse** of a right triangle is the side opposite the right angle.

Circle and **circumference** are defined in Art. 192.

A **radius** is a line drawn from the center of a circle to any point of the circumference.

Parallel lines are lines in a plane surface which do not meet, however far they be produced.



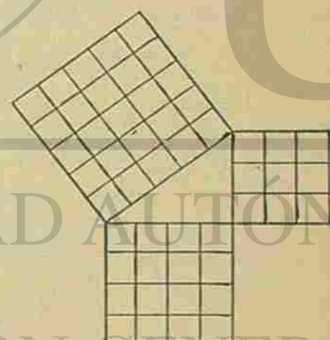
CIRCLE.

323. Formulas for mensuration of lines.

1. *The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides*,

or, denoting the hypotenuse by h, and the other two sides by a and b

$$h^2 = a^2 + b^2,$$
and $$a^2 = h^2 - b^2,$$
whence $$h = \sqrt{a^2 + b^2}$$
$$a = \sqrt{h^2 - b^2}.$$



Hence, given any two sides of a right triangle, the third side may be computed without the labor of measuring it.

Ex. Find the hypotenuse of a right triangle of which the two other sides are 50 ft. and 60 ft.

$$\sqrt{50^2 + 60^2} = 78.102^+$$
$$\therefore \quad \text{Hypotenuse} = 78.102^+ \text{ ft.}$$

2. *Circumference of a circle = diameter × 3.1416 (approx.).*

($\frac{22}{7}$ is sometimes used instead of 3.1416, though it is not quite so accurate).

Hence, also, *diameter* $= \dfrac{\textit{circf.}}{3.1416} =$ *circf.* × .3183 (*approx.*).

Let the pupil measure the diameter and circumference of a silver dollar, and show that the *circf.* = *diam.* × 3.1416 (*approx.*). Let him measure other circles, as a dinner plate, wagon-wheel, etc., similarly.

EXERCISE 158.

In the following examples a and b represent the legs of a right triangle, and c, the hypotenuse.

1. Given $a = 8$, $b = 15$, find c.
2. Given $b = 35$, $c = 37$, find a.
3. Given $c = 29$, $a = 21$, find b.
4. Given $b = 28$, $a = 45$, find c.
5. Given $a = 112$, $c = 113$, find b.
6. Given $c = 73$, $b = 55$, find a.
7. Given $a = 24$, $b = 143$, find c.
8. Given $b = 780$, $c = 901$, find a.
9. Given $c = 1105$, $a = 561$, find b.

Find correctly to 3 decimal places, the remaining side, when:

10. $a = 5$, $b = 8$.
11. $b = 2$, $c = 11$.
12. $c = 43$, $a = 34$.
13. $a = 92$, $b = 65$.

Find the circumference of each circle, when:

14. Radius = 7.
15. Diameter = 46.
16. Radius = $8\frac{1}{2}$.
17. Diameter = $13\frac{1}{3}$.
18. Radius = 74.6.
19. Diam. = 175.4.

Find the diameter of the circle, when:

20. Circumference = 40.
21. Circumference = 375.
22. Circum. = 57.3.
23. Circum. = 103.8.

24. A ladder 25 ft. long stands against the side of a house, and with its foot 7 ft. from the wall. How high is the top of the ladder?

25. A field 156 rds. long and 133 rds. wide is cut by a path running diagonally across it. Find the length of the path.

26. A flag pole was broken 16 ft. from the ground, and the top struck 63 ft. from the foot of the pole. How long was the pole?

27. Two rafters 20.5 ft. long meet at the ridge of a roof 4.5 ft. above the level of the walls. How wide is the house?

28. A ladder 65 ft. long stands in the street; if it fall on one side, it touches a point on that house 16 ft. above the pavement; but on the other side the point it touches is 56 ft. above the pavement. How wide is the street?

29. If the diameter of a pipe is $8\frac{1}{2}$ in., what is its circumference? What is the diameter of another pipe, whose circumference is $8\frac{1}{2}$ in.?

30. The diameter of the earth is about 7920 miles. How many miles is it around the earth?

31. A rope is wound spirally around a cylindrical mast 2 ft. in diameter and 60 ft. high, the spires being 1 ft. apart. How long is the rope?

II. Mensuration of Plane Areas.

324. Definitions.—Triangle and circle have already been defined.

An **equilateral triangle** is one which has all its sides equal.

A **quadrilateral** is a portion of a plane bounded by four straight lines.

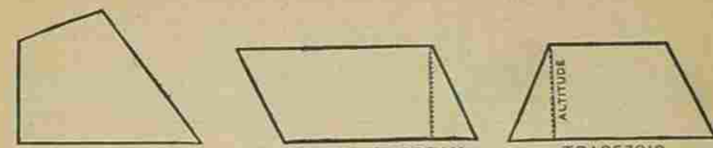
A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A **trapezoid** is a quadrilateral which has two and only two of its sides parallel.

The **altitude** of a parallelogram or trapezoid is the perpendicular distance between parallel sides.

A **rectangle** is a parallelogram whose angles are right angles.

A **square** is a rectangle whose sides are equal.



QUADRILATERAL. PARALLELOGRAM. TRAPEZOID.

A **polygon** is a portion of a plane surface bounded by straight lines. A polygon of *three* sides is called a **triangle**; of *four* sides, a **quadrilateral**; of *five* sides, a **pentagon**; of *six* sides, a **hexagon**, etc.

A **regular polygon** is one in which the sides are all equal, and the angles are all equal.

The **perimeter** of a polygon is the sum of the lengths of its sides.

A **unit of area** is a square, each side of which is a linear unit, as a square inch, or a square yard.

The **area of a plane figure** is the *number* of square units which it contains (see also Art. 175).

325. Formulas for areas of plane figures.

1. Area of a **triangle** = $\frac{1}{2}$ *base* × *altitude.*

2. " " **parallelogram** = *base* × *altitude.*

3. " " **trapezoid** = $\frac{1}{2}$ *sum of parallel sides* × *altitude.*

4. " " **circle** = *radius squared* × 3.1416.

5. " " **circular ring** = $(R^2 - r^2) \times 3.1416$, where R and r are the radii of the two circles.

When the *three sides* of a triangle are given instead of the base and altitude,

6. Area of a **triangle** = $\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c denote the three sides, and $s = \frac{1}{2}(a+b+c)$.

7. Area of **equilateral triangle** = $\frac{a^2\sqrt{3}}{4}$, where a denotes one of the sides.

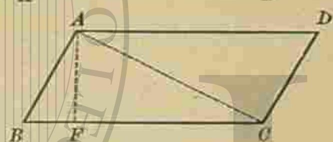
It should be clearly understood that when we speak of multiplying one line by another (as the base by the altitude), we mean that the *number* of linear units in one line is to be multiplied by the *number* of linear units in the other line.

It has been shown (Art. 175) that the product of the number of linear units in the base of a rectangle by the number of linear units in the altitude equals the number of units of area in the rectangle. Thus,

$$7 \text{ in.} \times 5 \text{ in.} = 35 \text{ sq. in.}$$

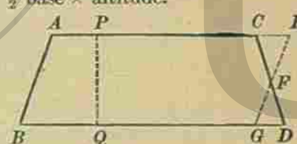
To obtain the area of a parallelogram, it is shown in geometry that the triangle FCD = triangle EAB,

$\therefore$ area $ABCD$ = area of rectangle $AEFD = AD \times DF$ = base $\times$ altitude.



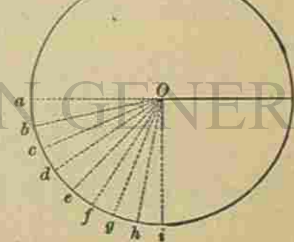
To obtain the area of a triangle, it is shown that triangle ABC = triangle ADC.

$$\therefore \text{area triangle } ABC = \tfrac{1}{2} \text{ area } ABCD.$$
$$= \tfrac{1}{2} BC \times AF.$$
$$= \tfrac{1}{2} \text{ base} \times \text{altitude}.$$

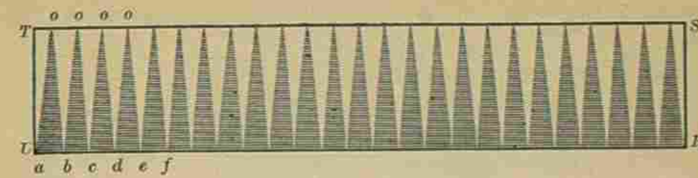


To obtain area of a trapezoid, $ABDC$, we take F, the middle point of CD, and draw EG parallel to AB, and produce AC to meet it at E, and prove triangle ECF = triangle FGD (hence, $CE = GD$).

$$\therefore \text{area } ABCD = \text{area } ABGE \text{ (a parallelogram).}$$
$$= BG \times PQ.$$
$$= \tfrac{1}{2}(AC + BD) \times PQ.$$



In order to understand the formula for obtaining the area of a circle, it will be useful to regard the circle as split up into parts as in the figure opposite; and then conceive the parts oab, obc, ocd, etc., as arranged in the figure on the next page.



The smaller the parts into which the circle is divided, the more nearly will their bases, when taken thus, approximate to a straight line, and their areas taken together = $\frac{1}{2}$ rectangle $RSTU$, whose base is the circumference of the circle and altitude its radius.

$$\therefore \text{area of circle} = \tfrac{1}{2} \text{ circumference} \times R$$
$$= \tfrac{1}{2} \times 2R \times 3.1416 \times R = R^2 \times 3.1416.$$

NOTE.—The student should carefully observe that the determination of all the above areas is made by first measuring certain straight lines, and computing the area from the lengths obtained. This is much more expeditious than any direct counting of the units of area, which is indeed often impossible.

Ex. Find area of a triangle whose sides are 13, 14, 15.

SOLUTION.

Here $a = 13$, $b = 14$, $c = 15$. Hence, $s = 21$
$s - a = 8$
$s - b = 7$
$s - c = 6$

$$\therefore \text{area} = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{7056} = 84.$$

EXERCISE 159.

Find the area of:

1. A rectangle 8 yards long and 11 ft. 8 in. wide.
2. The walls and ceiling of a room $17\frac{1}{2} \times 16\frac{2}{3} \times 8\frac{1}{3}$ feet.
3. A parallelogram whose base is 30 yd. and alt. 20 ft.
4. A straight rectangular street 7 mi. long and $2\frac{1}{2}$ rd. wide.
5. A page $8\frac{2}{3}$ in. long and $4\frac{1}{4}$ in. wide.
6. A triangle on base of 18 in. and alt. 15 in.
7. A triangular field whose alt. is 40 yd. and base 45 rd.
8. A trapezoid whose bases are 60 and 75 feet and alt. is 15 yd.

9. A trapezoid whose bases are 3 mi. and 400 rd. respectively, and altitude is 80 rd.

10. A circle whose radius is 6 in.

11. A circle whose diam. is 10 rds.

12. A circle whose circumference is 80 ft.

13. A triangle whose sides are 9, 10, 17 in.

14. A triangle whose sides are 12, 17, 25 ft.

15. A triangle whose sides are 13, 30, 37 yds.

16. A triangle whose sides are 20, 37, 51 rds.

17. A triangle whose sides are 25, 63, 74 mi.

18. An equilateral triangle whose sides are each 5 in.

19. An equilateral triangle whose sides are each 80 rds.

20. A circular ring whose two diameters are 28 and 16 ft.

21. A circular race-track is 3 rds. wide and placed around and just inside a field whose radius is 63 rds. Find area of the track in acres.

22. What is the land in a river-bed worth at $60 an acre, if the river increases from 6 to 60 rds. in width and is 20 miles long? (Trapezoid.)

23. A farm in shape of a triangle whose sides are 140, 143, 157 rods was sold at $85 an acre. Find the value of the farm.

24. A barn is 48 feet wide and 90 feet long. At the corner its height is 20 ft., but at the middle the height to the peak is 38 ft. Find (*a*) the area of the end; (*b*) the length of the rafters; and (*c*) the entire exterior surface of the barn.

III. Mensuration of the Surfaces of Solid Figures.

326. Definitions.—A **solid** is that which has length, breadth, and thickness.

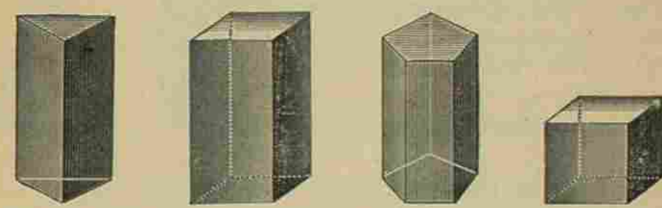
A **prism** is a solid bounded by two equal and parallel polygons called bases, and by parallelograms (which together form the **lateral surface**).

The **altitude** of a prism is the perpendicular distance between the bases.

Prisms are **triangular, quadrangular, pentagonal,** etc.,

according as their bases are triangles, quadrilaterals, pentagons, etc.

A **regular prism** is one which has regular polygons for its bases.



TRIANGULAR PRISM. QUADRANGULAR PRISM. PENTAGONAL PRISM. CUBE.

A **right prism** is one in which the other faces are perpendicular to the bases.

An ordinary box is a right rectangular prism.

A **cube** is a prism bounded by squares.

A **pyramid** is a solid bounded by a polygon called the base, and by triangles meeting at a point called the vertex.

The triangles which meet at the vertex taken together form the *lateral surface.*

The **altitude** of a pyramid is the perpendicular distance from the vertex to the base.

A pyramid is **triangular, quadrangular, pentagonal,** etc., according as the base is a triangle, quadrilateral, pentagon, etc.

A **regular pyramid** has a regular polygon for its base, and the triangles bounding the pyramid all equal.

PYRAMID.

The slant height of a regular pyramid is the perpendicular distance from the vertex to one side of the base.

A **cylinder** is a solid formed by the revolution of a rectangle about one of its sides as an axis. Hence, a cylinder has two circles for bases.

A **cone** is a solid formed by the revolution of a right triangle about one of its sides as an axis. Hence, a cone has a circle for its base.

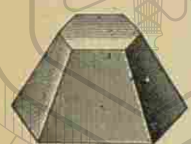
CYLINDER.

CONE.

The altitude of a cone is the perpendicular distance from the vertex to the base. The **slant height** is the distance from the vertex to any point in the circumference of the base.

The **frustum of a pyramid** is the portion of the pyramid intercepted between the base and a plane parallel to the base.

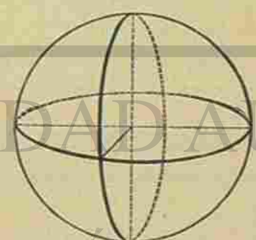
The **frustum of a cone** is the portion of a cone intercepted between the base and a plane parallel to the base.



FRUSTUM OF PYRAMID.

FRUSTUM OF CONE.

A **sphere** is a solid bounded by a curved surface, every point of which is equally distant from a point within called the center.



The **radius** of a sphere is a line drawn from the center to any point of the surface. The **diameter** is a line passing through the center and terminated by the surface.

327. Formulas for areas of surfaces of solids.

1. *Lateral surface of a* **right prism** = *perimeter of base* × *altitude.*

2. *Convex surface of a* **cylinder** = *circf. of base* × *alt.* = $2\pi RH$ (where $\pi = 3.1416$, R = radius of base, H = altitude).

3. *Lateral surface of a* **regular pyramid** = $\frac{1}{2}$ *perimeter of base* × *slant height.*

4. *Convex surface of a* **cone** = $\frac{1}{2}$ *circf. of base* × *slant height* = $\pi R L$ (where L = slant height).

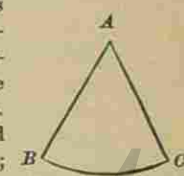
5. *Lateral surface of* **frustum of regular pyramid** = $\frac{1}{2}$ *sum of perimeters of bases* × *slant height.*

6. *Convex surface of* **frustum of cone** = $\frac{1}{2}$ *sum of circumferences of bases* × *slant height* = $\pi (R + r) L$.

7. *Surface of a* **sphere** = 3.1416 × *diameter squared.*

$$= \pi D^2, \text{ or } 4\pi R^2.$$

These formulas (except 7) are derived from those given in Art. 325. Thus the lateral surface of a right prism is composed of rectangles, all having the same altitude; that of a regular pyramid is composed of equal triangles; that of a frustum of a regular pyramid of equal trapezoids. Also the convex surface of a cylinder unrolled forms a rectangle; of a cone forms a portion of a circle, called a sector, as in the Fig. ABC, its area being $\frac{1}{2}\, BC \times AC$, which is determined in the same way that the area of a circle is obtained; the convex surface of a frustum of a cone equals the difference between two sectors.



The student should read at this point the note to Art. 325.

EXERCISE 160.

Find the area of the lateral surface of:

1. A right prism 10 in. high on square base, 3 in. on a side.

2. A right prism 8 ft. high, and on an octagonal base 9 in. on each side.

3. A regular pyramid on a hexagonal base 5 in. on a side, and of slant height of 10 in.

4. A regular pyramid on pentagonal base, 7 ft. on a side, and slant height = 19 yds.

5. A frustum of a triangular pyramid, each side of the lower base being 6 ft., and of the upper base being 5 ft., and with slant height of 8 ft.

6. A cylinder of revolution 7 ft. long, the radius of whose base is 3 ft.

7. A cone of revolution on base of radius 8 ft., and whose slant height is 40 ft.

8. A pipe 18 in. through and a mile long.

9. A cone whose radius is 6 in. and slant height is a yard.

10. The frustum of a cone of revolution, if the radii of the bases are 7 and 17 in. respectively, and the slant height is 20 in.

11. Find area of surface of a sphere whose radius is 3 ft.

12. Find area of surface of a sphere whose diameter is 19 in.

13. At 12¢ a sq. ft., what is the cost of painting a pyramidal spire, whose base is a hexagon of 9 ft. on a side and slant height is 90 ft.?

14. What will it cost to paint a cylindrical water-tower at 20¢ a sq. yd., if the diameter of the tower is 10 ft. and its height is 80 ft.?

15. Compute the cost of gilding a dome in the shape of a hemisphere, whose radius is 18 ft., at $1.75 a sq. yd.

16. A post 40 ft. long, in the shape of the frustum of a cone, is 10 in. thick at one end and 18 in. at the other. Find its *entire* superficial area.

IV. Mensuration of Solids.

328. Definitions.—Beside the definitions given in Art. 326, it should be recalled (see Arts. 178, 179) that a *unit of volume* is a cube, each edge of which is a linear unit, as a cubic inch, or a cubic yard; and that the *volume* of a solid is the number of cubic units which the solid contains. Thus the volume of a room is the *number* of cubic feet which it contains.

329. Formulas for volumes of solids.

1. *Volume of a* **prism** = *area of base* × *altitude.*

2. *Volume of a* **rectangular prism** = *length* × *breadth* × *thickness.*
3. *Volume of a* **cube** = *cube of its edge.*
4. *Volume of a* **cylinder** = *area of base* × *altitude* = $\pi R^2 H$.
5. *Volume of a* **pyramid** = $\frac{1}{3}$ *area of base* × *altitude.*
6. *Volume of a* **cone** = $\frac{1}{3}$ *area of base* × *altitude* = $\frac{1}{3}\pi R^2 H$.
7. *Volume of a* **frustum of pyramid**

= $\frac{1}{3}$ *altitude* × (*sum of areas of bases + square root of their product*)

= $\frac{1}{3} H(B + b + \sqrt{Bb})$ (when H = alt., B, b = areas of bases).

8. *Volume of a* **frustum of cone** = same as in 7.

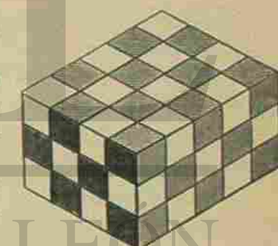
= $\frac{1}{3} H\pi (R^2 + r^2 + Rr)$ where R and r are radii of bases.

9. *Volume of a* **sphere** = $\frac{1}{3}$ *surface* × *radius,*

$$= \tfrac{4}{3}\pi R^3.$$

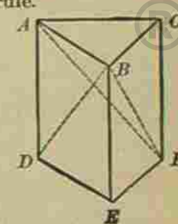
It should be remarked again that the student needs to study solid geometry, in order to understand fully the reasons for these formulas.

It will be of service, however, to recall (see Art. 179) that in a rectangular prism the volume, or number of cubic units, is equal to the number of linear units in the three edges multiplied together.



It is also to be observed that a rectangular prism may be conceived as divided into two equal triangular prisms with equal bases and the same altitude. Hence, the volume of each will equal half the volume of the rectangular prism, or the volume of a triangular prism = *base* × *alt.*

Any other prism may be split up into triangular prisms and its volume obtained by the same rule.



A cylinder may be conceived as determined by a prism with an infinite number of sides.

Any triangular prism, as $ABCDEF$, may be separated into three equivalent pyramids, for $B - DEF = F - ABC$ (or $B - ACF$) $= B - ADF$.

∴ Volume of pyramid $B - DEF = \frac{1}{3}$ prism $ABCDEF$.

The sphere may be regarded as an aggregate of very small pyramids with their common vertex at the center of the sphere, and the sum of their bases approximating to the surface of the sphere. Hence, the volume of the sum of the volumes of the pyramids will be determined by $\frac{1}{3}$ the product of the surface of the sphere × its radius.

The student should again read the note to Art. 325, and state how it applies to the mensuration of volumes.

EXERCISE 161.

Find the volume of:

1. A prism on base of 10 sq. ft. and whose height is 12 ft.
2. A rectangular prism whose dimensions are 8 × 9 × 10 ft.
3. A room whose dimensions are 15 ft. 3 in. × 13 ft. 4 in. × 11 ft. 6 in.
4. A 12-in. cube. A 13-ft. cube.
5. A pyramid 9 ft. high whose base is 100 sq. ft.
6. A pyramid whose alt. is 28 ft. and base is 60 sq. yds.
7. A cylinder whose radius is 9 in. and alt. is 10 ft.
8. A piece of wire $\frac{1}{4}$ in. thick and 75 yds. long.
9. A sphere of radius 5 in. One of radius 7 ft.
10. A sphere whose diameter is 11 in.
11. The frustum of a pyramid whose bases are 32 and 50 sq. ft. and alt. is 9 ft.
12. The frustum of a cone 6 ft. high, the radii of whose bases are 6 ft. and 8 ft.
13. How many cubic feet of water in a cylindrical water-tank 10 ft. in diameter and 80 ft. high? How many gallons?
14. How many cu. in. in a glass shaped in the frustum of a cone $3\frac{1}{2}$ in. high, if the diameters of the base and top are 2 and 3 in. respectively?
15. How much larger is a 4-inch cube than a 4-in. sphere?
16. From a 7-ft. cube of granite the greatest possible sphere was cut out. How many cu. ft. of stone were removed? What was the area of the surface of the sphere?
17. Supposing a drop of water to be a sphere having $\frac{1}{4}$ in. diameter. How many drops of rain in a cylindrical pail 20 in. deep and 8 in. in diameter? How many such drops in a gallon?
18. If a bushel-measure in form of a cylinder is 18 inches in diameter, how deep is it? If it is 18 inches deep, what is its diameter?
19. Into a cylindrical water-tank 13 ft. in diameter and standing on end, an iron globe 10 ft. in diameter is sunk. How far will the surface of the water rise?
20. A heap of wheat in shape of a cone is 8 ft. deep and the diameter of the base is 15 feet. How many bushels in the heap?
21. A regular pyramid is 40 ft. high and stands on an equilateral triangle for base whose sides are each 6 ft. Find its volume. Find its slant height and lateral area.

V. Lines, Areas, and Volumes of Similar Figures.

330. Definitions.—**Similar surfaces** are those which have the same shape. Thus, any two squares are similar plane figures.

Similar solids are solids which have the same shape. Thus, any two cubes, or two spheres, are similar.

331. Properties of Similar Figures.—In any two similar figures

I. *Any two* **corresponding lines** *have the same ratio as any other two corresponding lines;*

II. *The* **areas** *of any two similar figures are to each other as the* **squares** *of any two corresponding lines;*

III. *The* **volumes** *of any two similar solids are to each other as the* **cubes** *of any two corresponding lines.*

Let the pupil illustrate these principles by drawing two squares with edges 2 in. and 5 in. respectively, and comparing their areas; and by drawing, or forming, two cubes with edges 2 in. and 5 in. respectively, and determining the number of cubic inches in each figure.

It is to be observed that the comparison of surfaces and solids of the same shape is made to depend again on the measurement of straight lines and computations from them (see Art. 325, note).

Ex. If a pipe 1 in. in diameter discharges 50 gal. in a minute, how much will a pipe 2 in. in diameter discharge?

The quantity discharged by a pipe is in proportion to the area of the section of the pipe, and hence, in proportion to the square of its diameter. Hence,

$$1^2 : 2^2 = 50 : (\quad)$$

$$\text{or No. gals. required} = \frac{50 \times 4}{1} = 200.$$

EXERCISE 162.

1. One of two similar triangles contains 135 sq. in. If its base is 15 in., what is the area of the other whose base is 18 in.?

2. Two sides of a polygon are 27 and 32 inches. In a similar polygon the less of the two corresponding sides is 18 in. What is the length of the other?

3. A polygon whose base is 12 ft. contains 62 sq. ft. What is the area of a similar polygon whose base is 42 ft.?

4. If the area of a circle, whose radius is 5 in., is 78.54 sq. in., find the area of a circle whose radius is 7 in. Prove your answer correct.

5. If a cylinder whose alt. is 8 ft. has a convex surface of 44 sq. ft., what is the convex surface of a similar cylinder whose alt. is 20 ft.?

6. The volume of a solid is 52 cu. in. and one side is 4 in. Find the volume of a similar solid if a corresponding side is 6 in.

7. The volume of a solid is 400 cu. ft. and one side is 12 ft. Find the volume of a similar solid if the corresponding side is 21 ft.

8. If the sides of two squares are as 2 : 3, what is the ratio of their areas? If the edges of two cubes are as 3 : 5, what is the ratio of their volumes?

9. Two spheres have radii equal to 7 and 9 inches respectively. What is the ratio of their circumferences? Of the areas of their surfaces? Of their volumes?

10. A man whose coal-bin is of a certain size, builds another having each dimension twice as great. How much lumber would he require compared with the first bin? How many times as much coal will it hold?

11. The volumes of two similar solids are 297 cu. in. and 704 cu. in. If the shortest side of the less is 3 in., what is the shortest side of the other?

12. The areas of two similar triangles are 324 and 1444 sq. ft. respectively. If the base of the greater is 14 ft., what is the base of the less?

13. If there are 300 yards in a 4-in. ball of yarn, how many yards will there be in a 6-in. ball? In a 2-in. ball?

14. If it costs $250 to paint a certain house, how much will it cost to paint another, all of whose dimensions are double those of the first?

15. If the planet Jupiter has 11 times the diameter of the earth, how do their surfaces compare? How do their volumes compare?

16. How many rods in the radius of a circle twice as large as another which contains 160 sq. rds.?

17. What is the ratio of the depths of similar quart and peck measures? A peck and bushel measure?

18. If a grindstone 18 in. in diameter costs $4, what ought another cost having the same thickness but 24 in. in diameter?

19. If a 3-in. roll of butter is worth 60 cents, what is a 5-in. roll worth?

20. If a person 5 ft. 6 in. tall ought to weigh 150 lbs., what should a person 6 ft. tall weigh?

CHAPTER XX.

METRIC SYSTEM.

332. The **metric system** of weights and measures is a system based on a certain unit of length called the *meter*, all other units of the system being derived from the meter by use of the decimal scale.

The **meter** is the length of a certain platinum bar kept in the archives of the International Bureau of Weights and Measures at Lèvres (approximately 39.37 inches).

It was intended to make the meter equal $\frac{1}{10000000}$ part of the distance from earth's equator to its pole, but the relation is not exact, owing to a slight error in the original computation.

333. Derived Units.—Other units are derived from the meter taken as the fundamental unit, in two different ways:

(1) By use of 10 as a multiplier or divisor.

The size of the units derived thus is indicated by the prefixes, **deka-**, **hekto-**, **kilo-**, **myria-**, meaning 10, 100, 1000, 10000, respectively; and by **deci-**, **centi-**, **milli-**, meaning $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, respectively.

It is to be observed that the decimal *divisions* of the unit are expressed by *Latin* prefixes; the decimal *multiples* by *Greek* prefixes.

(2) By taking a certain portion of space or matter, determined by the metric unit of length, and making a new unit of this portion of space or matter.

Thus the volume of 1 cubic decimeter = the *liter;*
the volume of 1 cubic meter = the *stere;*
the weight of 1 cubic centimeter of water = the *gram.*

These new units, taken as primary units, may in turn be multiplied or divided in the decimal scale; hence, we have

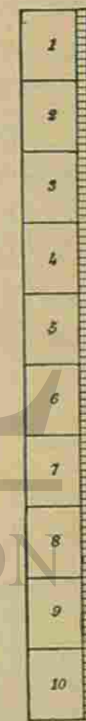
the various tables for metric weights and measures which are about to be given.

In these tables, only those units printed in **black letters** are much used in practice (just as in U. S. money only the dollar and cent are much used), the other units serving for computation, or other theoretic purposes merely.

334. I. **Measures of Length.**—The primary unit of length, as has been said, is the meter.

TABLE OF LENGTH.

10 **millimeters** (mm.)	= 1 **centimeter** (cm.).
10 centimeters	= 1 decimeter (dm.).
10 decimeters	= 1 **meter** (m.).
10 meters	= 1 dekameter (Dm.).
10 dekameters	= 1 hektometer (Hm.).
10 hektometers	= 1 **kilometer** (Km.).
10 kilometers	= 1 myriameter (Mm.).



1 Decimeter.

EXERCISE 163.

ORAL.

How many:

1. Meters in a kilometer? In a Dm.? In 5 Hm.?
2. Centimeters in a meter? In a Hm.? In 8 dm.?
3. mm. in a cm.? In 3 m.? In 5 dm.?
4. cm. in 800 mm.? In 4 m.? In $\frac{1}{2}$ m.?
5. dm. in 25 m.? In 500 cm.? In 12 Dm.?
6. m. in 300 cm.? In 8000 dm.? In 15 Hm.?
7. m. in 3.5 Dm.? In 3.5 dm.? In 7 cm.?
8. cm. in 1.2 dm.? In 1.2 mm.? In 11 m.?
9. Hm. in 7 m.? In 70 Km.? In 77 cm.?
10. Km. in 300 Hm.? In 56 Dm.? In 8 m.?
11. The pupil may now state his own rule for changing metric quantities from one denomination to a higher, and his other rule for changing to a lower unit. A thorough mastery of this process by sufficient drill renders all the rest of metric system questions comparatively easy.

335. II. **Measures of Area.**—The unit of area is the square meter.

TABLE OF SQUARE MEASURE.

100 sq. millimeters (sq. mm.)	= 1 **sq. centimeter** (sq. cm.).
100 sq. cm.	= 1 sq. decimeter (sq. dm.).
100 sq. dm.	= 1 **sq. meter** (sq. m.).
100 sq. m.	= 1 sq. dekameter (sq. Dm.).
100 sq. Dm.	= 1 sq. hektometer (sq. Hm.).
100 sq. Hm.	= 1 **sq. kilometer** (sq. Km.).
100 sq. Km.	= 1 sq. myriameter (sq. Mm.).

336. III. **Measures of Land Surface.**—The ratio, 100, between two successive units of square measure is too great for many practical purposes, as in measuring *land*. Hence, a new unit, the *are*, is selected, which is increased or decreased with 10 as a scale.

Square Centimeter.

The unit of land measure is the **are** (pronounced "air"), which equals 100 square meters.

TABLE OF LAND MEASURE.

10 centares (ca.)	= 1 **deciare** (da.).
10 deciares	= 1 are (a.).
10 ares	= 1 dekare (Da.).
10 dekares	= 1 **hektare** (Ha.).

EXERCISE 164.

ORAL.

How many:

1. Sq. m. in a sq. Hm.? In a sq. Km.? In an are?
2. Sq. cm. in a sq. m.? In a sq. dm.? In 7 sq. m.?
3. Centares in an are? In a Da.? In 15 a.?
4. Ares in 25 Ha.? In 500 ca.? In 7 ca.?
5. Ares in ½ Ha.? In 63 Da.? In 63 da.?
6. Ha. in 3 a.? In 303 a.? In 36 ca.?
7. Sq. dm. in 16 sq. m.? In 16 sq. cm.? In 106 sq. cm.?
8. Sq. m. in 2 sq. Dm.? In 22 sq. dm.? In 22 sq. cm.?
9. Sq. m. in 5 a.? In 75 a.? In 83 Ha.?
10. a. in 15 Ha.? In 35 ca.? In 35 Da.?

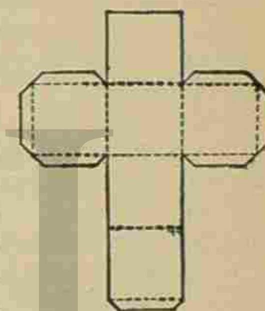
(To be continued by the teacher.)

337. IV. **Measures of Volume.**—The **cubic meter** is the unit of volume.

TABLE OF CUBIC MEASURE.

1000 cubic millimeters	= 1 **cubic centimeter** (cu. cm.).
1000 cu. cm.	= 1 cubic decimeter (cu. dm.).
1000 cu. dm.	= 1 **cubic meter** (cu. m.).

Let the student form a cubic centimeter by cutting out a piece of pasteboard of the shape indicated in the figure, cutting it half through at the dotted lines, then folding together in the shape of a cube and pasting with mucilage.



338. V. **Measures of Wood, etc.**—The ratio, 1000, between two successive units in the above table is too great for many practical purposes, as in measuring *wood*. Hence, a new unit, the *stere*, is taken, which is multiplied and divided according to the scale of 10.

The unit of wood measure is the **stere** (pronounced "stair"), which equals one cubic meter.

TABLE OF WOOD MEASURE.

10 millisteres (ms.)	= 1 **centistere** (cs.).
10 centisteres	= 1 decistere (ds.).
10 decisteres	= 1 **stere** (s.).

EXERCISE 165.

ORAL.

How many:

1. Cu. cm. in a cu. m.? In 16 cu. dm.? In 38 cu. mm.?
2. Cu. cm. in 6 cu. mm.? In 3 cu. m.? In 75 cu. dm.?
3. Cu. m. in 76 cu. dm.? In 768 cu. cm.? In 500 cu. mm.?

4. cs. in 1 stere? In 3 ds.? In 35 ds.?
5. St. in 3 ds.? In 75 cs.? In 8 cs.? In 800 ds.?
6. Cu. m. in 53 st.? In 25 ds.? In 6 cs.?
7. St. in 14 cu. dm.? In 240 cu. cm.? In 240 cs.?
8. Let the pupil construct a cu. dm. and a cu. cm. from pasteboard. Let him clearly see how the lengths of their edges compare; how the areas of their faces compare; and how their volumes compare.

339. VI. **Measures of Capacity.**—The **liter** (pronounced "leeter") is the unit of capacity. It equals a cubic decimeter; that is, the volume of a cube whose edge is $\frac{1}{10}$ of a meter.

TABLE OF CAPACITY.

10 milliliters (ml.)	= 1 **centiliter** (cl.).
10 centiliters	= 1 deciliter (dl.).
10 deciliters	= 1 **liter** (l.).
10 liters	= 1 dekaliter (Dl.).
10 dekaliters	= 1 **hektoliter** (Hl.).
10 hektoliters	= 1 kiloliter (Kl.).
10 kiloliters	= 1 myrialiter (Ml.).

EXERCISE 166.

ORAL.

How many:
1. Liters in a Kl.? In a Hl.? In 5 Dl.? In 75 dl.?
2. Dl. in 7 Kl.? In 40 Hl.? In 7 l.? In 58 dl.?
3. Liters in a cubic meter? In 206 cu. cm.?
4. cl. in 3 l.? In 1 Hl.? In 55 ml.?
5. Hl. in 25 Kl.? In 6 l.? In 44 Dl.? In 325 cl.?
6. dl. in 3 Dl.? In 15 l.? In 77 cl.? In 156 ml.?
7. l. in 30 cu. dm.? In 6 cu. cm.? In 45 cu. m.?
8. Let the pupil construct a liter out of pasteboard.

340. VII. **Measures of Weight.**—The **gram** is the unit of weight. It is the weight of a cubic centimeter of distilled water at its greatest density; that is, when at a temperature of 39.2° Fahrenheit.

TABLE OF WEIGHT.

10 milligrams (mg.)	= 1 **centigram** (cg.).
10 centigrams	= 1 decigram (dg.).
10 decigrams	= 1 **gram** (g.).
10 grams	= 1 dekagram (Dg.).
10 dekagrams	= 1 hektogram (Hg.).
10 hektograms	= 1 **kilogram** (Kg.).
10 kilograms	= 1 myriagram (Mg.).
10 myriagrams	= 1 quintal.
10 quintals	= 1 metric ton.

EXERCISE 167.

ORAL.

How many:
1. Grams in a Kg.? In 5 Hg.? In 87 dg.?
2. cg. in 4 g.? In 31 Dg.? In 7 mg.? In 70 g.?
3. dg. in 15 g.? In 17 Hg.? In 36 mg.? In 360 g.?
4. Kg. in 50 g.? In $\frac{1}{2}$ g.? In 7 Dg.? In 4 dg.?
5. Grams in weight of 1 cu. m. of water? In 7 cu. dm. water?
6. Kg. in 30 cu. m. water? In 16 cu. Km. of water?
7. cg. in 1 cu. cm. of water? In 13 cu. dm. of water?

NOTATION, NUMERATION, REDUCTION.

341. The methods of notation and numeration used in the metric system have been indicated in general already. With respect to **notation** it should be further remarked that

(1) Of the abbreviations used, those for units larger than the primary units begin with capital letters; the others with small letters.

(2) The place occupied by each unit with reference to the decimal point (when the scale is 10) should be fixed in mind. Thus, for the meter,

Mm.	Km.	Hm.	Dm.	m.	dm.	cm.	mm.
5	8	7	2	3	4	8	6

(3) In writing a metric number for which the scale is 100, the number of units of each denomination must occupy two

places (by the aid of zeroes if necessary); when the scale is 1000, three places. Thus, 35 sq. Km. 8 sq. Hm. 17 sq. m. is written 35080017 sq. m.

With respect to **numeration** or *reading* metric numbers, it should be remarked that a metric number may be read in either of two ways:

(1) *By specifying each unit.*

Thus, 37.56 m. may be read as "3 dekameters 7 meters 5 decimeters 6 centimeters," or

(2) *By reading the entire number as an abstract number, and then affixing the name of the primary unit.* According to this method the above number would read "thirty-seven and fifty-six hundredths meters."

342. Reduction.—*A metric number is reduced to a lower denomination by moving the decimal point to the right, one place for each reduction of the unit when the scale is 10; two places when the scale is 100; three places when the scale is 1000.*

Thus, 6.728 m. = 672.8 cm.

A metric number is reduced to a higher denomination by moving the decimal point to the left, correspondingly to the above rule.

Thus, 8596.2 mm. = 8.5962 m.

It is often an advantage, instead of making a single direct reduction, to make two steps of the reduction, by

(1) Converting the number of *given* units into *primary* units, and

(2) Converting the number of *primary* units into *required* units.

Ex. Convert 938765.23 dm. into kilometers.

We first change decimeters into meters, and then convert meters into kilometers.

Hence, 938765.23 dm. = 93876.523 m. = 93.876523 Km.

EXERCISE 168.

Reduce

1. 7.324 Km. to dm. To Dm. To cm.

2. 36.08 Dg. to Kg. To cg. To dg.

3. 712.45 sq. m. to sq. cm. To sq. Hm. To Ha.

4. 503.217 dl. to ml. To Dl. To Kl.

5. 55.171 ca. to Ha. To a. To sq. Dm.

6. .025 cu. m. to cu. cm. To l. To dst.

7. 5.3 st. to dst. To cu. dm. To st.

8. 12.345 Km. to Dm. To dm. To mm.

9. 3267.1 cg. to Dg. To Kg. To mg.

10. 106.73 dl. to ml. To Dl. To Kl.

11. 8.3 cu. cm. to cu. m. To l. To cu. mm.

12. 46.71 Dl. to cu. m. To cu. cm. To dl.

13. 500.7 Hg. to mg. To dg. To Kg.

14. 375.5 a. to Ha. To ca. To sq. Dm.

15. 40 Ha. to a. To sq. m. To ca. To sq. Dm.

16. 345.75 m. to cm. To Hm. To Km. To mm.

17. 863200 sq. cm. to sq. m. To Ha. To sq. Dm.

18. 385 l. to cl. To cu. m. To cu. Km.

19. How many Kgs. in the weight of 75 l. of water? Of 1 cu. m. of water? Of 170 cu. mm. of water? Of 35 cu. cm. of water? Of 11 Kl. of water?

20. How many l. of water are needed to weigh 75 Kg.? To weigh 5000 Kg.? To weigh 30 g.? To weigh 58.7 cg.? To weigh 138.75 Dg.?

21. Change 42.3 Hl. to dl. To cu. cm. To cu. m. Find its weight (if water) in g. In Kg.

OPERATIONS WITH METRIC NUMBERS.

343. Addition.—Metric numbers may be added in the same way as any other decimal numbers. It is necessary in all cases to reduce to the same denomination the metric numbers which are to be added.

Ex. Add 37 cm., 36.26489 Hm., 3 Km.

We reduce to meters all the numbers to be added.

37 cm. =	0.37 m.
36.26489 Hm. =	3626.489 m.
3 Km. =	3000. m.
	6626.859 m., *Sum.*

344. Subtraction.

Ex. Subtract 387.92 cg. from 5.827 Dg.

Reducing both numbers to grams and subtracting.

5.827 Dg. = 58.27 g.
387.92 cg. = 3.8792 g.
54.3908 g., *Difference.*

345. Multiplication.

Ex. Find the area of a rectangular field which is 0.5 Km. long and 27.8 m. wide.

0.5 Km. = 500 m.
Area = 27.8 × 500 sq. m. = 13900.0 sq. m.
= 1.39 Ha., *Area.*

346. Division.

Ex. 1. Divide 21.856 Dm. by 3.2 cm.

21.856 Dm. = 218.56 m.
3.2 cm. = .032 m.
218.56 ÷ .032 = 6830, *Quotient.*

Ex. 2. How high must a box be in order that it may hold 30 liters, if it is 50 cm. long and 2 dm. wide?

Since 1 l. = 1 cu. dm.,
30 l. = 30 cu. dm. = volume of the box.

Since 50 cm. = 5 dm.,
the area of base of the box = 5 × 2 sq. dm. = 10 sq. dm.
Height of the box = volume ÷ area of base
= 30 ÷ 10 = 3.
∴ Height = 3 dm.

EXERCISE 169.

Add:

1. 3.6 m. + 45 cm. + .06 Km. + 3.2 Dm.
2. 7 a. + 120 ca. + .08 Ha.
3. 9 cu. dm. + 300 cu. mm. + 50.6 cu. m.
4. 4.81 dg. + 325.1 mg. + 14.78 Dg. + 1.31 Kg.
5. 43 cl. + 7.1 Hl. + 305 ml. + 2.5 Kl. + 27.8 l.
6. 75.38 Ha. + 438.1 a. + 9587 ca.
7. 9.03 m. + 903 mm. + 9030 cm. + 90.3 Dm. + 0.903 Km.

8. 307.5 dg. + 48.091 g. + 385.61 mg. + 7 Kg. + 9.9 Dg.
9. 17 cu. dm. + 385 cu. cm. + 4128 cu. mm. + 30.9 cu. m.
10. 14 l. + 1.403 Kl. + 378.12 cl. + 99 Hl. + 14 dl.
11. 77 cu. cm. + 32.3 l. + 9.5 dl. + 307.5 cu. dm. + 4 Dl.
12. 38 sq. Dm. + 19.3 Ha. + 1435 sq. m. + 281.5 a.

Subtract:

13. 7.3 cm. from .08 Km.
14. .46 Hl. from 8769.1 cl.
15. 5.7 sq. dm. from .09 Ha.
16. 3.57 g. from 2.538 Kg.

17. What is the difference between 7 Hm. 5 Dm. 3 m. 4 cm. and 8 Km. 3 Hm. 4 dm. 2 mm.?

18. What is the difference between 8 Kl. 5 l. 7 dl. 4 ml. and 7 Hl. 2 Dl. 1 l. 5 cl.?

19. From 57.3128 Kl. take 4875.625 cu. cm.

Multiply:

20. 3 Dm. 5 m. 4 cm. by 5.

21. 7 Kg. 3 Hg. 2 Dg. 7 dg. 8 cg. by 35.

22. Find the total length of 19 poles, each 7 m. 4 cm. 3 mm. long.

23. Find the entire weight of 40 casks, each weighing 4 Kg. 7 Dg. 5 g.

24. Find the total area of 32 fields, each containing 36 Ha. 8 a.

25. Find the entire capacity of 16 cans, each containing 4.037 Hl. Change result to cu. meters.

Find the area of a surface:

26. 3.02 m. × 25 cm.
27. 94.5 Km. × 6.8 m.
28. 3.6 mm. × 7.5 Hm.
29. 8.8 dm. × 2.4 Dm.

Divide:

30. 1800 dl. by 90 Dl.
31. 540 Ha. by 6 a.
32. 3.8 cu. m. by 1.9 cu. mm.
33. 77 sq. mm. by 5 sq. Dm.

34. 17 Km. 4 Dm. 5 m. 6 cm. 5 mm. by 5.
35. 130 Ha. 9 a. 4 ca. by 8.
36. 28 Kg. 9 Hg. 4 dg. 3 mg. by 25.

37. A field containing 56 Ha. is 35 Dm. long. Find its width. If it had been 26 Dm. 5 cm. long, find its width.

38. How many cu. m. in a box 30.6 dm. × 204 cm. × 0.5 Dm.?

39. How many liters in a box 3 m. × 8 dm. × 5 cm.?

40. A tank 12 m. × 75.8 dm. × 1.05 Dm. is full of water. Find its weight in Kg.

41. A vat is 4 m. × 36 dm. × 250 cm. Find its capacity in Kl. Find in Kg. the weight of water it will contain.

42. A box 15 dm. long and 80 cm. wide contains a cu. m. How deep is it?

43. A room is 5.2 m. long, 4.5 m. wide, 3.2 m. high. Find cost of plastering it at 42 cents per sq. m. Find No. of Kl. of air in the room.

44. A bin 7.4 m. × 3.6 m. × 2.5 m. will contain how many Hl. of grain? How many steres of wood?

45. What cost a pile of wood 12.3 m. long, 5.2 m. wide, and a Dm. high, at $3.50 a stere?

46. A cellar is 12.4 m. × 9.6 m. × 8.5 m. How many cu. m. of earth were removed in digging it? How many steres of wood might be piled in it? How many liters of water would it contain? What does this volume of water weigh in kilograms? How many Hl. of corn would the cellar hold?

47. How many liters in a cube whose edges are all 20 cm.?

48. A tank contains 9600 Kg. of water. It is 32 m. long and 1.2 m. deep. How wide is it?

49. A bar of metal is 3 m. × 2.8 m. × 1.5 m. and weighs 4.5 times as much as an equal volume of water. Find the weight of this bar (Kg.).

A certain vat is 14 m. × 10.8 m. × 9.5 m.

50. Find its capacity in steres. In liters. In Kl.

51. Find the weight of water it will contain.

52. Find the weight of tar it will contain, if tar is 1.8 as heavy as water.

53. Find the value of wheat it will hold at $2.40 a Hl.

54. Find the total area of its 6 faces in ares.

55. Find the edge of a cubical cistern of same contents.

56. If a piece of ore weighs 45 Kg., and of this 675 g. are silver, what per cent. of the ore is silver?

57. If a certain wine contains 12.8% alcohol, and 2.1% extract, the rest being water, what % is there of water? How many Kg. of water are there in 50 l. of the wine? How many cu. cm. of extract?

58. Divide 0.005 cu. dm. by 0.02, and express the quotient as a decimal of a cu. m.

59. A bin which holds 70 Hl. stands on a base 26 dm. × 2 m. How high is it?

METRIC EQUIVALENTS.

347. Direct Equivalents.—The following table shows the relation between different important metric units and the units of weight and measure in common use in the United States and Great Britain. The table should be committed to memory, the equivalents printed in black letters being of especial importance.

1 meter	**= 39.37 in. = 1.1 yd.**	**($39\frac{3}{8}$ in. approx.).**
1 kilometer	**= .6214 mile**	**($\frac{5}{8}$ mi. approx.).**
1 sq. meter	= 1.196 sq. yd.	($1\frac{1}{5}$ sq. yd. approx.).
1 hektare	**= 2.471 acres**	**($2\frac{1}{2}$ A. approx.).**
1 cu. meter	= 1.308 cu. yd.	($1\frac{1}{3}$ cu. yd. approx.).
1 stere	= .2759 cord	($\frac{3}{11}$ cd. approx.).
1 liter	**= 1.057 liquid quart**	**($1\frac{1}{16}$ l. qt. approx.).**
1 liter	**= .9081 dry quart**	**($\frac{9}{10}$ d. qt. approx.).**
1 liter	= 61.022 cu. in.	(61 cu. in. approx.).
1 hektoliter	= 2.8376 bushels	($2\frac{5}{6}$ bush. approx.).
1 hektoliter	= 26.417 gallons	($26\frac{2}{5}$ gal. approx.).
1 gram	= 15.432 grains	($15\frac{2}{5}$ gr. approx.).
1 kilogram	**= 2.2046 pounds av.**	**($2\frac{1}{5}$ lb. av. approx.).**
1 metric ton	**= 1.1023 ton**	**($1\frac{1}{10}$ T. approx.).**

348. The **inverse equivalents**, showing the value of units in use in the United States and Great Britain in terms of the metric units, are also often useful.

1 yd.	=	.9144 m.	($\frac{9}{10}$ m.).
1 mi.	=	1.609 Km.	($1\frac{3}{5}$ Km.).
1 sq. yd.	=	.8361 sq. m.	($\frac{5}{6}$ sq. m.).
1 acre	=	.4047 Ha.	($\frac{2}{5}$ Ha.).
1 cu. yd.	=	.7645 cu. m.	($\frac{3}{4}$ cu. m.).
1 cord	=	3.624 st.	($3\frac{3}{5}$ st.).
1 liquid qt.	=	.9463 l.	($\frac{16}{17}$ l.).
1 dry qt.	=	1.101 l.	($1\frac{1}{9}$ l.).
1 bush.	=	.3524 Hl.	($\frac{6}{17}$ Hl.).
1 grain	=	.0648 g.	($\frac{5}{77}$ g.).
1 lb. av.	=	.4536 Kg.	($\frac{5}{11}$ Kg.).
1 ton	=	.9072 T.	($\frac{10}{11}$ T.).

Ex. 1. Reduce 30 m. to feet.

$$30 \text{ m.} = 39.37 \text{ in.} \times 30$$
$$= \frac{30 \times 39.37}{12} \text{ ft.} = \frac{393.7}{4} \text{ ft.} = 98.425 \text{ ft.}$$

Ex. 2. Find the area in hektares and also in acres of a field 50 Dm. long and 175 m. wide.

$$50 \text{ Dm.} = 500 \text{ m.}$$
$$\text{Area} = 175 \times 500 \text{ sq. m.} = 87500 \text{ sq. m.}$$
$$= 8.75 \text{ Ha. (since 10000 sq. m.} = 1 \text{ Ha.)}$$
$$= 2.471 \text{ A.} \times 8.75$$
$$= 21.62125 \text{ A., } \textit{Result.}$$

EXERCISE 170.

Change:

1. 5 mi. to Km.
2. 13 bu. to Hl.
3. 56 A. to Ha.
4. 45 gal. to l.
5. 20 cd. to st.
6. 13 cu. yd. to cu. m.
7. 60 lb. to Kg.
8. 50 Hl. to bu.
9. 25 Hm. to rds.
10. 30 m. to ft.
11. 32 st. to cd.
12. 425 a. to A.
13. 126 l. to gal.
14. 325 l. to pk.

15. 100 Kg. to cwt.
16. 40 cu. m. to bu.
17. 9 Kl. to bu.
18. 72 rd. to Hm.
19. 25 lb. to g.
20. 5 cu. m. to cu. ft.
21. 95 a. to sq. rd.
22. 8 t. to Kg.
23. 7 l. to pk.
24. 109 cu. cm. to cu. in.
25. 40 sq. m. to sq. ft.
26. 30 gal. to dl.
27. 4 a. to sq. yd.
28. 16 mm. to in.
29. 85 ml. to pts. (oats).
30. 43 pt. to cl. (milk).

31. 2 mi. 45 rd. to Km.
32. 4 bu. 3 pk. 3 qt. to Hl.
33. 3 gal. 2 qt. 1 pt. to l.
34. 40 A. 100 sq. rd. 6 sq. yd. to Ha.
35. 3 T. 15 cwt. 40 lb. to Kg.
36. 7.53 Ha. to acres and lower denominations.
37. 18.32 Hl. to bushels and lower denominations.
38. 7528 Kg. to tons and lower denominations.
39. 87.538 l. to gal. and lower denominations.
40. 24.25 Km. to miles and lower denominations.

Find the weight of:

41. 17 l. of water in pounds.
42. 28 Kl. of water in tons.
43. 6 cu. m. of water in tons.
44. 1000 gal. of water in Kg.
45. 25 cu. yd. of water in Hg.
46. 1 cu. in. of water in g.

47. How many cu. yds. in a room 3 Dm. × 25 m. × 120 dm.

48. How many tons of water will a cistern 26 m. × 20 m. × 155 dm. contain?

49. A box contains 35.2 l., and its lid is 0.44 m. × 25 cm. What is its height in inches?

50. A vat is 4.5 m. long and 18 cm. deep. It will hold 605.88 lbs. of water. What is its width in decimeters?

51. If I buy 360 bu. of wheat at \$.95 a bushel, and sell it at \$2.95 a hektoliter, how much do I gain?

52. A bin is 19 ft. × 12 ft. × 5 ft. How many Hl. will it contain?

53. How many acres in a field 75 m. long and 64 m. wide?

54. Atmospheric pressure is about 1 Kg. per sq. cm. How many pounds is that to the square foot?

55. From the datum that a liter of water weighs a Kg., compute the weight of a cubic foot of water in pounds.

56. If a train travels 50 miles an hour, how many meters is it moving each second?

57. Change your height and weight to equivalent metric units.

58. How many Kg. will a standard bbl. of water weigh?

59. A cubic inch of a certain material weighs $2\frac{1}{4}$ oz. How many Kg. in the weight of a block of it 11 yds. × 25 ft. × 16 ft.?

60. The circumference of a wheel is 4 m. 5 mm. How many revolutions will it make in 20 miles?

61. Explain how to construct the table of inverse equivalents (§ 348) from the table of direct equivalents (§ 347). By the use of only one table, what two rules will enable one to make any changes whatever from either system to the other?

62. The inside measurements of a cubical vessel are each 23 inches. How many liters of water will it contain? What will this water weigh in pounds?

63. How many quarts of milk will a box 3 dm. × 24 cm. × 15 cm. contain? How many quarts of corn?

64. Bought a farm of 220 acres at $60 an acre. For what must I sell it per Ha. to gain 20%?

65. Bought 560 bushels of grain at $1.70 per Hl. and sold it at 90 cents per bushel. What was my total gain and my gain %?

66. How many gallons in a tank 4.6 m. × 3 m. × 25 dm.?

67. A road 84 Km. long is 12 m. wide. What is the land worth at $15 an acre?

68. Bought wood at $4.40 a cord and sold it at $1.50 a stere. Did I gain or lose? What %?

69. The scale of a map is 1 to 80000. The distance between two cities on the map is 15 cm. How many miles between the cities?

70. On a map which is drawn on the scale of half an inch to the mile, 2 cities are 30 cm. apart. How many Km. are they really apart? How many miles?

71. When milk sells for 8 cents a quart, what is a Kl. of it worth?

72. If a pound of butter is worth 5 qts. of milk, how many liters of milk should a Kg. of butter be worth?

73. If a sq. rod of land is worth a stere of wood, how many Ha. of the same land are worth 500 cords of the same wood?

74. A cistern is 6 m. × 54 dm. × 4.5 m. Determine

- (*a*) how many liters of water it will contain.
- (*b*) what they will weigh in tons.
- (*c*) how many gallons it will contain.
- (*d*) how many bushels it will contain.
- (*e*) how many steres of wood can be piled into it.
- (*f*) how many ares in the sum of all its faces.
- (*g*) how many rods in the sum of all its edges.

349. The **specific gravity** of a body is the ratio between the weight of the body and the weight of an equal bulk of water.

Since a kilogram is the weight of a liter, or cubic decimeter, of water, the metric system affords peculiar advantages in dealing with problems relating to specific gravity.

Ex. A bar of iron 5 dm. long, 4 cm. wide, and 10 mm. thick, has a specific gravity of 7.8. Find its weight in kilograms and pounds.

4 cm. = 0.4 dm., 10 mm. = 0.1 dm.
Volume of bar = 5 × 0.4 × 0.1 cu. dm.
= 0.2 cu. dm.

Weight of an equal volume of water = 0.2 Kg.
" " the bar of iron = 0.2 Kg. × 7.8 = 1.56 Kg.
= 1.56 lb. × 2.2046 = 3.439 lb. } *Result.*

EXERCISE 171.

1. A block of stone contains 50 cu. dm. What is its weight (sp. gr. 6.6)? Answer in Kg. and in lbs.

2. An irregular stone (sp. gr. 6.5) weighs 11.7 Kg. How many cu. m. in its volume? How many cu. ft.?

3. The sp. gr. of lead is 11.3; how many tons of lead in a bar 4 m. × 3 m. × 2 m.?

4. How many cu. ft. in the wood of a brush-heap (sp. gr. 0.7), which weighs 27.3 Kg.?

5. What is the weight of a granite shaft (sp. gr. 3.2) 8 m. × 75 dm. × 620 cm.? (Answer in metric tons.)

6. A bar of iron (sp. gr. 7.8) is 12 ft. × 3 ft. × 2 ft. Find its weight in Kg.

7. A coil of gold wire (sp. gr. 19.3) weighs a ton. How many cu. dm. in the coil? How many cu. in.?

8. A sheet of zinc weighs 300 Kg. (sp. gr. 7). How many cu. dm. in the zinc?

9. A bar of steel 35 m. long and 2.6 sq. dm. on the end, weighs how many pounds? (Sp. gr. 7.8.)

10. If a bar of metal containing 86 cu. dm. weighs 1267.64 lbs., what is its sp. gr.?

11. If a cubic meter of copper weighs 19580 lbs., find its sp. gr.

12. If a cubic meter of cork weighs 660 lbs., find its sp. gr.

13. A tank 8 m. × 6 m. × 45 dm. contains 178.2 tons of oil. What is the sp. gr. of the oil?

14. A cubic yard of a certain substance weighs as much as a cubic meter of water. What is its sp. gr.?

15. A liter of a certain liquid weighs as much as a gallon of water. Find its sp. gr.

16. A cubic foot of ice weighs 27.84 Kg. Find its sp. gr.

17. What is the sp. gr. of silver, when 12 cu. dm. of silver weigh 277.2 lbs.?

18. What is the sp. gr. of air when the air in a room 7.2 m. × 6.5 m. × 35 dm. weighs 212.94 Kg.?

19. State carefully and correctly a rule for each case:

(*a*) given volume and sp. gr., to find weight.

(*b*) given volume and weight, to find sp. gr.

(*c*) given weight and sp. gr., to find volume.

20. A vessel when empty weighs 2.5 Kg., and when full of mercury (sp. gr. 13.5) it weighs 121502.5 Kg. Find the capacity of the vessel.

21. An empty vessel weighs 3.3 lbs., and when full of milk weighs 22.1078 lbs. How many liters will the vessel contain? (Sp. gr. milk 1.03.)

22. A vessel holding a Dl. weighs when empty 2.4 Kg., and when full of oil 22 lbs. Find sp. gr. of the oil.

23. Find the weight in grams of a pint of sulphuric acid (sp. gr. = 1.8.)

24. If a bar of copper (sp. gr. 8.9) is 1.2 m. × 0.4 cm. × 0.6 mm., find its weight in ounces. In grams.

25. If a bar is 3 dm. × 6.4 cm. × 55 mm., how many coins, each weighing 4.5 grams, can be made from it (sp. gr. = 12.3)?

26. If sulphuric acid has a sp. gr. of 1.84, how many liters are there in 14 Kg. of the acid?

27. A tank 2.5 m. × 12.6 dm. × 75 cm. is $\frac{2}{3}$ full of tar (sp. gr. 2.5). Find in pounds the weight of the tar.

28. A glass inkstand, in the form of a cube, 8 cm. on each edge, contains a dl. of ink. If glass is 3 times as heavy as water, what ought the empty inkstand weigh? (Ans. in Kg.)

29. A box 3.8 m. × 3.5 m. × 50 cm. contains a liquid 0.92 times as heavy as water. Find the weight of the liquid when the box is $\frac{3}{4}$ full. Find the total weight if the other part be filled with water.

30. If 100 cu. in. of air weigh 31 g., find the sp. gr. of air. Find also the weight in Kg. of a cubic meter of air.

31. A Kg. of gold (sp. gr. 19.36) is beaten into sheets 0.01 mm. thick. How many ares will it cover?

32. If sea-water is 2.8% salt, and has a sp. gr. of 1.025, how many Kg. of salt can be obtained from 1000 cu. m. of sea-water?

33. A rectangular block of an alloy is 3.4 m. $\times$ 16 dm. $\times$ 25 cm. If the sp. gr. of the alloy is 15, and the block contains 6.12 Kg. of brass, what per cent. of it is brass

34. A rectangular vessel, 10 cm. wide and 3 cm. deep, contains 3 Kg. of sea-water (sp. gr. 1.025). How long is the vessel?

35. A mixture is formed by diluting 1 l. of nitric acid with 2 l. of water. If the sp. gr. of the acid is 1.5, what is the weight of the mixture? What % of this weight is nitric acid?

36. It being given that a cu. in. equals 16.39 cu. cm., how many cu. in. of gold (sp. gr. 19.3) weigh 100 Kg.?

37. A tank containing a certain volume of liquid, which weighs .962 as much as an equal volume of water, is emptied at the rate of 25 l. per hour. At the end of 3 hours the tank contains 163.2 Kg. of the liquid. What was the weight in Kg. of the liquid in the tank at first? What was its volume in gallons?

38. What must be the height of a tank whose bottom is 1 m. 2 dm. $\times$ 7 dm., and which holds 414.54 Kg. of oil (sp. gr. 1.05)?

39. If 20 cu. cm. of iron weigh as much as 144 cu. cm. of water, what will be the weight in Kg. of an iron cube 21 cm. on each edge? Of a cubic meter of iron?

40. A block of wood is 0.1 m. $\times$ 0.15 m. and weighs 3 Kg. Find its 3d dimension (sp. gr. 0.8).

CHAPTER XXI.

ARITHMETICAL HISTORY.

HISTORY OF NUMERATION AND NOTATION.

350. Number Groups.—Arithmetic begins with the making of units into groups, and the dealing with number by some group method. The first grouping of units in almost all savage tribes is done by aid of the fingers, either of one hand or of both hands, or by use of all the fingers and the toes. Hence the first number groups were fives, tens, or twenties.

For instance, one South African tribe uses three persons for numeration purposes, the first to count units on his fingers, the second to count tens, and the third hundreds.

Five is the primary number group among tribes who do not count much beyond twenty, as those in North Siberia, in New Hebrides, and the Esquimaux; twenty was the primary group among the Phœnicians, Basques, Aztecs, and is so among most of the tribes of South America, and some of those of North America. Ten is the usual base among primitive peoples in the rest of the world. However, the Maories of New Zealand use eleven as a base (thus, for them, the symbols 13 would mean one 11 and 3 units, or 14).

The Indo-European races seem to have used twelve as a base to a great extent, owing probably to the fact that two, three, four, and six will all divide it exactly. Twelve is in fact the best practical base, but ten is now too well established to make possible a general change to twelve.

The ancient Babylonians used sixty as a base, for a reason which will be given later.

351. Number Words.—Owing to the difficulty which savage peoples have in forming and using abstract language, the number words used by them do not always correspond to the groups formed by them by the aid of their fingers and

33. A rectangular block of an alloy is 3.4 m. × 16 dm. × 25 cm. If the sp. gr. of the alloy is 15, and the block contains 6.12 Kg. of brass, what per cent. of it is brass

34. A rectangular vessel, 10 cm. wide and 3 cm. deep, contains 3 Kg. of sea-water (sp. gr. 1.025). How long is the vessel?

35. A mixture is formed by diluting 1 l. of nitric acid with 2 l. of water. If the sp. gr. of the acid is 1.5, what is the weight of the mixture? What % of this weight is nitric acid?

36. It being given that a cu. in. equals 16.39 cu. cm., how many cu. in. of gold (sp. gr. 19.3) weigh 100 Kg.?

37. A tank containing a certain volume of liquid, which weighs .962 as much as an equal volume of water, is emptied at the rate of 25 l. per hour. At the end of 3 hours the tank contains 163.2 Kg. of the liquid. What was the weight in Kg. of the liquid in the tank at first? What was its volume in gallons?

38. What must be the height of a tank whose bottom is 1 m. 2 dm. × 7 dm., and which holds 414.54 Kg. of oil (sp. gr. 1.05)?

39. If 20 cu. cm. of iron weigh as much as 144 cu. cm. of water, what will be the weight in Kg. of an iron cube 21 cm. on each edge? Of a cubic meter of iron?

40. A block of wood is 0.1 m. × 0.15 m. and weighs 3 Kg. Find its 3d dimension (sp. gr. 0.8).

CHAPTER XXI.

ARITHMETICAL HISTORY.

HISTORY OF NUMERATION AND NOTATION.

350. Number Groups.—Arithmetic begins with the making of units into groups, and the dealing with number by some group method. The first grouping of units in almost all savage tribes is done by aid of the fingers, either of one hand or of both hands, or by use of all the fingers and the toes. Hence the first number groups were fives, tens, or twenties.

For instance, one South African tribe uses three persons for numeration purposes, the first to count units on his fingers, the second to count tens, and the third hundreds.

Five is the primary number group among tribes who do not count much beyond twenty, as those in North Siberia, in New Hebrides, and the Esquimaux; twenty was the primary group among the Phœnicians, Basques, Aztecs, and is so among most of the tribes of South America, and some of those of North America. Ten is the usual base among primitive peoples in the rest of the world. However, the Maories of New Zealand use eleven as a base (thus, for them, the symbols 13 would mean one 11 and 3 units, or 14).

The Indo-European races seem to have used twelve as a base to a great extent, owing probably to the fact that two, three, four, and six will all divide it exactly. Twelve is in fact the best practical base, but ten is now too well established to make possible a general change to twelve.

The ancient Babylonians used sixty as a base, for a reason which will be given later.

351. Number Words.—Owing to the difficulty which savage peoples have in forming and using abstract language, the number words used by them do not always correspond to the groups formed by them by the aid of their fingers and toes. For language purposes they seem at first to use smaller groups of units.

Not a few tribes have no number words beyond two; they count "one, two, many." Others have a binary system, in that for four they use "two-two"; for six, "two-two-two," etc.

The Campos of Peru count to three; for four, they say "one and three"; for five, "two and three."

If a tribe has a number word for four, it is almost sure to go one step further and have a word for five, and thus reach a quinary system. Five is often expressed by the word for hand; six as "hand one"; seven as "hand two," etc.; ten as "both hands"; twelve as "two on the foot"; twenty as "the whole man"; sixty as "three men," etc.

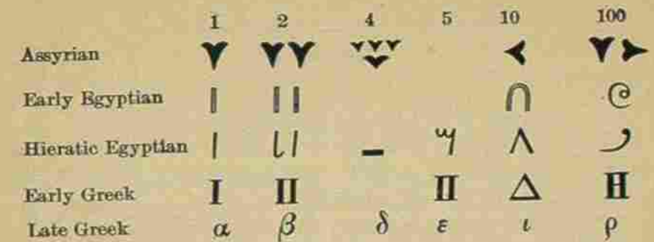
The number words "one, two, three," etc., no doubt originally had similar concrete meanings, but these were early lost, as it was an advantage, when number words were much used, to have purely abstract terms for them.

As civilization developed, primitive systems of numeration survived along with later systems, and mixed systems resulted. Thus in our use of "pair, brace, couple," the binary system survives along with our decimal system. In our use of score (as in "three score and ten," see also the French "quatre vingt" for eighty), the vigesimal system appears.

352. Number Symbols.—The first number symbols were fingers held up, or pebbles laid aside, or scratches made on some object, as wood or stone.

The Greeks, by using the separate joints of the fingers, could indicate numbers up to 10,000 on the hands. Proceeding from the little finger of the left hand through to the little finger of the right hand, the joints of the first three fingers denoted units; those of the next two fingers denoted tens; of the next two, hundreds; and of the last three, thousands. The Chinese to-day, by using the two sides and the front of each finger-joint as symbols, express 100,000 on the left hand alone.

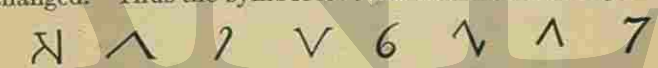
The following are illustrations of early written symbols used for numbers:



Number symbols are combined in these early systems in additive or multiplicative ways (thus, ▼▼ for "two" is an example of the additive use of the symbol for one; ▼➤ of the multiplicative use of symbols).

The Romans also use a subtractive principle in combining symbols, as in IV., IX., etc.

The positional (or exponential) system of written symbols was used to some extent by the Babylonians, but was rediscovered by the Hindoos, and the zero symbol invented, about 400 A. D. The figures 1, 2, 3, 4, etc., were originally the initial letters of the Hindoo words for the corresponding numeral adjectives, but the form of some of them has been much changed. Thus the symbol for 7 has had the following forms:



The Semitic peoples write from right to left, the Chinese from top to bottom, the Aryan peoples from left to right. Similarly, in writing numbers, each of these peoples as a rule follows its own order, putting the symbols for the largest groups first.

353. Higher Number Words.—The Hindoos used a separate name for each order of units; thus, they read 52965378196 as "5 kharva, 2 padwa, 9 vyarbada," etc. This system required the use of an unnecessarily large set of number words.

In modern times, the Italians grouped the digits into periods, sometimes of six, sometimes of three figures. The number given above would at one time have been read by them thus, "52 thousand thousand thousand, 965 thousand thousand, 378 thousand, 196."

The word *million* was invented by the Italians in the fourteenth century, and words *billion*, *trillion*, etc., by the French about the year 1500.

At that time figures were generally separated into periods of six figures each, hence, billion meant one million million, trillion meant one million billion, etc. These words continue to have these values in England, Germany, and the north of Europe generally. About the year 1750 it became the custom in France to divide figures into periods of three figures each; hence billion came to mean one thousand million, trillion one thousand billion, etc., which is the meaning now assigned to these words in the United States, France, and the south of Europe generally.

HISTORY OF ARITHMETICAL OPERATIONS.

354. Finger Reckoning.—The ancient Greeks, for example, had methods of performing addition, subtraction, etc., by a finger symbolism. The precise methods employed are not now understood, but there is a possibility that they may yet be worked out by a study of Greek monuments and literature. Finger reckoning was also much used in the Middle Ages in the monastic work of calculating the date of Easter, etc.

355. Abacus.—The method of counting by tens early led to the invention of the abacus.

This instrument had many different forms among different peoples, as the Egyptians, Chinese, Greeks, and Romans. The typical form is a rectangular frame containing parallel wires, on each of which are 9 buttons or counters. The counters on the wire to the extreme right (or left) represent units; those on the next wire represent tens, etc.

Let the teacher show the class an abacus, and how addition and subtraction are performed on it. Multiplication is performed by successive additions, and division by successive subtractions. Thus, to multiply 37 by 64,

$$37 \times 64 = (30 + 7)(60 + 4) = 30 \times 60 + 30 \times 4 + 7 \times 60 + 7 \times 4$$
$$= 1800 + 120 + 420 + 28.$$

Frequently a flat board covered with dust was used as an abacus. Lanes or columns were marked out in the dust, and in these pebbles were used as counters. Sometimes grooves were cut in a metallic plate, and movable buttons used in the grooves. Abacus reckoning was modified into reckoning with marks on horizontal lines, called counters. This was used in England as late as the seventeenth century, and Shakespeare makes allusions to it.*

Addition and subtraction, and even multiplication can be performed rapidly with the abacus, but its use has the serious disadvantage that no record is kept of the steps of the work.

356. Addition and Subtraction.—The Hindoos performed their arithmetical work upon a small board, on which they made large marks by a cane or brush. To save space, they erased a figure as soon as it had done its service. Hence, their methods of operation differed in some respects from those employed at present. The Hindoos usually performed addition and subtraction from left to right, and set the result above the numbers added or subtracted instead of below. Thus, to add 376 and 258, they would arrange the work as at the right, and say "3 and 2 = 5, set down 5 above 3; 7 and 5 = 12, erase 5 and put 6 in its place, set down 2; 8 and 6 = 14, erase 2 and put 3 in its place, and set down 4 above 6."

```
634, Sum.
376
258
```

The Arabs followed the same methods, except that they crossed out figures and wrote others above them, instead of erasing them. Thus, to subtract 275 from 653, they proceeded in the manner indicated at the right.

```
 37
488, Difference.
653
275
```

The present method of subtracting from right to left and setting results below came into use in Europe after the year 1200 A. D.

357. Multiplication was performed by the Hindoos in several different ways, of which the following two are the most representative. Ex. Multiply 157 by 62. The multiplier is written below and the product above the multiplicand. Thus, $6 \times 1 = 6$, set down 6 above 1; $6 \times 5 = 30$, erase 6 and put 9 in its place; set down 0 above 5; $6 \times 7 = 42$, etc.

```
   7
   6
 943
6024
 157
  62
```

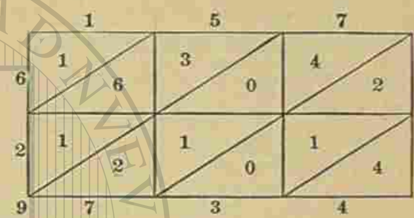
* Othello, act I., scene 1, line 31, "counter-caster." Cymb., V., 4, 167. As You Like It, II., 7, 63.

The Hindoos erased figures so that the result of their work would appear thus:

```
9734
157
62
```

The Arabs crossed out figures, leaving the work as given above.

The Hindoos also used a diagonal method of multiplication. The multiplication of 157 by 62 by this method is here given:



The method of multiplication commonly used at present is found in Pacioli (1494 A. D.), but it had been occasionally used long before.

A kind of multiplication called complementary multiplication, or sluggard's rule, was much used in the Middle Ages. In working with it the multiplication table was not needed beyond 5×5, but the method was tedious in operation. The principle on which it was based is as follows: If a and b represent digits, then,

$$a \times b = (10 - a)(10 - b) + 10(a + b - 10).$$

For example, $7 \times 6 = 3 \times 4 + 10(13 - 10) = 12 + 30 = 42$.

358. Division.—In all ancient mathematics we find no idea of a quotient. Division is performed by successive subtractions. After the Hindoos devised our present system of notation, they performed division by writing the divisor below the dividend, and setting down and erasing remainders above.

Ex. Divide 8479 by 36.

36 is contained in 84 twice; $2 \times 3 = 6$, 6 from 8 leaves 2, which set down above 8; $2 \times 6 = 12$, erase 2 and set down 1 above it, 2 from 4 leaves 2, which set down above 4, etc.

```
 11
134
2299
8479|235
36  |
```

The quotient is 235, with remainder 19.

As the Hindoos erased figures instead of scratching them, the result of their work would appear as follows:

```
  19|
8479|235
36  |
```

This "scratch" or "galley" method continued to be the favorite method of division in Europe till about the year 1700.

Our present method is given by Pacioli (1494), and is called by him the method "giving" (*i. e.*, bringing down one more figure after each subtraction), but he prefers the other method.

359. Factors. Primes.—The Greeks classified numbers in a great variety of ways; for example, as triangular, perfect, defective, excessive, etc.

The distinction of *odd* and *even* is due to Pythogoras (550 B. C.). Euclid discusses the properties of *prime* numbers (300 B. C.). Eratosthenes (200 B. C.) invented the method of determining primes, called the "sieve."

The Hindoos discovered the short way of determining whether a number is divisible by 3 or 9. They used this property of numbers in a method of verifying operations, called "casting out the nines," which is still used to some extent.

HISTORY OF FRACTIONS.

360. Fractions presented great difficulties to early peoples. An early Egyptian MS. (dating 1500 B. C.) shows that the Egyptians used only those fractions which have unity for a numerator.

This MS. gives tables by which other fractions can be reduced to these unit fractions. Thus, $\frac{2}{35} = \frac{1}{30} + \frac{1}{42}$. Fractions were written by writing only the denominator with a dot or special mark over it. By this method, even the addition of fractions was extremely complex.

The Babylonians used only those fractions which have 60 for a denominator (called sexagesimal fractions).

These fractions were expressed by writing the numerator a little to the right of its ordinary position, and omitting the denominator. The use of sexagesimal fractions survives in our present system of dividing a degree into 60 minutes, etc.

The Greeks used both the unit fractions of the Egyptians, and fractions of any numerator or denominator.

They indicated unit fractions by writing only the denominator with an accent, thus, ριδ′ means $\frac{1}{114}$. If the numerator was not unity, they wrote it with an accent, and the denominator twice, with a double accent. Thus, ιδ′ κγ″ κγ″ means $\frac{14}{23}$.

The Romans used only those fractions which have 12 for a denominator (called duodecimal fractions), and a few others derived from them, as $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{72}$, $\frac{1}{144}$, $\frac{1}{288}$.

The addition and subtraction of such fractions present no difficulty, but multiplication and division are extremely complex.

The Hindoos used fractions in general, writing the numerator above the denominator, with no line between. Thus, $\begin{smallmatrix}2\\3\end{smallmatrix}$ means $\frac{2}{3}$, and $\begin{smallmatrix}5\\2\\3\end{smallmatrix}$ means $5\frac{2}{3}$.

Methods of obtaining the L. C. D. of fractions are given by Tartaglia (1556 A. D.).

The method of dividing by a fraction by inverting the divisor and multiplying, is given by Stifel (1544).

361. Decimal Fractions.—Several close approaches were made to decimal fractions before they were finally invented. Thus, in the Middle Ages, the square root of 3 was extracted by annexing six zeroes to the 3, extracting the square root of 3000000, and writing the last three figures of the root over 1000, thus $1\frac{732}{1000}$; but the part of the root $\frac{732}{1000}$ was at once converted into sexagesimal fractions.

Rudolff (1525) divided a number by 1000 by marking off the last three figures with a comma.

Stevinus (Belgium, 1548–1620) was the inventor of decimal fractions. As he had no decimal point, however, his notation was clumsy. Thus he expressed 3.912 either as $\overset{0}{3}\overset{1}{9}\overset{2}{1}\overset{3}{2}$, or $3_{(0)}9_{(1)}1_{(2)}2_{(3)}$, and read it "3 and 9 primes 1 sekonde 2 terzes." etc.

Later 3.912 was written $3|\underline{912}$.

The decimal point was first used by **Pitiscus (Germany, 1612)**.

Decimal fractions at first were used in a very limited way, as in the calculation of interest. They did not come into general use till after the adoption of the metric system (1799).

HISTORY OF COMPOUND QUANTITIES.

362. The earliest **units of length** were taken from convenient parts of the human body, as the *digit* (a finger breadth), *palm*, *span*, *foot*, *cubit*, *ell*, the *fathom* (the extended arms). These units were convenient, being always at hand, but were not uniform enough when transactions were required to be exact. Later, the length of some natural object, as a *grain of* barley, became the unit of length. Finally, the length of some *piece of metal*, kept in the government archives, was used as a standard.

In very early times (in Egypt, Assyria, Canaan) two principal units of length, the *digit* and the *cubit*, were used.

The *foot* first came into general use in Greece and Rome, and from Rome it spread all over Europe. The Romans divided the foot into 12 "*unciae*" or *inches*.

In the year 1324, English law first defined the length of 3 barley-corns as equal to 1 inch, 12 inches = 1 foot, etc.

363. Of **units of weight**, the *pound* or *libra* originated in Rome, and from Rome was handed down to the various European peoples.

The Romans divided the libra into 12 *unciae* or *ounces* (thus the words ounce and inch each mean one-twelfth). The Greeks at times also divided the pound into 16 parts.

In the Middle Ages it became customary for merchants to make their profits in many cases, not merely by buying goods at one price and selling them at another, but by buying goods according to one kind of a pound (or other measure) and selling them by another, just as coal is now often bought by the long ton and sold by the short ton. In this way many different kinds of pound (and of other measures) arose, each trade or guild often having its own. Thus, the *Troy pound* was one used at a famous fair at the city of Troyes in France. Many changes and customs also arose which are now difficult to trace.

In the year 1266 English law fixed the weight of 32 barley-corns as equal to 1 pennyweight, 20 pennyweights = 1 oz., 12 oz. = 1 lb.

364. Of **units of capacity** a *bushel* (diminutive of box) measure was kept in the town hall at Winchester, the ancient

Saxon capital of England. This was the standard bushel in England till the year 1826, when the Imperial bushel of 2218.192 cu. in. was adopted by law. The Winchester bushel, however, continues to be standard in the United States.

365. Of **units of value** the *libra*, or pound of silver, was used in the Roman empire. From it are derived the *pound* of Great Britain, the *livre* of France, the *lira* of Italy, etc.

These were all originally of the same value, about \$15, but the currency of each country was debased by the government at different times, till in England the unit now has but $\frac{1}{3}$ its original value, in France, $\frac{1}{75}$, etc. *Sterling* means easterling, referring to the coinage of the Hanseatic League, to the east of Great Britain.

366. In **units of time**, the Babylonians divided the day into 24 hours, the hour into 60 minutes, and the minute into 60 seconds. The month is determined approximately by the time it takes the moon to go round the earth, $29\frac{1}{2}$ days.

The Babylonians divided the circle into 360 degrees for convenience in astronomical work, since 360 in a close approximation to the number of days in the year, and then divided the circumference into 6 equal parts of 60 degrees each, because they knew that a radius applied as a chord 6 times exactly completes a circumference. Hence, probably arose the whole system of sexagesimal notation.

367. The **metric system** was adopted in France in 1799. The theory of the system is that the meter is $\frac{1}{10000000}$ of a quadrant of the earth's circumference through Paris, though owing to an error in the calculation it is actually a very small fraction less. Hence, the meter as used must be taken as the distance between two marks on a bar of platinum kept in Paris.

The liter, gram, and other units are derived from the meter in the manner described in Chapter XX.

The metric system has been adopted in all countries of the civilized world except Great Britain and the United States. It is used in such countries as Mexico, Hayti, Congo Free State, etc.

HISTORY OF OTHER TOPICS AND PROCESSES.

368. Percentage and **Interest** were used among the Romans, but these took their modern form among the Italians (especially at Florence, where bookkeeping by double entry was also invented).

Many mistakes in computing discount were made, and the method of true discount was not established till about the year 1700.

Equation of Payments is treated by Tartaglia (1556).

Exchange was developed to its present form among the Dutch.

369. Proportion, or the **Rule of Three,** till early in the nineteenth century, was used to include almost all the operations of arithmetic except the fundamental ones, and that in a very mechanical and superficial way. At one time eleven different kinds of proportion were used. During the nineteenth century an intelligent method of analysis has gradually taken the place of the mechanical "Rule of Three."

Partnership problems occur in Ahmes' treatise (Egypt, 1500 B. C.).

370. Involution and **Evolution** were performed by the Hindoos much as at present.

For other details of the history of arithmetic, the student is referred to Cajori's History of Elementary Mathematics, and to Fink's Brief History of Mathematics (translated by Beman and Smith).

EXERCISE 172.

A MISCELLANEOUS EXERCISE.

1. Find, to 3 decimal places, the number of gallons in a bushel.

2. Divide twelve per cent. of four hundred sixty by two-thirds of seven and two-tenths.

3. Reduce 2.4637 years to lower denominations.

4. What is the value of:

$$37 \times 42 + 86 - (738 - 528 \div 4) + 19 \times 17 - 1300.$$

5. Change $\frac{5}{8}$ acre to lower denominations.

6. Compute: .01 of $\frac{3}{4} \times 200 \times .08\frac{2}{3} \div .035$.

7. Subtract the sum of $9\frac{3}{5}$, $8\frac{7}{8}$, $4\frac{9}{16}$ from the sum of $7\frac{5}{12}$, $8\frac{3}{16}$, $10\frac{1}{18}$.

8. Reduce 75 rd. 3 yd. 1 ft. 5 in. to inches.

9. Find $2\frac{2}{3}$% of $295.

10. Compute the interest of $270 at 4% for 3 yr. 15 days.

11. What will $3\frac{1}{4}$ acres of land cost, if $7\frac{1}{5}$ acres cost $\$655\frac{1}{2}$?

12. Find the loss per cent. when a horse which sold for $225 cost $325.

13. Find the H. C. F. and L. C. M. of 473, 516, 559.

14. At $\frac{7}{8}$ dollar each, how many books can be purchased with $\$17\frac{1}{2}$?

15. If a man can mow a lawn in 6 days and his boy can do it in 9 days, how many days will they both require to do it, working together?

16. Change the following fractions to other equivalent fractions having 72 for their denominator:

$$\frac{11}{12},\ \frac{7}{24},\ \frac{8}{9},\ \frac{35}{45},\ \frac{50}{16},\ \frac{21}{56},\ \frac{33}{108}, \text{ and } \frac{24}{27}.$$

17. If 9 be added to both terms of the fraction $\frac{13}{16}$, will the value of the fraction be increased or diminished?

18. If the divisor is $\frac{3}{4}$, the quotient $\frac{2}{3}$, and the remainner $\frac{1}{5}$, what was the dividend?

19. How many bushels of corn at $\$\frac{2}{3}$ a bushel will pay for $\frac{4}{5}$ barrel of flour at $\$6\frac{1}{3}$ a barrel?

20. A carpenter worked $23\frac{1}{3}$ days and paid $\frac{3}{7}$ of his earnings for board and other expenses. If he saved $\$53\frac{1}{3}$ in this time, what was his daily wage?

21. A and B can spade a garden in 5 days, but B alone could do it in 7 days. How long would A require?

22. Reduce 5 wk. 3 da. 11 hr. 16 min. to minutes.

23. Find the actual gain if a selling price of $1320 was a gain of 10%.

24. Add $8\frac{1}{4}$, $10\frac{5}{7}$, $12\frac{2}{3}$, $5\frac{13}{42}$, $9\frac{10}{21}$, $17\frac{5}{6}$.

25. Simplify $(2\frac{3}{4} \times 11\frac{3}{7}) \div (\frac{2}{3} \text{ of } 18\frac{3}{5} \times 1\frac{4}{7})$.

26. Find the least whole number that is exactly divisible by $4\frac{1}{5}$, $3\frac{1}{2}$, $4\frac{2}{3}$.

27. A certain ore contains $8\frac{1}{2}$% of metal. How much metal will be obtained from 75 tons of ore?

28. A man lost 10% in selling a carriage for $234. What should he have sold it for to gain 10%?

29. What is the difference between the $\sqrt[3]{10.01}$ and the $\sqrt{10.01}$ expressed in 3 decimal places?

30. Reduce 5.1735 mi. to lower denominations.

31. If $\frac{11}{12}$ of one line is $\frac{14}{15}$ of another, which line is the greater?

32. After I sold $\frac{2}{5}$ of my apple crop to one man and $\frac{3}{4}$ of the remainder to another there were 186 barrels left. How many barrels were there in the crop?

33. If a man can repair $\frac{5}{8}$ of a bridge in 10 days and his brother can repair $\frac{2}{3}$ of it in 6 days, how long would it require them both to repair the entire bridge, working together?

34. If the circumference of a wheel is 3 yd. 11 in., how many revolutions will it make in going a mile?

35. A man spent $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{5}$ of his money and had $2613 left. How many dollars did he spend altogether?

36. Of the earth's surface $39.87\frac{1}{2}$% lies in the Torrid Zone, and $25.91\frac{1}{2}$% lies in the Temperate Zones. What part lies in each Frigid Zone?

37. A tank's inside dimensions are 3 ft. 4 in. by 2 ft. 6 in. by 1 ft. 10 in. How many gallons of water will it contain?

38. How many bushels of grain will it hold?

39. Reduce 8 mi. 5 yd. 4 in. to inches.

40. Find the interest of $916.50 for 4 yr. 6 mo. 24 da. at 4%.

41. Write a list of the prime numbers between 200 and 300.

42. A merchant marks goods 20% above cost and sells them $12\frac{1}{2}$% below marking price. What is his per cent. of gain?

43. Two cities have longitude 90° 15′ 16″ W. and 30° 20′ 14″ E. respectively. What is their difference in time?

44. If a man buys stock at 40% discount and every three months receives a dividend of 2% on the par value of the stock, what annual rate of interest does he receive on his investment?

45. What sum of money put at interest for 7 yr. 6 mo. 12 da. at $3\frac{1}{2}$% will gain $1392.16 interest?

46. The expenses of a town for a year are $7324 and the balance in treasury is $696. There are 6862 polls to be assessed at $0.25 each, and taxable property amounting to $1965000. Besides the town tax there is a county tax of $1\frac{1}{4}$ mills and a state tax of $\frac{3}{4}$ mill on every dollar of taxable property. Mr. A. pays for 2 polls and has property worth $28970. Find his total tax.

47. How many rails will be required to fence a field 5456 yd. long and $4051\frac{2}{3}$ yd. wide, provided the fences are all straight, all 6 rails high, and the rails of equal length, and the longest that can be used without cutting any?

48. What is the smallest sum of money with which I can purchase either chairs at \$8 each, or desks at \$24, or tables at \$52, or couches at \$72?

49. A farmer planted $\frac{3}{8}$ acre on Monday, $\frac{2}{7}$ acre on Tuesday, $\frac{3}{5}$ acre on Wednesday, $\frac{1}{4}$ acre on Thursday, $\frac{4}{15}$ acre on Friday, and the rest of his 2-acre lot on Saturday. Find on which day he planted the most ground and on which day the least.

50. A owned $\frac{5}{7}$ of a store and sold $\frac{2}{3}$ of his share to B, who sold $\frac{3}{5}$ of what he bought to C. C, in turn, sold $\frac{4}{5}$ of his purchase to D. What part of the entire store did each then own?

51. If I paid \$40 an acre for some land, how much must I ask for it, that I may abate 25% from my asking price and still gain 30% on the cost?

52. Divide $\dfrac{\frac{3}{4} \times (\frac{4}{5})^2 \text{ of } 4\frac{1}{5}}{\frac{2}{5} \text{ of } (2\frac{1}{4})^3 \times (2\frac{1}{2})^4}$ by $\dfrac{(\frac{4}{7})^2 \times 8\frac{3}{4}}{\frac{10}{11} \text{ of } (1\frac{2}{3})^3 \times (3\frac{3}{4})^3}$.

53. The front wheel of a wagon was 11 ft. in circumference and the rear wheel was 13 ft. A screw in the tire of each was uppermost when the wagon started, and when it stopped the same screws were uppermost again for the 633d time. How many miles had the vehicle traveled?

54. A real estate agent sold $5\frac{5}{9}$ acres at $\$138\frac{3}{5}$ each; $12\frac{5}{8}$ acres at $\$118\frac{3}{8}$ each; and $20\frac{11}{12}$ acres at \$123.60 each. Find the number of acres sold, the aggregate price, and the average price per acre.

55. If I buy a lot and it increases in value each year at the rate of 50% over the value of the previous year for 5 years and then is worth \$9000, how much did it cost?

56. What is the value of

$$\{6\tfrac{1}{4} \times (\tfrac{2}{5})^2 + \tfrac{2}{3} \text{ of } \tfrac{3}{4}\} \div \{18\tfrac{1}{5} - 14\tfrac{1}{3} + \tfrac{7}{24} \div (\tfrac{5}{6})^2 \times \tfrac{4}{7}\}\,?$$

57. A farmer having a triangular piece of land, the sides of which are 481 ft., 1144 ft., and 1469 ft., wishes to enclose it with a fence having panels of the greatest possible uniform length. What will be the length of each panel?

58. What number is that from which if $11\frac{2}{3}$ be subtracted, $\frac{3}{5}$ of the remainder is $110\frac{4}{5}$?

59. A woman at her death left her son \$11640, which was $\frac{1}{2}$ of $\frac{3}{4}$ of her wealth. He at his death left $\frac{3}{5}$ of his portion to his daughter. What part of her grandmother's estate did the daughter receive? (Compute this fractional part two distinct ways.)

60. An agent wishing to sell a house and lot asked 40% more than it cost. But he finally sold it for 20% less than his asking price, thereby gaining \$4896. How much did the house and lot cost? What was his asking price? What was the selling price?

61. Reduce 3 lb. 8 oz. 19 pwt. 6 gr. to grains.

62. Which is the greater, $\sqrt{50}$ or $\sqrt[3]{346}$?

63. Divide 38 mi. 100 rd. 5 yd. 2 ft. by 6.

64. A man bought wood for $\$287\frac{3}{7}$ and coal for $\$384\frac{5}{8}$ and oil for $\$76\frac{5}{13}$. He sold the wood for $\$327\frac{4}{5}$ and the coal for $\$375\frac{1}{6}$ and the oil for $\$88\frac{7}{10}$. How much did he gain on all?

65. A merchant bought 3 hhd. of molasses, each containing 63 gal., at 40 ¢ a gal. and paid \$6.75 for freight and cartage. Allowing 4% for waste and 5% of sales for bad debts and 2% of the remainder for collecting, what must he charge per gallon in order to make 27% on the entire cost?

66. Find the G. C. D. of 2538, 4089, 4324.

67. If cloth $1\frac{1}{2}$ yd. wide require $8\frac{2}{3}$ yd. in length for a suit of clothes, how many yd. in length will cloth $\frac{2}{3}$ yd. wide require for same suit?

68. Reduce 207958 in. to higher denominations.

69. Collect: $\dfrac{0.7 \text{ of } 4.5 \times 6.8}{0.17 \text{ of } 4.2 \times 9} + \dfrac{3.9 \times 5.7 \times 1.6}{0.64 \text{ of } 1.9 \times 13}$.

70. Of a certain ore 6% is iron and 45% is rock, the rest being conglomerate. In a car of ore weighing 20 tons, how much is iron and how much rock?

71. Compute the compound interest on \$2560 for 3yr. 5 mo. 10 da. at 3%.

72. Find the G. C. D. of 2680, 1541, 2211.

73. A man owning 80% of a store sold $\frac{2}{3}$ of his share for $\$6781\frac{4}{5}$; what was the value of the entire store at same rate?

74. Change 16 wk. 5 da. 9 hr. 40 min. 20 sec. to seconds.

75. A square field contains 3 acres. Find the length of each side in rods.

76. What is the edge of a cubical box that will just contain a bushel? Of another that will exactly contain a gallon?

77. If $\frac{5}{8}$ lb. of sugar be worth $\frac{1}{4}$ lb. of butter, and butter be worth $\$\frac{3}{8}$ per pound, how many pounds of sugar will \$75 buy?

78. I gained $33\frac{1}{3}$% in selling a gray horse, and with the money bought a black horse which I sold for \$120, losing 20%. On the two horses did I gain or lose? What per cent.?

79. A can do a piece of work in 15 days; A and B together can do it in 10 days; A and C can do it in 6 days. How many days will B and C require, working together?

ANSWERS.

Exercise 4.

1. 76 men.
2. 97 mi.
3. 559 hr.
4. 978 boys.
5. $8778.
6. 74799.
7. 988899.
8. 986936.
9. 978889.
10. 91 men.
11. 142 pens.
12. 123 pages.
13. 135 balls.
14. 1185 books.
15. $11923.
16. 121291.
17. 1041400.
18. 1293154.
19. 920901.
20. 155554.
21. 952 ft.
22. 1781 lines.
23. 1653 rd.
24. 1469 mk.
25. $146991.
26. 130755.
27. 1620506.
28. 1657686.
29. 199379.
30. 267411.
31. 2364.
32. 2727.
33. 26182.
34. 22545.
35. 288231.
36. 359012.
37. 278954.
38. 388149.
39. 3722639581.
40. 373197661105.
41. 40814.
42. 341011.
43. 53140.
44. 1352375.
45. 98553.
46. 80459.
47. 96604.
48. 88500.
49. 92338.
50. 91606.

Exercise 6.

1. 605 bu.
2. 167 A.
3. $4305.
4. 2600.
5. 1640 A.
6. $1130.
7. $10363.
8. 2619 mi.
17. 53011.
18. 75686.
19. $149174.
20. 265780 sq. mi.
21. 3784219.
22. 4457114.

Exercise 7.

1. 42 men.
2. 222 boys.
3. 220 balls.
4. 11 mi.
5. 504.
6. 104 men.
7. 502 boys.
8. 413 balls.
9. 413 mi.
10. 203.
11. 34.
12. 15.
13. 18.
14. 69.
15. 182.
16. 115.
17. 579.
18. 168.
19. 176.
20. 392.
21. 557.
22. 93.
23. 3234.
24. 859.
25. 5149.

26. 13028.
27. 18308.
28. 19590.
29. 3750667.
30. 131062348.
31. 1099077733354.
32. 24468929016658.
33. 3808473092159.
34. 11910410694969481.
35. 219571835427077356.
36. 4197533343580235802.

Exercise 8.

2. $11.
3. $153.
4. 805 yd.
5. 1395 bu.
6. Lost $19.
7. $1142.
8. 42494 bu.
9. 508 yr.
10. $8.
11. 2554 men.
12. $7381.
13. $9160.
14. 83 yr.; 157 yr.
15. 5714.
16. $9.
17. $614555.
18. $614.
19. $546.
20. $5958.
21. 10.
22. 28.
23. 66.
24. 21.
25. 26.
26. 95.
27. 252.
28. 199.
29. 19.
30. 151.
31. 377.
32. 4479.
33. 50.
34. 72.
35. 73.
36. 134051.
37. 167307.
38. 198.
39. 7545.
40. 3882.
41. 1941.
42. 10588.
43. 26.
44. 17.
45. 119.
46. 85.
47. 27.
48. 522.
49. 289.
50. 1773.
51. 1125.
52. 527.
53. 7191.
54. 46704.

Exercise 9.

20. 19.
21. 43.
22. 19.
23. 28.
24. 52.
25. 12.
26. 17.
27. 58.
28. 42.
29. 74.
30. 25.
31. 62.
32. 40.
33. 133.
34. 72.
35. 68.
36. 3.
37. 3.
38. 25.
39. 38.
40. 45.
41. 2.
42. 0.
43. 4.
44. 100.
45. 62.
46. 53.
47. 10.
48. 4.
49. 28.
50. 20.
51. 14.
52. 32.
53. 51.
54. 60.
55. 49.
56. 3.
57. 52.
58. 22.
59. 34.
60. 30.
61. 10.
62. 37.
63. 81.

Exercise 10.

1. 72 hr., 120 hr., 168 hr., 216 hr.
2. 224 lb., 504 lb., 336 lb.
3. 264 ft., 462 ft., 528 ft.

4. 784 lb., 980 lb., 1764 lb.
5. 1095 da., 1825 da., 2555 da.
6. $2900, $4350, $5800.
7. 1792 boys.
8. 3512 girls.
9. 4296 ft.
10. 8645 men.
11. $16818.
12. $54024.
13. $35334.
14. 296149 yd.
15. 54078 ft.
16. 117648 in.
17. 13112 min.
18. 248808 rd.
19. 8274.
20. 15036.
21. 233945.
22. 472182.
23. 376012.
24. 54537.
25. 36536.
26. 34227.
27. 43225.
28. 121383.
29. 2806032.
30. 2585160.
31. 5259186.
32. 723765442616.
33. $70.
34. 94 ct.
35. $500.
36. $1575.
37. $215.
38. $9085.

Exercise 11.

1. $1200.
2. $2075.
3. 3168 boys.
4. 5888 pens.
5. 21170 da.
6. $25929.
7. $42364.
8. 46125 ft.
9. 127484 da.
10. 86140 da.
11. $247000.
12. $344634.
13. 238150.
14. 469476.
15. 241152.
16. 565785.
17. 1603550.
18. 32005260.
19. 28597105.
20. 15312780.
21. 28804616.
22. 88564080.
23. 156530435.
24. 203058102.
25. 87831900.
26. 576267915.
27. 272916080.
28. 234262788.
29. 66988960.
30. 170421097.
31. 181000512.
32. 214307272.
33. 137606352.
34. 273922943.
35. 416312090.
36. 362530048.
38. 8760 hr., 52560 hr., 657000 hr.
39. $96950.
40. 748000 yd., 1267200 yd.
41. 63360 in., 1584000 in.
42. 353685 da.
43. 25612 lines.
44. 358568 wds.
45. 4032 mi.
46. The latter; 8939.
47. 3489863.

Exercise 12.

1. 15480000.
2. 148396.
3. 6676950.
4. 5640.
5. $64855.
6. 124 mi.
7. 3960 mi.
8. 47600 ap's.
10. 2623; 10492 trees.
11. 252 pages.
12. Mr. Dash owes Mr. Blank $24.
13. 2812992 lb.
14. 6 mi.
15. $15555.
16. $2450.
17. The latter.
18. 25578.
19. $1110, $740, $1850.
20. $40460.
21. $1429.
22. $28470.
23. If selling price exceeds cost. If cost exceeds selling price.
24. $2888.
26. Last is greatest; second is least.
27. 88; 932.

Exercise 14.

31. 142 in.
32. 295 ft.
33. 238 da.
34. 296 men.
35. 159 T.
36. 369 men.
37. 86 da.
38. 275 pk.
39. 157 yd.
40. 129 ft.
41. 7562.
42. 3357.
43. 7329.
44. 9524.
45. 2857.
46. 65277.
47. 37672.
48. 15432.
49. 72469.
50. 87893.
51. 61731.
52. 82787.
53. 47784.
54. 43219.
55. 892787.
56. 62159.
57. 31421.
58. 45367.
59. 41235.
60. 57384.
61. Q. = 404508, R. = 2.
62. 8090795.
63. 508097.
64. 807904.
65. 2088099
66. 1908207.
67. 9504203.
68. 760055.
69. 507804.
70. 809516.
71. Q. = 2804761, R. = 1.
72. 12013908.
73. 2496, 18901, 447039, 209048.
74. 868; Q. = 102850, R. = 2; 1505293, 2057613.
75. 35069, 84055, 150621, 1024609.
76. 50368, 708602, 4300213, 1667003.
77. 7507, 439080, 18922594, 1251027.
78. 13393, 640261, 742084, 1975308.
79. 4762, 1011558; Q. = 175780, R. = 3; 2890103.
80. 65044, 338201, 423947, 8750151.
81. 2638; 112483; Q. = 5569273, R. = 1; Q. = 7791495, R. = 1.
82. 51626.
83. 53380.
84. 6374.
85. 5456.
86. 77513.
87. 479201.
88. 3256, 4071, 370543, 924608.
89. 2357, 50813, 74032, 3901506.
90. $1239.
91. 286 wk.
92. 788 bbl.
93. 780 wk.
94. 5280 ft.
95. 4075 bu.
96. 3899 sq. yd.
97. 12936.
98. 715540.
99. 2227.
100. 40275.
101. 5468.
102. 342.
103. 22009.
104. 42165.
105. Q. = 3926, R. = 7.
106. Q. = 2418, R. = 1.
107. Q. = 202405, R. = 3.
108. Q. = 60086, R. = 1.
109. Q. = 62411, R. = 4.
110. Q. = 1675, R. = 8.
111. Q. = 16048, R. = 3.
112. Q. = 4937, R. = 10.
113. Q. = 2255, R. = 20.
114. Q. = 21034, R. = 10.
115. Q. = 445, R. = 70.
116. Q. = 1371, R. = 66.
117. Q. = 267, R. = 50.
118. Q. = 4881. R. = 150.
119. Q. = 3255, R. = 80.
120. Q. = 239, R. = 70.
121. Q. = 3592, R. = 300.
122. Q. = 153, R. = 570.
123. Q. = 738, R. = 184.
124. Q. = 70, R. = 792.
125. Q. = 430, R. = 650.
126. 306.
127. 3934.
128. 1766.
129. 86306.
130. 181961.

Exercise 15.

1. 28.
2. 36.
3. 17.
4. 28.
5. 47.
6. 28 ft.
7. 46 T.
8. 49 min.
9. 84 hr.
10. 64 pages.
11. 85.
12. 116.
13. Q. = 75, R. = 16.
14. 178.
15. 256.
16. 327.
17. 361.
18. 426.
19. 559.
20. 748.
21. 205.
22. 307.
23. 409.
24. 507.
25. 601.
26. 1032.
27. Q. = 2043, R. = 32.
28. 4028.
29. 5304.
30. 7009.
31. 16.
32. 18.
33. 56.
34. 37.
35. 29.
36. 173.
37. Q. = 2140, R. = 9.
38. 327.
39. 463.
40. 514.
41. 623.
42. 528.
43. Q. = 546, R. = 35.
44. 846.
45. 503.
46. 605.
47. 807.
48. 504.
49. 407.
50. 609.
51. 804.
52. 1234.
53. 3124.
54. 5205.
55. 6032.
56. 6507.
57. Q. = 7037, R. = 399.
58. 8106.
59. 9027.
60. 4213, 5608.
61. 3255, 7016.
62. 3631, 7803.
63. 5298, 6309.
64. 7035, 8704.
65. 78.
66. 93.
67. 27.
68. 63.
69. 308.
70. 530.
71. Q. = 313, R. = 14.
72. Q. = 140, R. = 238.
73. Q. = 80, R. = 13.
74. Q. = 68, R. = 828.
75. Q. = 4875, R. = 1611.
76. Q. = 1672, R. = 5813.
77. 89; 132 da.
78. 68; 190 bu.
79. 83 sq. in.
80. 205 bbl.
81. 2054 yr. 290 da.
82. 125000 da.
83. 500 sec.
84. 7208 sh.
85. $235.
86. 80 ch.
87. 907.
88. 4307.
89. 14 yd.
90. 12 T.
91. 234 oxen.
92. $36.
93. 2723.
94. 6175.
95. 9268.
96. Q. = 2304, R. = 1862.
97. Q. = 6507, R. = 1444.
98. Q. = 9506, R. = 4822.
99. 10960.
100. 5804.
101. 9603.
102. 4007.
103. 6009.
104. 8007.
105. Q. = 43662, R. = 2575.
106. 73048.
107. Q. = 790136, R. = 64304.
108. 50441783.
109. 4000901.
110. 49.
111. 65.
112. 1.
113. 32.
114. 32.
115. 146.
116. 95.

Exercise 16.

1. 37.
2. 62.
3. 85.
4. 94.
5. 131.
6. 246.
7. 209.
8. 356.
9. 418.
10. 538.
11. 627.
12. 818.
13. 399.
14. 427.
15. 473.
16. 737.
17. 813.
18. 863.
19. Q. = 9, R. = 31.
20. Q. = 19, R. = 15.
21. Q. = 12, R. = 23.

22. Q. = 17, R. = 47.	25. Q. = 29, R. = 113.	28. Q. = 2, R. = 96.	31. Q. = 20, R. = 163.
23. Q. = 8, R. = 7.	26. Q. = 32, R. = 73.	29. Q. = 21, R. = 95.	32. Q. = 16, R. = 233.
24. Q. = 25, R. = 71.	27. Q. = 24, R. = 143.	30. Q. = 18, R. = 233.	33. Q. = 13, R. = 169.

Exercise 17.

1. 11167, 11628 votes.	4. 30 and 60; 15 and 75.	9. 11 and 121; 6 and 126.
2. John, 78; William, 39; George, 13.	8. 360 girls; 120 boys, 12 teachers.	10. A, 600; B, 300; C, 150; D, 50.
3. 80, 20.		11. 27629 and 28700 votes.

Exercise 18.

1. 8900.	10. 73000.	19. 7469.	28. 106900.
2. 15900.	11. 243000.	20. 29008.	29. 193500.
3. 12100.	12. 557250.	21. 42912.	30. 698000.
4. 19425.	13. 754.	22. 72324.	31. 13740000.
5. 12150.	14. 1152.	23. 314370.	32. 14400000.
6. 36625.	15. 1554.	24. 713852.	33. 33180000.
7. 28000.	16. 3233.	25. 1761237.	34. 64070000.
8. 59150.	17. 3354.	26. 2375460.	35. 15650000.
9. 55200.	18. 6460.	27. 3474545.	

Exercise 19.

1. 3045.	12. $95\frac{91}{375}$.	23. $966\frac{25}{81}$.	34. 913.
2. 3575.	13. $140\frac{132}{625}$.	24. $467\frac{4}{83}$.	35. $950\frac{28}{97}$.
3. 2667.	14. $65\frac{31}{35}$.	25. $211\frac{33}{85}$.	36. $1007\frac{67}{98}$.
4. $1915\frac{1}{2}$.	15. $390\frac{3}{25}$.	26. $544\frac{53}{87}$.	37. $147\frac{463}{499}$.
5. 858.	16. $108\frac{34}{175}$.	27. $829\frac{59}{92}$.	38. $187\frac{21}{111}$.
6. 1260.	17. 34.	28. $593\frac{2}{19}$.	39. $256\frac{831}{998}$.
7. 10872.	18. 48.	29. $663\frac{19}{33}$.	40. 311800.
8. 5264.	19. 56.	30. $745\frac{85}{241}$.	41. 444400.
9. 10280.	20. $72\frac{52}{79}$.	31. 314.	42. 345000.
10. 9512.	21. $90\frac{18}{43}$.	32. $436\frac{23}{28}$.	43. 3017000.
11. $454\frac{6}{25}$.	22. $121\frac{24}{59}$.	33. $894\frac{9}{49}$.	

Exercise 20.

1. $5060.	3. $15690; $3186.	5. $8568.
2. $2000; $150.	4. 12 hr.; 540 mi.; 10 hr.	6. 27981.

7. Q. = 3485, R. = 424.	8. 1005415082.	10. 42953119.	12. 1763.
	9. 696772.	11. 12.	13. 21.

Exercise 22.

1. 3.	10. 5.	19. 4.	26. Q. = 3, R. = 24480.
2. 4.	11. 3.	20. 2.	
3. 2.	12. 6.	21. 102.	27. Q. = 4, R. = 9744.
4. 8.	13. 2.	22. $2\frac{3}{7}$.	
5. 4.	14. 16.	23. 3.	28. Q. = 2, R. = 199584.
6. $8\frac{2}{5}$.	15. 3.	24. Q. = 13, R. = 280.	
7. 15.	16. 8.		29. Q. = 1, R. = 424536.
8. 6.	17. $1\frac{20}{21}$.	25. Q. = 1, R. = 12600.	
9. 18.	18. 27.		

Exercise 23.

1. 70¢; $2.25.	7. $341.	13. $900.	19. 18 yd.
2. 56¢.	8. 17649 lb.	14. $216.	20. 32 bu.
3. $3.78.	9. 34000 qt.	15. 49 men.	21. 9 firkins.
4. $1411.	10. $3250.	16. 54 da.	22. 4 rolls.
5. $575.	11. $20.	17. 300 ft.	23. 45 bales.
6. 304 lb.	12. $60.	18. 30 lb.	

Exercise 24.

1. 6.	6. 3.	11. 12.	16. 32.	21. 10.
2. 12.	7. 24.	12. 18.	17. 4.	22. 24.
3. 15.	8. 42.	13. 24.	18. 54.	23. 18.
4. 18.	9. 6.	14. 27.	19. 8.	24. 48.
5. 16.	10. 8.	15. 30.	20. 14.	

Exercise 25.

1. 5.	6. 9.	11. 15.	16. 7.	21. 15.
2. 4.	7. 11.	12. 21.	17. 11.	22. 13.
3. 6.	8. 12.	13. 27.	18. 16.	23. 36.
4. 8.	9. 17.	14. 31.	19. 17.	24. 29.
5. 9.	10. 13.	15. 57.	20. 8.	25. 37.

Exercise 26.

1. 24.	7. 30.	13. 840.	19. 540.	25. 1800.
2. 45.	8. 144.	14. 5040.	20. 2376.	26. 5040.
3. 50.	9. 630.	15. 3780.	21. 504.	27. 4620.
4. 90.	10. 360.	16. 2310.	22. 360.	28. 24570.
5. 168.	11. 2592.	17. 4004.	23. 600.	29. 198450.
6. 140.	12. 1680.	18. 4620.	24. 3696.	30. 39270.

Exercise 27.

1. 7656.
2. 16320.
3. 7084.
4. 13650.
5. 16200.
6. 20800.
7. 201348.
8. 686880.
9. 13680.
10. 4480.
11. 1980.
12. 11340.

Exercise 28.

1. 12.
2. 900.
3. 2.
4. 2100.
5. 17.
6. 39442.
7. 6 and 6048.
8. 8 and 480.
9. 42 and 3780.
10. 21 and 65520.
11. 3 and 39780.
12. 7 and 767340.
13. 3 ft.
14. 6 yd.
15. 60 qt.
16. 120 ft.
17. 11 ft.
18. 13 bu.
19. L. C. M. = the product of the two numbers.
20. An indefinite number.
21. 20.

Exercise 31.

1. $\frac{17}{4}$ in.
2. $\frac{23}{3}$ ft.
3. $\frac{39}{7}$ mi.
4. $\frac{58}{9}$ rd.
5. $\frac{83}{16}$ yd.
6. $\frac{117}{11}$ gal.
7. $\frac{112}{12}$ qt.
8. $\frac{84}{5}$ yr.
9. $\frac{220}{9}$.
10. $\frac{328}{13}$.
11. $\frac{261}{8}$.
12. $\frac{262}{15}$.
13. $\frac{243}{2}$.
14. $\frac{241}{4}$.
15. $\frac{1554}{5}$.
16. $\frac{1301}{12}$.
17. $\frac{235}{4}$.
18. $\frac{527}{8}$.
19. $\frac{731}{10}$.
20. $\frac{1353}{5}$.
21. $\frac{3860}{11}$.
22. $\frac{10513}{15}$.
23. $\frac{6506}{21}$.
24. $\frac{2239}{30}$.
25. $\frac{2642}{36}$.
26. $\frac{19205}{48}$.
27. $\frac{5220}{17}$.
28. $\frac{3992}{347}$.
29. $\frac{1399}{75}$.
30. $\frac{1999}{100}$.
31. $\frac{14501}{503}$.
32. $\frac{9647}{222}$.

Exercise 32.

1. $6\frac{1}{7}$ qt.
2. $7\frac{1}{3}$ mi.
3. $18\frac{3}{4}$ in.
4. $24\frac{4}{5}$ da.
5. $23\frac{2}{7}$ wk.
6. $19\frac{5}{9}$ gal.
7. $20\frac{3}{4}$ ft.
8. \$$29\frac{3}{16}$.
9. \$$22\frac{19}{20}$.
10. \$$58\frac{7}{15}$.
11. \$$35\frac{14}{25}$.
12. \$$68\frac{8}{11}$.
13. $53\frac{5}{13}$.
14. $32\frac{1}{17}$.
15. $26\frac{26}{27}$.
16. $18\frac{4}{33}$.
17. $14\frac{18}{35}$.
18. $21\frac{14}{41}$.
19. $10\frac{37}{45}$.
20. $18\frac{2}{51}$.
21. $15\frac{55}{61}$.
22. $25\frac{25}{39}$.
23. $36\frac{21}{55}$.
24. $48\frac{52}{63}$.
25. $60\frac{2}{80}$.
26. $102\frac{63}{73}$.
27. $94\frac{76}{101}$.
28. $75\frac{105}{127}$.

Exercise 33.

1. $\frac{4}{7}$.
2. $\frac{5}{9}$.
3. $\frac{5}{9}$.
4. $\frac{1}{8}$.
5. $\frac{2}{7}$.
6. $\frac{5}{9}$.
7. $\frac{5}{9}$.
8. $\frac{9}{11}$.
9. $\frac{6}{11}$.
10. $\frac{6}{7}$.
11. $\frac{7}{8}$.
12. $\frac{4}{5}$.
13. $\frac{5}{7}$.
14. $\frac{6}{7}$.
15. $\frac{6}{11}$.
16. $\frac{8}{11}$.
17. $\frac{6}{25}$.
18. $\frac{7}{11}$.
19. $\frac{26}{35}$.
20. $\frac{9}{14}$.
21. $\frac{11}{17}$.
22. $\frac{11}{15}$.
23. $\frac{15}{16}$.
24. $\frac{11}{25}$.
25. $\frac{12}{17}$.
26. $\frac{11}{14}$.
27. $\frac{8}{9}$.
28. $\frac{62}{65}$.
29. $\frac{2}{5}$.
30. $\frac{6}{7}$, $\frac{1}{2}$.
31. $\frac{4}{5}$, $\frac{6}{7}$.
32. $\frac{2}{5}$ yr.
33. $\frac{51}{80}$ ton.
34. $\frac{3}{4}$ mi.
35. $\frac{5}{7}$.
36. $\frac{11}{13}$.
37. $\frac{3}{80}$.
38. $\frac{2}{9}$.
39. $\frac{7}{10}$.
40. $\frac{35}{43}$.
41. $\frac{21}{25}$.
42. $\frac{15}{19}$.

Exercise 34.

1. $\frac{9}{12}$, $\frac{10}{12}$.
2. $\frac{4}{24}$, $\frac{9}{24}$.
3. $\frac{25}{30}$, $\frac{14}{30}$.
4. $\frac{8}{12}$, $\frac{9}{12}$.
5. $\frac{36}{45}$, $\frac{40}{45}$.
6. $\frac{55}{60}$, $\frac{52}{60}$.
7. $\frac{25}{80}$, $\frac{44}{80}$.
8. $\frac{156}{455}$, $\frac{437}{455}$.
9. $\frac{70}{90}$, $\frac{63}{90}$.
10. $\frac{6}{12}$, $\frac{8}{12}$, $\frac{3}{12}$.
11. $\frac{2}{6}$, $\frac{3}{6}$, $\frac{5}{6}$.
12. $\frac{18}{24}$, $\frac{4}{24}$, $\frac{9}{24}$.
13. $\frac{45}{72}$, $\frac{16}{72}$, $\frac{66}{72}$.
14. $\frac{25}{60}$, $\frac{8}{60}$, $\frac{14}{60}$.
15. $\frac{28}{48}$, $\frac{15}{48}$, $\frac{2}{48}$.
16. $\frac{272}{288}$, $\frac{276}{288}$, $\frac{135}{288}$.
17. $\frac{45}{30}$, $\frac{20}{30}$, $\frac{6}{30}$.
18. $\frac{21}{84}$, $\frac{36}{84}$, $\frac{14}{84}$.
19. $\frac{8}{36}$, $\frac{45}{36}$, $\frac{33}{36}$.
20. $\frac{88}{154}$, $\frac{126}{154}$, $\frac{539}{154}$.
21. $\frac{315}{1155}$, $\frac{616}{1155}$, $\frac{220}{1155}$.
22. $\frac{720}{3120}$, $\frac{832}{3120}$, $\frac{975}{3120}$.
23. $\frac{80}{300}$, $\frac{36}{300}$, $\frac{105}{300}$.
24. $\frac{120}{264}$, $\frac{99}{264}$, $\frac{220}{264}$.
25. $2\frac{8}{12}$, $1\frac{9}{12}$, $4\frac{2}{12}$.
26. $2\frac{10}{40}$, $5\frac{4}{40}$, $7\frac{15}{40}$.
27. $\frac{210}{360}$, $\frac{100}{360}$, $\frac{165}{360}$, $\frac{156}{360}$.
28. $\frac{1}{3}$.
29. $\frac{2}{3}$.
30. $\frac{2}{9}$.
31. $\frac{7}{13}$.
32. $\frac{7}{12}$.
33. $\frac{9}{15}$.
34. $\frac{13}{27}$ and $\frac{11}{24}$; $\frac{13}{40}$ and $\frac{15}{35}$.
35. Yes.
36. Yes.
37. $\frac{13}{25}$.
38. Yes; no; yes.

Exercise 35.

1. $\frac{11}{12}$.
2. $1\frac{5}{18}$.
3. $1\frac{7}{60}$.
4. $1\frac{1}{5}$.
5. $1\frac{19}{24}$.
6. $1\frac{13}{30}$.
7. $1\frac{7}{10}$.
8. $1\frac{1}{4}$.
9. $1\frac{14}{45}$.
10. $2\frac{5}{18}$.
11. $1\frac{211}{240}$.
12. $2\frac{2}{5}$.
13. $2\frac{37}{105}$.
14. $1\frac{49}{54}$.
15. $4\frac{1}{120}$.
16. $4\frac{113}{114}$.
17. $3\frac{188}{225}$.
18. $3\frac{43}{216}$.
19. $4\frac{1}{6}$.
20. $4\frac{103}{1320}$.

Exercise 36.

1. $\frac{10}{21}$.
2. $\frac{13}{40}$.
3. $\frac{4}{15}$.
4. $\frac{16}{39}$.
5. $\frac{29}{204}$.
6. $1\frac{1}{3}$.
7. $1\frac{46}{77}$.
8. $\frac{19}{24}$.
9. $\frac{37}{60}$.
10. $\frac{19}{55}$.
11. $\frac{1}{3}$.
12. $\frac{431}{1716}$.
13. 2.
14. $1\frac{7}{16}$.
15. $1\frac{1}{6}$.
16. $2\frac{17}{30}$.

Exercise 37.

1. $6\frac{1}{6}$.
2. $7\frac{1}{12}$.
3. $12\frac{1}{2}$.
4. $8\frac{13}{24}$.
5. $8\frac{11}{45}$.
6. $12\frac{31}{36}$.
7. $9\frac{29}{72}$.
8. $12\frac{1}{7}$.
9. $15\frac{1}{3}$.
10. $24\frac{3}{16}$.
11. $15\frac{37}{42}$.
12. $21\frac{4}{15}$.
13. $26\frac{82}{225}$.
14. $18\frac{19}{24}$.
15. $32\frac{247}{360}$.
16. $25\frac{137}{144}$.

Exercise 38.

1. $\frac{1}{3}$.
2. $\frac{1}{36}$.
3. $\frac{4}{15}$.
4. $\frac{7}{72}$.
5. $\frac{11}{42}$.
6. $\frac{103}{150}$.
7. $\frac{47}{210}$.
8. $\frac{9}{110}$.
9. $\frac{47}{350}$.
10. $\frac{52}{75}$.
11. $\frac{5}{72}$.
12. $\frac{185}{192}$.
13. $\frac{11}{21}$.
14. $\frac{17}{72}$.
15. $\frac{165}{187}$.
16. $\frac{17}{72}$.
17. $\frac{25}{98}$.
18. $\frac{84}{175}$.
19. $\frac{39}{152}$.
20. $\frac{37}{162}$.
21. $\frac{109}{576}$.
22. $\frac{461}{525}$.
23. $\frac{287}{891}$.
24. $\frac{17}{200}$.
25. $\frac{187}{2964}$.
26. $1\frac{457}{1428}$.
27. $\frac{1059}{1144}$.
28. $\frac{175}{374}$.
29. $\frac{503}{1476}$.
30. $\frac{97}{1584}$.

Exercise 39.

1. $2\frac{2}{21}$.
2. $\frac{3}{10}$.
3. $1\frac{19}{24}$.
4. $2\frac{1}{12}$.
5. $3\frac{5}{12}$.
6. $\frac{1}{12}$.
7. $\frac{3}{40}$.
8. $\frac{4}{77}$.
9. $\frac{5}{72}$.
10. $\frac{1}{210}$.
11. $1\frac{1}{3}$.
12. $1\frac{1}{12}$.
13. $2\frac{6}{55}$.
14. $1\frac{1}{24}$.
15. $1\frac{1}{30}$.
16. $\frac{7}{10}$.
17. $3\frac{19}{24}$.
18. $2\frac{9}{20}$.
19. $\frac{29}{33}$.
20. $\frac{29}{35}$.
21. $1\frac{2}{3}$.
22. $1\frac{3}{4}$.
23. $3\frac{7}{12}$.
24. $1\frac{1}{2}$.
25. $1\frac{1}{3}$.
26. $8\frac{4}{9}$.
27. $4\frac{9}{11}$.
28. $4\frac{9}{35}$.
29. $16\frac{71}{120}$.
30. $2\frac{61}{72}$.
31. $4\frac{112}{135}$.

Exercise 41.

1. $\frac{11}{12}$.
2. $\frac{11}{40}$.
3. $\frac{17}{18}$.
4. $1\frac{2}{3}$.
5. $2\frac{1}{6}$.
6. 6.
7. $2\frac{11}{20}$.
8. $2\frac{3}{20}$.
9. $\frac{2}{3}$.
10. $7\frac{13}{18}$.
11. $6\frac{13}{24}$.
12. $1\frac{17}{24}$.
13. $3\frac{7}{80}$.
14. $5\frac{41}{48}$.
15. $4\frac{2}{3}$.
16. $2\frac{5}{24}$.
17. $7\frac{4}{9}$.
18. $11\frac{5}{6}$.
19. $\frac{1}{10}$.
20. $92\frac{1}{6}$ ft.
21. $\$38\frac{3}{40}$.

Exercise 42.

1. $2\frac{2}{5}$.
2. 8.
3. $7\frac{1}{2}$.
4. $1\frac{7}{8}$.
5. $2\frac{6}{7}$.
6. $6\frac{2}{5}$.
7. $11\frac{1}{4}$.
8. $\frac{3}{4}$.
9. $\frac{2}{3}$.
10. $\frac{21}{55}$.
11. $\frac{2}{5}$.
12. $\frac{9}{11}$.
13. $\frac{20}{21}$.
14. $\frac{27}{28}$.
15. $\frac{1}{4}$.
16. $\frac{1}{2}$.
17. $\frac{1}{7}$.
18. $\frac{1}{2}$.
19. $\frac{1}{3}$.
20. $\frac{98}{111}$.
21. $\frac{3}{28}$.
22. $11\frac{1}{3}$.
23. $10\frac{1}{2}$.
24. $17\frac{1}{2}$.
25. $50\frac{2}{3}$.
26. $52\frac{1}{2}$.
27. $\frac{6}{11}$.
28. $\frac{8}{11}$.
29. $\frac{5}{9}$.
30. $\frac{16}{87}$.
31. $\frac{8}{15}$.
32. $\frac{5}{8}$.
33. $\frac{1}{8}$.
34. $21\frac{2}{3}$.
35. $7\frac{37}{63}$.
36. $\frac{17}{28}$.
37. $12\frac{61}{115}$.
38. $35\frac{5}{7}$.
39. $\frac{2}{3}$.

Exercise 43.

1. 57.
2. 87.
3. 33.
4. 129.
5. $22\frac{1}{2}$.
6. $15\frac{3}{5}$.
7. $42\frac{8}{11}$.
8. $219\frac{1}{3}$.
9. $586\frac{2}{3}$.
10. $944\frac{4}{5}$.
11. $672\frac{3}{11}$.
12. 11.
13. $6\frac{2}{3}$.
14. 49.
15. 18.
16. 302.
17. $469\frac{1}{3}$.
18. 949.
19. $8\frac{1}{4}$.
20. $18\frac{4}{5}$.
21. 52.
22. $14\frac{4}{5}$.
23. 132.
24. $4\frac{1}{2}$.
25. $202\frac{2}{3}$ sq. ft.
26. $202\frac{1}{2}$ sq. ft.
27. 175 sq. ft.
28. $299\frac{1}{4}$ sq. ft.
29. $627\frac{1}{2}$ sq. ft.
30. $527\frac{1}{3}$ sq. ft.
31. 820 sq. ft.
32. 276 sq. ft.
33. $345\frac{1}{12}$ sq. ft.
34. $360\frac{3}{8}$ sq. ft.

Exercise 45.

1. $\$33\frac{3}{4}$.
2. $\$115\frac{1}{2}$.
3. $\$15\frac{3}{8}$.
4. $\$87\frac{1}{2}$.
5. $360.
6. $\$1.55\frac{1}{4}$.
7. $\$821\frac{2}{3}$.
8. $30.
9. $\$277\frac{1}{2}$.
10. $\$1658\frac{1}{2}$.
11. $\$6330\frac{3}{4}$.
12. $164\frac{1}{3}$ sq. ft.
13. $2508\frac{2}{3}$ sq. ft.
14. $\frac{1}{4}$, $\frac{3}{8}$.
15. $\frac{2}{5}$.
16. 702 ft.
17. $\frac{11}{18}$.
18. $\frac{35}{54}$.
19. $3\frac{47}{60}$ yd.

Exercise 46.

1. $\frac{4}{15}$.
2. $\frac{5}{29}$.
3. $\frac{4}{33}$.
4. $\frac{13}{105}$.
5. $\frac{13}{315}$.
6. $\frac{2}{99}$.
7. $\frac{5}{363}$.
8. $\frac{5}{18}$.
9. $\frac{4}{25}$.
10. $\frac{8}{135}$.
11. $\frac{3}{1604}$.
12. $\frac{5}{808}$.
13. 6.
14. $\frac{3}{4}$.
15. $1\frac{3}{7}$.
16. $\frac{18}{35}$.
17. $1\frac{1}{5}$.
18. $1\frac{1}{4}$.
19. $1\frac{1}{5}$.
20. $\frac{10}{11}$.
21. $\frac{2}{5}$.
22. $21\frac{7}{10}$.
23. $15\frac{3}{7}$.
24. $35\frac{13}{35}$.
25. $42\frac{5}{6}$.
26. $76\frac{3}{4}$.
27. $67\frac{5}{6}$.
28. $41\frac{7}{18}$.
29. $4\frac{2}{7}$.
30. $11\frac{2}{3}$.
31. $4\frac{2}{3}$.
32. $8\frac{1}{4}$.
33. $6\frac{3}{4}$.
34. $\frac{5}{7}$.
35. $\frac{8}{11}$.
36. $\frac{69}{73}$.
37. $8\frac{79}{82}$.
38. $2\frac{515}{603}$.
39. $3\frac{431}{539}$.

Exercise 47.

1. $\frac{3}{4}$.
2. $\frac{1}{2}$.
3. $\frac{4}{7}$.
4. $\frac{8}{9}$.
5. $2\frac{2}{3}$.
6. $2\frac{5}{14}$.
7. $\frac{50}{51}$.
8. $7\frac{31}{52}$.
9. $1\frac{2}{3}$.
10. $\frac{5}{18}$.
11. $2\frac{4}{19}$.
12. $7\frac{1}{2}$.
13. 28.
14. $2\frac{6}{11}$.
15. 1.
16. $\frac{6}{11}$.
17. 8.
18. $\frac{9}{16}$.
19. $8\frac{7}{37}$.
20. $\frac{416}{2745}$.
21. $3\frac{31}{33}$.
22. $\frac{2}{3}$.
23. $\frac{3}{4}$.
24. $\frac{3}{4}$.
25. $\frac{3}{32}$.
26. $\frac{2}{7}$.
27. $\frac{23}{24}$.
28. $\frac{4}{7}$.
29. $\frac{8}{17}$.
30. $6\frac{1}{8}$.
31. $\frac{4}{5}$.
32. $3\frac{61}{105}$.
33. $3\frac{25}{168}$.
34. $\frac{25}{69}$.
35. $2\frac{4}{13}$.
36. $3\frac{3}{7}$.
37. $3\frac{3}{7}$.
38. $18\frac{1}{2}$.
39. $2\frac{2}{3}$.
40. 18.
41. $1\frac{6}{7}$.
42. $\frac{2}{9}$.
43. $6\frac{1}{2}$.

Exercise 48.

1. $\frac{2}{35}$.
2. $\frac{1}{30}$.
3. $\frac{1}{110}$.
4. $\frac{5}{24}$.
5. $\frac{5}{24}$ T.
6. $\frac{1}{144}$.
7. $\frac{7}{60}$.
8. $\frac{1}{63}$.
9. $\frac{6}{385}$.
10. $\frac{15}{56}$.
11. $2\frac{1}{2}$.
12. $2\frac{2}{3}$.
13. 18.
14. 18.
15. $37\frac{1}{3}$.
16. $1\frac{1}{3}$.
17. 1.
18. $11\frac{1}{5}$.
19. 440.
20. $12\frac{1}{2}$.

Exercise 49.

1. 22¢; $9\frac{7}{20}$¢.
2. $\$40\frac{1}{2}$.
3. $50\frac{2}{3}$ yd.
4. $5\frac{3}{14}$ T.
5. $\$201\frac{1}{5}$.
6. $24\frac{3}{7}$ lb.
7. $5831\frac{1}{4}$ lb.
8. $2438\frac{1}{3}$ cu. in.
9. $325\frac{3}{5}$ in.
10. $8\frac{2}{3}$ rolls.
11. $11\frac{41}{44}$ l.

Exercise 50.

1. 546 p.
2. $8550.
3. $100977.
4. 5300 mi.
5. $\$30\frac{1}{3}$.
6. $\$41\frac{2}{3}$.
7. $\$95\frac{1}{3}$.
8. $214\frac{1}{2}$ cu. in.
9. $23\frac{3}{7}$ c.
10. $466\frac{1}{14}$ h.
11. $\$83\frac{5}{12}$.
12. $62\frac{1}{3}$ lb.
13. $\$\frac{3}{4}$.
14. $165.
15. 48 m.
16. 80 A.
17. 360 p.
18. 112 ft.
19. 30 m.; 6 m.
20. 497 A.; 213 A.
21. 372 p.
22. $12600.

Exercise 51.

1. Latter; $\frac{1}{3}$.	*30.* $2\frac{16}{21}$.	*58.* $1\frac{22225}{33312}$.	*86.* $490\frac{5}{8}$ A.
2. Latter; $\frac{1}{90}$.	*31.* $12\frac{7}{9}$.	*59.* $\frac{41}{264}$.	*87.* $1\frac{79}{81}$.
3. $1\frac{11}{12}$.	*32.* 9.	*60.* $\frac{3}{11}$.	*88.* $\$2401\frac{38}{45}$.
4. $38\frac{7}{9}$.	*33.* $22\frac{1}{2}$.	*61.* $7\frac{48}{77}$.	*89.* $100\frac{25}{32}$ qt.
5. $23\frac{10}{21}$.	*34.* $13\frac{1}{3}$.	*62.* $1\frac{1}{2}$.	*90.* $\$7\frac{1}{2}$ per yd.
6. $62\frac{27}{165}$.	*35.* 4.	*63.* $1\frac{13}{30}$.	*91.* $\$45937\frac{1}{2}$.
7. $26\frac{19}{48}$.	*36.* 316.	*64.* $2\frac{7}{13}$.	*92.* $\frac{1}{72}$.
8. $53\frac{113}{120}$.	*37.* 224.	*65.* $3\frac{5}{24}$.	*93.* $1\frac{25}{57}$.
9. $19\frac{119}{135}$.	*38.* $1\frac{11}{15}$.	*66.* $1\frac{28}{181}$.	*94.* $1\frac{3}{17}$.
10. $2\frac{29}{63}$.	*39.* $\frac{14}{15}$.	*67.* $3\frac{13}{43}$.	*95.* $6\frac{6}{7}$.
11. $1\frac{1}{12}$.	*40.* $2\frac{7}{10}$.	*68.* $4\frac{17}{38}$.	*96.* $\$6\frac{3}{4}$.
12. $\frac{3}{5}$.	*41.* $1\frac{2}{11}$.	*69.* $\frac{6}{11}$.	*97.* Increased; $\frac{21}{176}$.
13. $1\frac{28}{161}$.	*42.* $1\frac{1}{11}$.	*70.* $1620.	*98.* Dim'd; $\frac{21}{80}$.
14. $9\frac{3}{56}$.	*43.* $14\frac{2}{3}$.	*71.* $1\frac{25}{27}$.	*99.* $138\frac{10}{33}$.
15. $\frac{193}{247}$.	*44.* $\frac{4}{35}$.	*72.* $9\frac{1}{3}$.	*100.* 528 A.
16. $1\frac{83}{168}$.	*45.* $\frac{4}{15}$.	*73.* $4\frac{8}{25}$.	*101.* $\$15\frac{17}{25}$.
17. $\frac{593}{1518}$.	*46.* $\frac{1}{12}$.	*74.* $78\frac{11}{12}$.	*102.* $17\frac{1}{10}$ c.
18. $1\frac{611}{820}$.	*47.* $\frac{1}{6}$.	*75.* $7\frac{1}{3}$.	*103.* $6\frac{73}{360}$ da.
19. $6\frac{442}{777}$.	*48.* $\frac{34}{35}$.	*76.* $\frac{13}{40}$; $\$32\frac{1}{2}$.	*104.* $\$27\frac{11}{15}$.
20. $\frac{3514}{4455}$.	*49.* $\frac{3}{14}$.	*77.* $\frac{8}{35}$.	*105.* $9\frac{3}{8}$ T.
21. $3\frac{202}{945}$.	*50.* $1\frac{1}{2}$.	*78.* 262.	*106.* Dim'd; $\frac{13}{168}$.
22. $4\frac{7}{8}$.	*51.* $\frac{15}{44}$.	*79.* 75.	*107.* $90; $70.
23. $9\frac{7}{10}$.	*52.* $\frac{31}{35}$.	*80.* 60 boys.	*108.* 265; $35\frac{1}{3}$.
24. $3\frac{3}{28}$.	*53.* $\frac{2}{5}$.	*81.* 90 w. balls.	*109.* Last, $7\frac{1}{5}$ da.
25. $1\frac{3}{10}$.	*54.* $\frac{2}{21}$.	*82.* 625 sheep.	*110.* $9\frac{9}{14}$ da.
26. $24\frac{37}{110}$.	*55.* $3\frac{3}{5}$.	*83.* $3\frac{1}{3}$ mi. an hr.	*111.* Last, 3 da.
27. $3\frac{1}{3}$.	*56.* $\frac{8}{15}$.	*84.* 650 T.	*112.* $3\frac{3}{25}$ hr.
28. $6\frac{1}{2}$.	*57.* $7\frac{4}{5}$.	*85.* 90 A.	
29. $4\frac{1}{6}$.			

Exercise 53.

1. $198.27.	*4.* $134.98.	*7.* 373.039.
2. 108.506 in.	*5.* $490.18.	*8.* 74.4964 mi.
3. 65.5051 sq. yd.	*6.* 3150.356.	*9.* 8020.528964.

Exercise 54.

1. $4.14.	*6.* 221.5119.	*11.* 1.54.	*16.* 271.395.	*21.* $12.77.
2. $13.446.	*7.* 9.5243.	*12.* .296.	*17.* .9.	*22.* .001265.
3. 32.77 in.	*8.* .8293.	*13.* .0933.	*18.* .007.	*23.* $128.48.
4. 17.127.	*9.* .17.	*14.* .0102.	*19.* 9.999.	*24.* $84.94.
5. 47.18.	*10.* .18.	*15.* 63.662.	*20.* 1.989797.	*25.* 399.441.

Exercise 55.

1. 41.5.	*9.* .4.	*13.* 100.2001.	*17.* .24.	*21.* $3906.28.
2. 4.2.	*10.* .4.	*14.* .09018009.	*18.* 4.90038.	*22.* 272.85.
3. .38.	*11.* .01.	*15.* 108.	*19.* .616.	*23.* 20.3186.
4. 1.56.	*12.* .1.	*16.* .484.	*20.* .145435.	*24.* $1236.02.
5. .616.				

Exercise 56.

1. 2.5.	*11.* 3.	*21.* 1000.	*31.* 112.	*40.* 21.4.
2. .015.	*12.* 3.	*22.* .0001.	*32.* .00112.	*41.* 808.08.
3. .18.	*13.* .9.	*23.* 500.	*33.* .0005.	*42.* 800808.
4. .8.	*14.* .7.	*24.* 4.	*34.* 32.	*43.* .0164.
5. .0011.	*15.* 12.	*25.* 800.	*35.* 31.5.	*44.* 6.
6. .0019.	*16.* .28.	*26.* .7.	*36.* 2.51.	*45.* .3737 +.
7. .279.	*17.* .1.	*27.* 3100.	*37.* .1012.	*46.* 1.3808 +.
8. 8.	*18.* 10.	*28.* 17700.	*38.* 205200.	*47.* 3.3061 +.
9. 30.	*19.* .01.	*29.* 1000.	*39.* 43.6.	*48.* .1327 +.
10. .3.	*20.* .09.	*30.* .0019.		

Exercise 57.

1. $\frac{4}{5}$.	*5.* $\frac{11}{40}$.	*9.* $\frac{5}{8}$.	*13.* $\frac{3}{200}$.	*17.* $\frac{5}{8}$.	*21.* $\frac{1}{12}$.	*25.* $6\frac{1031}{2500}$.
2. $\frac{1}{4}$.	*6.* $\frac{3}{8}$.	*10.* $\frac{7}{8}$.	*14.* $\frac{1}{3}$.	*18.* $\frac{1}{16}$.	*22.* $1\frac{2}{5}$.	*26.* $10\frac{91}{800}$.
3. $\frac{3}{25}$.	*7.* $\frac{1}{40}$.	*11.* $\frac{37}{40}$.	*15.* $\frac{2}{3}$.	*19.* $\frac{1}{6}$.	*23.* $2\frac{11}{25}$.	*27.* $7\frac{2}{3125}$.
4. $\frac{1}{8}$.	*8.* $\frac{3}{40}$.	*12.* $\frac{3}{500}$.	*16.* $\frac{1}{6}$.	*20.* $\frac{1}{7}$.	*24.* $5\frac{2}{25}$.	*28.* $1\frac{7}{80}$.

Exercise 58.

1. .5.	*2.* .85.	*3.* $.16666\frac{2}{3}$.	*4.* $.23333\frac{1}{3}$.
.25.	1.45.	$.28571\frac{3}{7}$.	.0225.
.75.	1.3125.	$.55555\frac{5}{9}$.	$.01833\frac{1}{3}$.
.8.	4.72.	$.63636\frac{4}{11}$.	$.00385\frac{5}{7}$.
.125.	.234375.	$.61538\frac{6}{13}$.	$.00411\frac{1}{9}$.
.625.	.484375.	$.88235\frac{5}{17}$.	$1.98684\frac{4}{19}$.
.4375.	1.703125.	$.21052\frac{12}{19}$.	$.02736\frac{4}{11}$.
.6875.	3.3375.	$1.06666\frac{2}{3}$.	*5.* .25; .4; $.14\frac{2}{7}$.
.32.	1.8875.	$3.26666\frac{2}{3}$.	*6.* $.33\frac{1}{3}$; .4; .08.
.84.		$5.47619\frac{1}{21}$.	*7.* $.66\frac{2}{3}$; .4; .125.
.59375.			*8.* $.66\frac{2}{3}$; .375;
.96875.			.5625.

Exercise 59.

1. 11.205.	*4.* $7.5691\frac{2}{3}$.	*7.* 35.27862.	*10.* $.06586\frac{1}{3}$.
2. 4.7665.	*5.* 599.4.	*8.* $.0007781\frac{1}{4}$.	*11.* 147.
3. 1.8057.	*6.* 80.799924.	*9.* .225127504.	*12.* .00065.

13. 34000.
14. .000004.
15. 1670000.
16. 20000.
17. .0000001.
18. 100000000.
19. .0009125.
20. 32080.
21. 17.674.
22. 30 bu.
23. $3\frac{4}{17}$.

Exercise 60.

1. $161.27.
2. $61.47.
3. $71.68.
4. $90.22.
5. $69.75.
6. $5.33.
7. $2.06.
8. $3386.12½.
9. $9.91⅔.
10. $2.12½.
11. 13 ch.
12. 28.5 A.
13. 41 p.
14. 49 sh.
15. $91.35.
16. $253.19¼.
17. $5.43⅜.
18. $.50.
19. $372.50.
20. $6.75.
21. $18.
22. $23.33.
23. $39.60.
24. $62.67.
25. $59.50.
26. $11.88.
27. $62.25.
28. $51.33.
29. $53.33.
30. $9.
31. $28.88.
32. $15.33.
33. $12.
34. $7.63.
35. $9.40.
36. $6.33.
37. $68.25.

Exercise 61.

1. $8.73.
2. $32.96.
3. $10.435.
4. $9.03.
5. $169.20.
6. $27.04.
7. $151.02.
8. $436.20.

Exercise 62.

1. $77.36.
2. $622.79.
3. Dealer owes 32½¢.
4. $374.64.
5. $487.33.
6. $337.50.
7. $1653.75.

Exercise 64.

1. 117930 oz.
2. 10080 lb.
3. 193615 oz.
4. 25211 oz.
5. 91790 lb.
6. 3 T. 15 cwt. 40 lb. 10 oz.
7. 7 T. 8 cwt. 9 lb. 5 oz.
8. 6 T. 12 cwt. 80 lb.
9. 10 T. 8 cwt. 5 oz.
10. 9 T. 70 lb. 15 oz.
11. 2317766 oz.
12. 19 cwt. 28 lb. 6 oz.
13. 64 f.
14. 48 men.
15. 44 b.
16. 9 T. 17 cwt. 95 lb.

Exercise 66.

1. 1556 gr.
2. 3738 gr.
3. 32889 gr.
4. 50,412 gr.
5. 46,222 gr.
6. 3018 pwt.
7. 2 lb. 7 oz. 10 pwt. 16 gr.
8. 10 oz. 13 pwt. 5 gr.
9. 3 lb. 11 oz. 14 pwt.
10. 5 lb. 6 oz. 21 gr.
11. 8 lb. 16 pwt. 10 gr.
12. 4 lb. 8 oz. 5 pwt. 9 gr.
13. 2 lb. 10 oz. 10 pwt.; 11 medals.

Exercise 67.

1. 2628 gr.
2. 21490 gr.
3. 11855 gr.
4. 2916 gr.
5. 33940 gr.
6. 57792 gr.
7. 10 ℥ 5 ʒ 2 ℈ 8 gr.
8. 3 ℥ 4 ʒ 11 gr.
9. 1 lb. 6 ℥ 1 ℈ 5 gr.
10. 3 lb. 3 ʒ 2 ℈.
11. 4 lb. 1 ℥ 6 ʒ 17 gr.
12. 7 lb. 2 ℈ 10 gr.

Exercise 69.

1. 737 ft.
2. 12637½ ft.
3. 19978 ft.
4. 21131 ft.
5. 7306 in.
6. 721 in.
7. 160566 in.
8. 237620 in.
9. 5 yd. 2 ft. 10 in.
10. 8 rd. 3 yd. 1 ft. 6 in.
11. 2 mi. 220 rd. 2 yd. 2 ft.
12. 6 mi. 5 yd. 1 in.
13. 4 mi. 125 rd. 2 ft. 8 in.
14. 7 mi. 305 rd. 2 yd. 2 ft. 3 in.
15. 1125 l.
16. 24000 l.
17. 5 l.
18. 2464 rails.
19. $121893.75.
20. 2300 panels.

Exercise 71.

1. 1674½ sq. ft.
2. 102456 sq. ft.
3. 130903 sq. ft.
4. 10900 sq. in.
5. 138373 sq. in.
6. 25095024 sq. in.
7. 3 A. 88 sq. rd. 6 sq. yd.
8. 60 sq. rd. 25 sq. yd. 7 sq. ft.
9. 24 sq. rd. 2 sq. yd. 8 sq. ft. 80 sq. in.
10. 30 sq. yd. 5 sq. ft. 125 sq. in.
11. 2 A. 112 sq. rd. 21 sq. yd. 3 sq ft.
12. 93 sq. rd. 9 sq. yd. 2 sq. ft. 6 sq. in.
13. 6400 sq. ch.; 50 sq. ch.
14. 2 sq. mi. 220 A.

Exercise 73.

1. 102152 cu. in.
2. 155 cu. ft.
3. 960 cu. ft.
4. 474 cu. ft.
5. 3 cu. yd. 15 cu. ft. 525 cu. in.
6. 75 cu. yd. 21 cu. ft.
7. 127 cu. yd. 26 cu. ft.
8. 4 cu. yd. 20 cu. ft.; 91⅛ cords.

Exercise 75.

1. 380 pt.
2. 779 pt.
3. 689 pt.
4. 685 pt.
5. 277 qt.
6. 831 qt.
7. 27 gal. 3 qt. 1 pt. 2 gi.
8. 3 bu. 2 pk. 4 qt. 1 pt.
9. 5 bbl. 20 gal. 2 qt.
10. 4 bu. 3 pk. 7 qt. 1 pt.
11. 3 bbl. 8 gal. 1 qt. 3 gi.
12. 25 bu. 1 pk. 6 qt. 1 pt.
13. 7276½ cu. in.
14. 14.93 cu. ft.
15. 67.2 cu. in.; 57¾ cu. in.
16. 9.31 nearly.
17. $1.88.
18. $10.50.
19. 48 gal. 1 pt.

Exercise 77.

1. 326 far.
2. 9949 far.
3. 16115 far.
4. 70735 far.
5. $7; 28¢.
6. $34; 52¢.

7. 12*s.* 7*d.* 1 far.
8. £2 10*s.* 8*d.*
9. 15*s.* 8*d.* 3 far.
10. £3 14*s.* 5*d.* 1 far.
11. £8 16*s.* 11*d.* 1 far.
12. £10 3*d.* 2 far.
13. £16 13*s.* 3 far.
14. $63.26.
15. $3.65.
16. $50.12.
17. $320.82.
18. $28.23.
19. $16.79.
20. £100.
21. £24.
22. £120.
23. $9.80.

Exercise 79.

1. $38.60.
2. $36.89.
3. $11.67.
4. $22.05.
5. 388.6 fr., or 315.13 mk.
6. 639.63 fr., or 518.7 mk.
7. 5061.14 fr., or 4104.2 mk.
8. $2.32.
9. $7.62.

Exercise 80.

1. 6830 min.
2. 885990 min.
3. 1304320 min.
4. 2714991 min.
5. 19009545 sec.
6. 27291638 sec.
7. 15 hr. 55 min. 30 sec.
8. 64 da. 19 hr. 24 min.
9. 7 yr. 121 da. 16 hr.
10. 1 da. 11 hr. 35 min. 40 sec.
11. 2 yr. 19 hr. 8 min.
12. 9 yr. 307 da. 7 hr.
13. 2912430 sec.
14. 7053 hr.
15. 78 da.
16. 196 da.
17. 159 da.
18. 85 da.
19. 188 da.
20. 316 da.
21. 187 da.
22. 308 da.
23. Fall.

Exercise 82.

1. 127045″.
2. 540050″.
3. 738640″.
4. 1188186″.
5. 5° 50′ 26″.
6. 75° 40″.
7. 110° 37′ 14″.
10. 13635″.
11. 69.173 mi.
12. 1.153 mi.

Exercise 83.

1. 880 fathoms.
2. 5⅓ ft.; 3¾ ft.
5. 1728 u.; 15 doz.
6. $4.32.
7. 1440 p.; 4320 p.
8. 480 s.; 960 s.
9. 3 ba. 3 bun. 1 r. 6 q.
10. 2⅓¢.
11. 147 bun.
12. ⅐.
13. Barley; $19.71.

Exercise 84.

1. 8748 oz.
2. 4056231 min.
3. 146896 in.
4. 31459″.
5. 151231¾ sq. ft.
6. 13383 sh.
7. 15956 lb.
8. 3770095 sec.
9. 1661 pt.
10. 1883 pt.
11. 171632 cu. in.
12. 27372 sq. yd.
13. 26861 in.
14. 45010 gr.
15. 91491 in.
16. 55556 gr.
17. 3 mi. 73 rd. 4 yd. 1 ft. 6 in.

18. 3 A. 65 sq. rd. 15 sq. yd.
19. 7 T. 18 cwt. 76 lb. 4 oz.
20. 8 bu. 3 pk. 2 qt. 1 pt.
21. 15 da. 10 hr. 39 min. 40 sec.
22. 9 bbl. 21 gal. 2 qt. 1 pt.
23. 7 lb. 8 ℥ 4 ʒ 2 ℈ 16 gr.
24. 3 cu. yd. 20 cu. ft. 508 cu. in.
25. 66° 35′ 28″.
26. 8 lb. 11 oz. 16 pwt. 20 gr.
27. 307 rd. 1 yd. 2 ft. 9 in.
28. 146 sq. rd. 27 sq. yd. 6 sq. ft.
29. 7310 p.
30. 1710 times.
31. 41 da. 5½ hr.
32. 448.
33. $6624.
34. 1 mi. 130 rd.
35. $12523½.
36. 480 sacks.
37. 150 sacks.

Exercise 85.

1. 20 cwt. 50 lb. 2 oz.
2. 16 yr. 70 da. 20 hr. 55 min. 1 sec.
3. 51 lb. 15 pwt. 15 gr.
4. 27 lb. 4 ℥ 4 ʒ 1 ℈ 19 gr.
5. 38 bu. 2 pk. 7 qt.
6. 47 bbl. 11 gal. 3 gi.
7. £26 6*s.* 11*d.* 2 far.
8. 39 cu. yd. 25 cu. ft. 1435 cu. in.
9. 112 mi. 285 rd. 2 yd. 1 ft. 7 in.
10. 32 A. 103 sq. rd. 18 sq. yd. 1 sq. ft. 15 sq. in.
11. 44 T. 5 cwt. 59 lb. 4 oz.
12. 24 mi. 232 rd. 3 yd. 2 ft. 3 in.
13. 44 yr. 14 da. 22 hr. 47 min. 12 sec.
14. 114° 25′ 41″.
15. 42 A. 132 sq. rd. 2 sq. yd. 1 sq. ft. 117 sq. in.
16. 42 lb. 6 ℥ 2 ℈ 16 gr.

Exercise 86.

1. 8 bbl. 4 gal. 3 qt. 1 pt.
2. 3 T. 15 cwt. 97 lb. 2 oz.
3. 2 lb. 9 ℥ 7 ʒ 1 ℈ 12 gr.
4. 13 cu. yd. 17 cu. ft. 538 cu. in.
5. 8 gal. 2 qt. 1 pt. 1 gi.
6. 8° 37′ 35″.
7. 11 mi. 186 rd. 4 yd. 1 ft. 3 in.
8. 36 A. 118 sq. rd. 14 sq. yd. 6 sq. ft. 65 sq. in.
9. 103° 25′ 17″.
10. 54° 20′ 14″.
11. 30 mi. 270 rd. 1 yd. 9 in.
12. 3 bbl. 15 gal. 2 qt. 2 gi.
13. 68 mi. 153 rd. 3 yd. 2 ft. 2 in.
14. 6 A. 135 sq. rd. 5 sq. yd. 1 sq. ft. 62 sq. in.
15. 13 T. 3 cwt. 50 lb. 1 oz.
16. 44° 13′ 10″.
17. 1 mi. 174 rd. 5 yd. 1 in.
18. 51 sq. rd. 21 sq. yd. 3 sq. ft. 6 sq. in.
19. 9 mo. 12 da.
20. 8 mo. 10 da.
21. 4 yr. 4 mo. 7 da.
22. 67 yr. 9 mo. 22 da.
23. 23 yr. 29 da.
24. 26 yr. 6 mo. 18 da.
25. 4 yr. 9 mo. 27 da.
26. 85 yr. 9 mo. 2 da.
27. Jan. 10, 1908.
28. Oct. 22, 1782.
29. May 17, 1881.
30. Mar. 26, 1903; 50 yr. 1 mo. 23 da.; 23 yr. 11 mo. 14 da.

Exercise 87.

1. £52 14*s.* 10*d.* 2 far.
2. 58 T. 9 cwt. 78 lb. 12 oz.
3. 17 bbl. 14 gal. 3 qt. 2 gi.
4. 125 yr. 157 da. 11 hr. 25 min. 20 sec.
5. 360 lb. 4 ℥ 7 ʒ 17 gr.
6. 121 mi. 136 rd. 5 yd. 2 ft. 8 in.
7. 142 mi. 102 rd. 3 yd. 6 in.
8. 79 A. 12 sq. rd. 30 sq. yd. 1 sq. ft. 63 sq. in.
9. 89 A. 114 sq. rd. 9 sq. yd. 3 sq. ft. 40 sq. in.
10. 109° 26′ 15″.
11. 1078 bu. 3 pk. 1 qt. 1 pt.
12. 491 T. 5 cwt. 30 lb.
13. 29 A. 56 sq. rd. 28 sq. yd. 4 sq. ft.
14. 21 mi. 300 rd. 1 ft. 3 in.

Exercise 88.

1. 13 bu. 3 pk. 7 qt. 1 pt.
2. 3 yr. 214 da. 17 hr. 8 min. 41 sec.
3. 9 T. 17 cwt. 48 lb. 15 oz.
4. 15 gal. 1 pt. 3 gi.
5. 5 mi. 121 rd. 2 yd. 1 ft. 7 in.
6. 8 mi. 75 rd. 2 ft. 9 in.
7. 4 A. 130 sq. rd. 27 sq. yd. 8 sq. ft. 5 sq. in.
8. 5 A. 155 sq. rd. 8 sq. ft. 140 sq. in.
9. 6 lb. 5 ℥ 5 ʒ 1 ℈ 7 gr.
10. 171.
11. 540.
12. 18.
13. 720.
14. 8 hr.
15. 384 nearly.
16. 480.

Exercise 90.

1. 21° 17′ 30″.
2. 75° 10′ 30″.
3. 118° 57′ 15″.
4. 142° 59′ 45″.
5. 2 hr. 52 min. 40 sec.
6. 5 hr. 10 min. 41 sec.
7. 48 min. 31 sec.
8. 4 hr. 38 min. 14 sec.
9. 76° 20′.
10. 162° 28′.
11. 2 hr. 48 min. 34 sec.; 48 min. 34 sec. past 2 P.M.; 11 min. 26 sec. past 7 A.M.
12. 19 sec. past 5 P.M.; 29 min. 41 sec. past 1 P.M.
13. 44° 35′ 50″.
14. 123° 44′ W.
15. 32 min. 59 sec. past 9 A.M.
16. 75° 10′ W.
17. 44 min. 3 sec. past 6 A.M.; 15 min. 57 sec. past 6 P.M. of day previous.
18. 2° 20′.
19. 122° 42′ E.
20. 137° 4′ 15″; 9 hr. 8 min. 17 sec.
21. 82° 23′ 45″ W.
22. 18° 3′ 30″ E.

Exercise 91.

1. 280 rd.
2. 18 cwt. 33 lb. $5\frac{1}{3}$ oz.
3. 10 oz. 13 pwt. 8 gr.
4. 5*s.* 11*d.* 1 far.

5. 108 da. 3 hr. $33\frac{1}{3}$ min. 20 sec.
6. 20′ 50″.
7. 5 sq. yd. 7 sq. ft. 135 sq. in.
8. 2 gal. 1 qt. $1\frac{1}{2}$ gi.
9. 8 ℥ 6 ʒ 1 ℈ 4 gr.
10. 26 cu. ft. 1080 cu. in.
11. 3 pk. 1 pt.
12. 137 sq. rd. 28 sq. yd. 1 sq. ft. $68\frac{8}{29}$ sq. in.
13. $3\frac{2}{25}$ min.
14. $\frac{8}{25}$ pt.
15. $28\frac{2}{5}$ sq. in.
16. $7\frac{3}{5}$ in.
17. $25\frac{1}{2}$ hr.
18. $6\frac{1}{5}$ pt.
19. $\frac{3}{16}$ da.
20. $\frac{57}{64}$ bu.
21. $\frac{13}{18}$ lb.
22. $\frac{7}{9}$ mi.
23. $\frac{8}{27}$ A.
24. $\frac{11}{16}$ gal.
25. $\frac{24}{25}$ wk.
26. $\frac{1809}{5000}$ T.
27. 107 rd. 1 yd. 6 in.
28. 203 da. 9 hr. 56 min. 30 sec.
29. 5 oz. 2 pwt. 22 gr.
30. 106 sq. rd. 28 sq. yd. 6 sq. ft.
31. 58 sq. rd. 11 sq. yd. 5 sq. ft. $88\frac{1}{2}$ sq. in.
32. 34 rd. 3 yd. 1 ft. 8 in.

Exercise 92.

1. 6 da. 3 hr.
2. 11 oz. 2 pwt.
3. 16 cwt. 64 lb. 12.8 oz.
4. 2 pk. 2 qu. .8 pt.
5. 269 rd. 2 yd. 1 ft. 3.12 in.
6. 67 sq. rd. 20 sq. yd. 5 sq. ft. 18.72 sq. in.
7. 6 ℥ 7 ʒ 17.76 gr.
8. 29 da. 16 hr. 11 min. 16.8 sec.
9. 10 cu. ft. 216 cu. in.
10. 14 rd. 3 yd. 2 ft. 10.56 in.
11. 14 gal. 1 pt. 1.6 gi.
12. 28 sq. rd.
13. 2 mo. 4 da. 2 hr. 16 min. 48 sec.
14. 45 rd. 11.88 in.
15. 3 mo. 21 da. 14 hr. 24 min.
16. .1305° +.
17. .6401 + mi.
18. .8276 + T.
19. .2854 lb.
20. .0107 + yr.
21. .65 bu.

Exercise 94.

1. 66 sq. rd. 9 sq. yd. 3 sq. ft. 33 sq. in.
2. 3.6716 w.
3. 26 da. 6 hr. 23 min. 42 sec.
4. $ 51.75.
5. 1 cu. yd. 5 cu. ft.; $\frac{125}{216}$ oz.
6. $16688\frac{4}{7}$ steps.
7. 9 hr. 45 min.
8. 21 mi. 192 rd.
9. $ 1.96.
10. 2 min. 30 sec. past 7 A.M.
11. 51 min. 27 sec. past noon.
12. $ 25.95.
13. $9\frac{2}{19}$ ¢.
14. $ 317.70.
15. $ 5527.79.
16. $\frac{11}{16}$.
17. 0.1.
18. 4 bu. 3 pk. 7 qt. $1\frac{3}{8}$ pt.
19. 45 da. 6 hr. 15 min.
20. 66 ft.
21. 3 mi. 45 rd. 1 yd. 2 ft. 1.2 + in.
22. July 2.
23. $ 50.54 +.
24. $537\frac{2}{3}$.
25. $6\frac{201}{313}$ da.

26. 84° 59′ 35″.
27. 12 cwt. 84 lb. $1\frac{1}{12}$ oz.
28. 70 bbl.
29. 1st, £1 1s. 5d. 2.56 far.
2d, 120 rd. 2 yd. 1 ft. 4.704 in.
30. 0.994+ lb.
31. 15 sq. ft.
32. 495000 cu. ft.
33. 97.
34. 5 yr. 334 da. 14 hr. 42 min. 45 sec.
35. 14 da. 16 hr. $33\frac{3}{4}$ min.
36. $3\frac{3}{26}$ ft.
37. 1.552+ ft.
38. $6\frac{2}{33}$ oz.
39. 0.118+ sq. mi.
40. 2 yr. 45 da. 1 hr. 25 min. 19.2 sec.
41. 41° 3′ 59″.
42. $7\frac{4}{5}$ ft.
43. $80\frac{2}{3}$ ft.
44. $3\frac{3}{11}$ hr.
45. 0.58+ mi.
47. 1 mi. 279 rd. 2 yd. 1 ft. 11.292 in.
48. $12\frac{3}{4}$.
49. 128° 14′ 32″. / 174° 33′ 11″.
50. 1415.5+ yr.; 147.4 yr.

Exercise 95.

1. 10 A.
2. 32 A.
3. $7\frac{1}{2}$ A.
4. $\frac{7}{22}$ A.
5. $22\frac{2}{3}$ sq. yd.
6. $1\frac{5}{11}$ sq. rd.
7. 63 A.
8. $25\frac{5}{9}$ sq. yd.
9. $1394\frac{10}{11}$ sq. rd.
10. 1 A. 106 sq. rd. 20 sq. yd. 1 sq. ft. 72 sq. in.
11. 78 A.
12. 140 sq. ft.
13. 1920 A.; 3200 A.; 320 A.; 240 A.; 336 A.
14. 1280 rd.; 960 rd.; 640 rd.
15. $23\frac{4}{27}$ sq. yd.
16. $1351\frac{1}{3}$ sq. ft.

Exercise 96.

1. 314.16 sq. ft.
2. 706.86 sq. ft.
3. 235.97 sq. yd.
4. 4860.404+ sq. rd.
5. 636.174 A.
6. 76.7 A. (about).
7. 19.7933 A.
8. 788.544+ A.
9. 13.18+ A.
10. .7854 sq. in.; 7.0686 sq. in.

Exercise 97.

1. $95.99. 2. $1232. 3. $743.32. 4. $509.44. 5. $417.60. 6. $5632.

Exercise 98.

1. 11.62\frac{2}{3}$. 2. $84.35. 3. $26.67. 4. $385.02.

Exercise 99.

1. 68 yd. 2. $64\frac{1}{3}$ yd. 3. 119 yd.; $113\frac{1}{3}$ yd. 4. $66.38. 5. Crosswise; $1.08.

Exercise 100.

1. 18 rolls. 2. $21.60. 3. $39.60. 4. $28.44.

Exercise 101.

1. 16; 8.
2. 150; $168\frac{3}{4}$.
3. 960.
4. 3240; 3240; 540.
5. $16.20.
6. $5.28.
7. $86.02.

Exercise 102.

1. 480 cu. in.; $\frac{5}{18}$ cu. ft.
2. $9\frac{3}{4}$ cu. yd.
3. 506.24 bu.
4. 3072 bu.
5. 2252.8 bu.
6. 3456 bu.
7. $36.56.

Exercise 103.

1. 93[illegible]$\frac{5}{7}$ gal.
2. $26660\frac{4}{7}$ gal.
3. $7899\frac{3}{7}$ gal.
4. 69120 gal.
5. $67863\frac{3}{11}$ gal.
6. $94254\frac{6}{11}$ gal.

Exercise 104.

1. $18\frac{2}{11}$ p.
2. 24 p.
3. $426\frac{2}{3}$ p.
4. $545\frac{5}{11}$ p.
5. $155\frac{5}{9}$ cu. yd.
6. 39920 cu. yd.
7. $16000.
8. $2295.
9. $1028.36.

Exercise 105.

1. 68740 br.
2. $1099.84.
3. $1838.59; $1216.
4. Cost, $1356.81; Laying, $266.52.
5. $81.90.

Exercise 106.

1. $3\frac{3}{13}$ T.; $3\frac{11}{13}$ T. 2. 56 T. 3. 1323 T. 4. $3499.20.

Exercise 107.

1. 28.2744 sq. in., 153.9384 sq. in., 201.0624 sq. ft., 1256.64 sq. rd., .070686 sq. in., 1.76715 sq. yd.
2. 302.8125 A.
3. Former.
4. 484 sq. in.; $\frac{4}{351}$ cu. yd.
5. 88 sq. rd., 14 sq. yd.
6. 840 cu. in.; 240 cu. in.
7. 2880 bd. ft.
8. $960.
9. 15018\frac{2}{3}$.
10. 1012.4 bu. seed; 806.4 bu. potatoes.
11. $15600.
12. $17.50.
13. $18.93.
14. 634\frac{2}{3}$.
15. $98.10.
16. $38.
17. 1134.12 sq. ft.
18. $108.78.
19. $523.15.
20. $73.81.

21. \$131.27.
22. \$273.55.
23. $2565\frac{2}{11}$ gal.
24. 8355 + lb.
25. \$89760.
26. 3.96 in.; .0000625.
27. \$38.12.
28. 1175.21 + bu.; 10940.26 gal.
29. \$37.44.
30. 273.67 sq. yd.
31. 107.42 bu.
32. $\$4693.33\frac{1}{3}$.
33. 1105.8432 sq. ft.
34. \$338.52.
35. \$24.35.
36. 263.18 lb.
37. 239.37 gal.
38. 56.55 cu. in.
39. $79\frac{3}{16}$ cu. ft.
40. $34644\frac{17}{32}$ lb.

Exercise 108.

1. 24 boys.
2. 10 days.
3. 7 lb.
4. \$75.60.
5. 45 bu.
6. 63.6 tons.
7. \$19.50.
8. \$3600.
9. 27 pupils.
10. $5\frac{1}{3}$ days.
11. 72 horses.
12. \$715.50.
13. Former.
14. 276 lambs.
15. 96 tons.
16. 87360 inh.
17. \$4060.
18. \$7820.
19. \$170.
20. 152 p.
21. 210 m.
22. 3 ft.
23. $\frac{5}{8}$ bu.
24. 18 tr.

Exercise 110.

1. 11 books.
2. 16 lambs.
3. 250 boys.
4. 240 lbs.
5. $7\frac{1}{3}$ mi.
6. 300 da.
7. $112\frac{1}{2}$ A.
8. 325 yr.
9. 348 T.
10. 1032 cities.
11. \$4.00.
12. \$0.51.
13. \$4.80.
14. $\$30\frac{2}{3}$.
15. \$52.50.
16. 21000.
17. 162 lb.
18. \$59.50.
19. $\frac{5}{12}$; worth $\$3333\frac{1}{3}$.
20. Equal.

Exercise 113.

1. 20%.
2. 35%.
3. $37\frac{1}{2}$%.
4. 8%.
5. $7\frac{1}{4}$%.
6. $232\frac{1}{7}$%.
7. $87\frac{1}{2}$%.
8. $2\frac{1}{2}$%.
9. $12\frac{1}{2}$% gain.
10. 70% gain.
11. 100% gain.
12. $33\frac{1}{3}$%; 75%; $62\frac{1}{2}$%.
13. $7\frac{3}{5}$%.
14. 8%.
15. 18%.
16. $37\frac{1}{2}$%.
17. 25% gain.
18. $3\frac{1}{5}$% gain.
19. $10\frac{1}{2}$% gain.
20. 80% gain.
21. $33\frac{1}{3}$% gain.
22. 25% gain.
23. 14% gain.
24. $41\frac{1}{3}$% loss.
25. 16% loss.
26. $\frac{1}{2}$%.
27. 0.67 + %.
28. 0.75 + %.
29. $0.8\frac{1}{3}$%.
30. $0.5\frac{5}{6}$%.
31. $0.5\frac{5}{9}$%.
32. 60%.
33. $66\frac{2}{3}$%.
34. 30%.
35. 36%.
36. $66\frac{2}{3}$%.
37. 80%.
38. $83\frac{1}{3}$%.
39. $87\frac{1}{2}$%.
40. 5%.
41. 55% boys.
42. 85%.
43. 24%.
44. 85%.
45. $72\frac{2}{15}$%.
46. 16% gain.
47. 20% gain.
48. Newsp., $33\frac{1}{3}$%.
49. Cattle; 80%.

Exercise 114.

1. \$2000.
2. 15200 bu.
3. 4600 gal.
4. 13750 da.
5. \$15800.
6. \$121.
7. \$970.
8. $28\frac{3}{4}$ T.
9. \$521.
10. \$316.
11. 1750.
12. 5200.
13. 600.
14. 2000.
15. 2900.
16. 1620.
17. 175.
18. 340.
19. 240 bu.
20. 300 da.
21. 2900 T.
22. $167\frac{1}{2}$ mi.
23. \$489.
24. \$3075.
25. 776.
26. $184\frac{1}{2}$.
27. 1144.
28. $1285\frac{5}{7}$.
29. 3800.
30. $8333\frac{1}{3}$.
31. \$50.
32. \$346.
33. $7\frac{1}{2}$.
34. \$9.60.
35. \$85.86.
36. 1.07325.
37. 128.
38. 250 da.
39. \$234.
40. $91\frac{1}{3}$.
41. $8\frac{1}{3}$ mi.
42. $2\frac{1}{2}$.
43. \$40.
44. \$284.
45. \$10.71.
46. \$315.
47. 67500.
48. \$2000.
49. 75 lb.; 108 lb.
50. 5¢.
51. 30¢.
52. 900 lambs.
53. 128 yd.
54. Lost \$5.00.
55. 568 mi.
56. \$198.
57. \$95.50.
58. \$15700.

Exercise 116.

1. 101.92 ft.; \$24.619; $0.65\frac{1}{3}$.
2. $66\frac{2}{3}$%; $37\frac{1}{2}$%; 5%; $133\frac{1}{3}$%.
3. 60%; 25%; 125%; $166\frac{2}{3}$%.
4. 700; 525; 200.
5. $266\frac{2}{3}$; 144; 40.
6. 64.96 A.; 7.511 mi.
7. $14\frac{1}{2}$%.
8. \$58.40.
9. $17.17\frac{1}{3}$ lb.
10. 15%.
11. \$71000.
12. \$5.75.
13. 125%.
14. 711.40.
15. \$866.88.
16. $1\frac{1}{3}$%; $12\frac{1}{2}$%; $14\frac{2}{7}$%; 25%; $8\frac{1}{3}$%; $33\frac{1}{3}$%; $2\frac{7}{9}$%; $4\frac{1}{6}$%.
17. $83\frac{1}{3}$%.
18. \$36800.
19. Increased 10.08 A. or 96%.
20. $36\frac{1}{2}$% gain.
21. 20%; 25%; 50%.
22. \$40000.
23. $47\frac{1}{2}$%.
24. \$2917.70.
25. .45; 1; 43.78; $\frac{1}{250}$.
26. 6; 33.2; $\frac{3}{500}$; $\frac{4}{375}$.
29. 92%.
30. \$262.50.
31. 25%.
32. \$1080.
33. \$75400.
34. 632.
35. $16\frac{2}{3}$% gain.
36. 14.208%; 14.247%.

Exercise 117.

1. 25%.
2. \$50.
3. \$900 cost.
4. \$2015.
5. Gained $6\frac{2}{3}$%.
6. James; \$40; 20%.
7. 30% loss.
8. Gained \$135.

9. Gained \$.30; $3\frac{3}{4}$%.
10. \$1.80.
11. Lost \$32; 16%.
12. \$1.14; \$1.20.
13. \$3.20.
14. \$420.

Exercise 119.

1. \$83.19.
2. \$5982.30.
3. \$1558.49.
4. Latter; \$499.20.
5. 32%.
6. \$430.92.

Exercise 120.

1. \$2.40; \$77.60.
2. \$12062.50.
3. 5%.
4. $3\frac{1}{2}$%.
5. \$2335.
6. \$2760.
7. \$223.59.
8. \$2341.50.
9. 278 b.
10. \$18.75.
11. \$3459.
12. \$3027.
13. \$3475.50.
14. \$10102.50.
15. 50 sh.

Exercise 121.

1. \$31.50; \$45.90; \$54.
2. \$253.50.
3. \$12.53; \$26.78; \$30.98; \$81.09; \$15.12; \$200.
4. .018.
5. .025.
6. \$22356.
7. .0085.
8. \$235.20.
10. Take $\frac{2}{3}$ of the results of example 3.

Exercise 122.

1. \$72.80.
2. \$198.80.
3. \$105.
4. \$486.52.
5. \$881.25; \$1092.75 \$1311.30.
6. 18%.
7. \$3.894.

Exercise 123.

1. \$112.95; \$125.50.
2. \$12540; \$540.
3. \$156.
4. \$6122.45.
5. .0175.
6. \$9846.
7. $.00\frac{21}{48}$.
8. \$69600; \$5400.

Exercise 124.

1. \$62.50; $4\frac{1}{6}$%.
2. First, $5\frac{5}{8}$%; Second, $5\frac{5}{9}$%.
3. First, $5\frac{2}{5}$ + %; Second, 6 + %.
4. \$25200.
5. \$60750.

Exercise 125.

1. \$31.
2. $16\frac{2}{3}$%.
3. \$39200.
4. \$5240.
5. $2\frac{1}{2}$%.
6. Lost; \$6.25; 4%.
7. $33\frac{1}{3}$%; 50%.
8. \$11400.
9. \$15345.

10. \$49.60.
11. \$17384.14.
12. 8%.
13. \$144.80.
14. $15\frac{5}{173}$%.
15. \$77.
16. 50%.
17. 28%.
18. Lost; 4%.
19. \$38.08.
20. \$1200.
21. \$$2733\frac{1}{3}$.
22. 50%.
23. 675.
24. 558 bu.
25. \$715.
26. \$30.50.
27. First; $4\frac{2}{3}$%.
28. Lad; 25%.
29. 12%.
30. $3\frac{2}{5}$% or \$21.76.
31. \$3000.
32. $16\frac{2}{3}$%.
33. 28 bu.; $7\frac{1}{4}$%.
34. Lost; \$5; 4%.
35. $88\frac{9}{10}$%; $11\frac{1}{10}$%; $12\frac{1}{2}-$%.
36. $6\frac{1}{4}$%.
37. $13\frac{1}{3}$% (about).
38. $44\frac{4}{5}$%.
39. \$1140.
40. Com. was $2\frac{1}{2}$%.
41. Com. was 3%.
42. \$2000.
43. $37\frac{1}{2}$%; $81\frac{1}{4}$%.
44. \$1.65.
45. 28%.
46. $19\frac{4}{5}$.
47. $23\frac{2}{11}$%.
48. 20%.
49. \$1664; \$4736.
50. $16\frac{2}{3}$%; 70%; $43\frac{3}{4}$%; $22\frac{1}{2}$%; $\frac{1}{3}$%; $32\frac{1}{2}$%; $166\frac{2}{3}$%; 450%.
51. $\frac{3}{400} = .0075$; $\frac{9}{1000} = .009$; $\frac{15}{200} = .075$. $3\frac{2}{5} = 3.4$, etc.
52. 40%.
53. 1680 T.
54. The advance; 20%.
55. $66\frac{2}{3}$%.
56. 40%.
57. \$168; $5\frac{85}{193}$%.
58. \$8500.
59. \$7500.
60. 1504 shares and \$8 over.
61. 233 sh.; \$58.25; \$4077.50.
62. 64% boys.
63. $33\frac{1}{3}$%.
64. $40\frac{10}{11}$%.
65. \$24.50.
66. \$5600.
67. \$30.
68. 25.51%.
69. \$55; 15+ %.

Exercise 126.

1. \$76.50.
2. \$60.50.
3. \$123.60.
4. \$19.53.
5. \$91.21.
6. \$1265.75.
7. \$82.77.
8. \$5.76.
9. \$709.50.
10. \$90.36.
11. \$1612.05.
12. \$236.78.

Exercise 128.

1. \$42.64.
2. \$81.75.
3. \$332.76.
4. \$406.55.
5. \$68.15.
6. \$25.68.
7. \$617.97.
8. \$438.86.
9. \$444.85.
10. \$19.10.
11. \$84.25.
12. \$34.46.
13. \$3403.13.
14. \$10848.19.
15. \$6457.24.
17. \$2854.52.
18. \$3784.45.
19. \$5.84.
20. \$0.73.

Exercise 129.

1. \$11.20.
2. \$46.80.
3. \$8.58.
4. \$27.07.
5. \$85.58.
6. \$194.78.
7. \$67.85.
8. \$27.71.
9. \$1.74.
10. \$.63.
11. \$1633.40.
12. $\frac{71}{72}$.

Exercise 130.

1. 5%.
2. 4%.
3. 6%.
4. 7%.
5. $4\frac{1}{2}$%.
6. 6%.
7. 3%.
8. 4%.
9. $5\frac{1}{2}$%.
10. 6%.
11. 4%.
12. $5\frac{1}{2}$%.
13. $3\frac{1}{2}$%.

Exercise 131.

1. 3 yr. 6 mo.
2. 5 yr. 3 mo.
3. 1 yr. 9 mo.
4. 2 yr. 10 mo.
5. 8 yr. 4 mo.
6. 3 yr. 2 mo. 15 da.
7. 5 yr. 8 mo. 20 da.
8. 7 yr. 6 mo. 24 da.
9. 3 yr. 3 mo. 10 da.
10. 2 yr. 9 mo. 18 da.
11. 9 yr. 10 mo. 12 da.
12. 5 yr. 7 mo. 25 da.

Exercise 132.

1. $460.
2. $78.40.
3. $55.
4. $288.
5. $416.
6. $25.20.
7. $82.
8. $151.20.
9. $95.
10. $216.

Exercise 133.

1. $375; $3.75.
2. $4240; $137.80.
3. $7800; $455.
4. $1440.

Exercise 134.

1. 5%.
2. $97.50.
3. 9 yr. 5 mo. 15 da.
4. $104.
5. 6 yr. 7 mo.
6. $3\frac{3}{5}$%.
7. $765.55.
   $760.09 +.
   $771.08.
8. $312.65.
9. 25 yr.
   20 yr.
   $16\frac{2}{3}$ yr.
   $14\frac{2}{7}$ yr.
   $12\frac{1}{2}$ yr.
10. 1 yr. 3 mo. 24 da.
11. 8%.
12. $2684.92.
13. $294.66.
14. 5%.
15. $491.96.
16. 14%.
17. 3 yr. 6 mo.
18. $1008.
19. $478.79.
20. 2 yr. 10 mo. 9 da.

Exercise 135.

1. Aug. 7, 1901; $366.
2. Aug. 8, 1893; $963.05.
3. Jan. 6, 1898; $706.13.
4. June 14, 1900; $536.22.
5. $3106.37.
6. $549.65.
7. $2838.50.
8. $6830.16.
9. $100.83.
10. $1707.58.
11. $4430.74.
12. $13402.84.

Exercise 136.

1. $288.94.
2. $682.71.
3. $997.89.
4. $566.22.
5. $2015.52.

Exercise 137.

1. $363.
2. $1325.35.
3. $422.98.

Exercise 138.

1. $1179; $1176.
2. $13.05; $13.49.
3. $790.67.
4. $14.73; $635.27.
5. $1386.10.
6. $5012.97.
7. Sept. 1, 1895; 48 da.; $12.80; $1587.20.
8. May 25, 1897; 54 da.; $21.32; $2009.68.
9. 3 mo. 9 da.; $15.94; $875.81.
10. 2 mo. 27 da.; $20.61; $1197.84.
11. Mar. 18, 1896; 79 da.; $23.88; $1790.07.
12. T. D. = $74.47; B. D. = $87.50.
13. $467.92.
14. $2312.90.

Exercise 139.

1. $900.
2. $3620.
3. $75000.
4. $833.33.

Exercise 140.

1. $53.88.
2. $477.57.
3. $174.65.
4. $148.51.
5. $393.89.
6. $106.16.
7. $579.77.
8. $895.75.
9. $60.14.
10. $854.80;
    $821.90;
    $863.

Exercise 141.

1. $110.25; $810.25.
2. $259.20; $1459.20.
3. $13.26; $103.26.
4. $28.28; $153.28.
5. $138.80; $888.80.
6. $.46.
7. 1st = 2d = $1560; $1793.27; $1776.

Exercise 142.

1. $902.25.
2. $2496.87.
3. $7619.
4. $99875.
5. $568.58.
6. $2971.25.
7. $2394.
8. (none printed)
9. $2490.66.
10. $1671.60.
11. $448.87 +.

Exercise 143.

1. $10957.50.
2. $30649.50.
3. £2750.
4. 11922.63 fr.
5. £1604.
6. $659.87.
7. $2740.79.
8. $1759.04.
9. $315.88.
10. 41680 fr.
11. 33160.62 mk.

Exercise 144.

1. 6 mo. 15 da. (about).
2. 6 mo. 23 da.
3. April 24.
4. Oct. 17.
5. 7 mo. 3 da.
6. 62 da.; Dec. 18.
7. 8 mo. 25 da.
8. Nov. 1.
9. Aug. 29.
10. 11 mo. 20 da.
11. 11 mo. 23 da.
12. 22 mo. 2 da.
13. 7 mo. 16 da.

Exercise 145.

1. $\frac{3}{7}$.
2. $\frac{3}{5}$.
3. $\frac{31}{32}$.
4. $\frac{10}{11}$.
5. $\frac{4}{5}$.
6. $\frac{11}{35}$.
7. $\frac{5}{7}$.
8. $\frac{15}{19}$.
9. $\frac{1711}{1350}$.
10. $\frac{4}{7}$.
11. 45.
12. 3.
13. 77.
14. 36.
15. 98.
16. $17\frac{1}{15}$.
17. $9\frac{1}{6}$.
18. $6\frac{77}{180}$.
19. $\frac{7}{9}$.
20. $22\frac{2}{25}$.

Exercise 146.

1. 16.
2. 96.
3. 48.
4. 10.
5. $4\frac{2}{3}$.
6. $12\frac{4}{7}$.
7. $48\frac{4}{15}$.
8. $11\frac{2}{3}$.
9. $12.42\frac{6}{13}$.
10. 2.85.
11. $11877.60.
12. 500 A.
13. $771\frac{3}{7}$ A.
14. 63 ft.
15. $133\frac{1}{2}$ ft.
16. $26\frac{2}{3}$ bu.
17. $135.
18. $97.01.
19. 8 da.
20. 30 da.
21. 16 men.
22. $1\frac{1}{2}$ yr.
23. $5000.
24. $3.
25. 51.
26. $24\frac{1}{2}$.
27. $6\frac{2}{9}$.
28. .0245+.
29. $23\frac{5}{9}$ hr.

Exercise 147.

1. $840.
2. $84\frac{2}{3}$ q.
3. $2820.
4. $7612\frac{1}{2}$ lb.
5. 14 da.
6. 384 bbl.
7. 55 da.
8. $58\frac{1}{3}$ da.
9. $2133\frac{1}{3}$ ft.
10. 400 lb.
11. $4\frac{4}{5}$ ft.
12. 108 men.

Exercise 148.

1. 3125, 4375, 5000, 7500.
2. 525, 875, 1925, 2975.
3. 5300, 10600, 15900.
4. 552, 828, 1104, 1380.
5. 550, 650, 800.
6. $12750, $15000, $18750.
7. $54\frac{18}{23}$, $73\frac{1}{23}$, $82\frac{4}{23}$.

Exercise 149.

1. $4.20, $5.40, $7.80.
2. $1600, $2050, $850.
3. $140, $160, $200.
4. $9600, $7200, $3200.
5. $640, $1420, $1240.
6. $5000, $4000, $3000.
7. $240, $200, $135.
8. $280, $192, $288.
9. $12.60, $18, $18.90.
10. $2700, $1620, $3780.

Exercise 151.

1. 676.
2. 21952.
3. 12769.
4. 262144.
5. 8615125.
6. 3.0625.
7. 12326.391.
8. 985.96.

9. .000054872.
10. .287496.
11. $2\frac{10}{27}$.
12. $12\frac{1}{4}$.
13. $181\frac{26}{27}$.
14. $76\frac{49}{64}$.
15. $364\frac{148}{343}$.
16. 50625.
17. 279841.
18. 24414.0625.
19. $31\frac{163}{216}$.
20. $1\frac{4641}{10000}$.
21. 432.
22. 1024.
23. $31\frac{1}{4}$.
24. 18.
25. $3\frac{3}{7}$.
26. $76\frac{9}{16}$.
27. 6834375.
28. $\frac{3}{25}$.
29. 21870.

Exercise 152.

1. 26.
2. 29.
3. 31.
4. 42.
5. 58.
6. 67.
7. 83.
8. 136.
9. 417.
10. 843.
11. 938.
12. 7403.
13. 8017.
14. 9091.
15. 9807.
16. 35047.
17. 46708.
18. 73594.
19. 7.86.
20. 83.4.
21. .907.
22. 1.324.
23. 37.68
24. .7118.
25. 3.4171.
26. 418.75.
27. 5.0809.
28. 6.324.
29. 2.828.
30. 5.567.
31. 4.147.
32. 3.674.
33. .948.
34. .866.
35. .836.
36. 2.291.
37. 1.825.
38. 15.684.
39. 5.479.
40. 2.723.
41. 5.1079.
42. 6.013.
43. 10.005.

Exercise 154.

1. 27.
2. 46.
3. 58.
4. 63.
5. 84.
6. 124.
7. 317.
8. 538.
9. 493.
10. 682.
11. 719.
12. 926.
13. 4.85.
14. .647.
15. 78.7.
16. 8.09.
17. .918.
18. 180.7.
19. 2.419.
20. 30.68.
21. 5.703.
22. .7908.
23. 1.817.
24. 2.289.
25. 3.072.
26. 1.650.
27. 4.641.
28. 2.683.
29. 4.319.
30. 3.036.
31. 4.013.
32. 8.004.
33. 3.715.
34. 2.180.

Exercise 156.

1. 24.
2. 36.
3. 42.
4. 48.
5. 63.
6. 75.
7. 96.
8. 135.
9. 162.

Exercise 157.

1. 24.
2. 47.
3. 83.
4. 95.
5. 186.
6. 328.
7. 23.
8. 36.
9. 1.76.
10. 1.95.
11. 2.06.
12. 2.15.
13. 2.09.
14. 2.71.
15. 4.03.
16. 1.03.
17. 2.56.
18. 1.28.
19. 1.96.
20. 1.78.
21. 6.96.
22. 2.660, .841, 2.018.
23. .622, .196, .087.
24. .968, .306, .027.
25. 1.84, .85, .42.
26. .41, .19, 2.66.
27. 1.13, .77, .54.

Exercise 158.

1. 17.
2. 12.
3. 20.
4. 53.
5. 15.
6. 48.
7. 145.
8. 451.
9. 952.
10. 9.4339.
11. 10.816.
12. 26.324.
13. 112.645.
14. 43.9824.
15. 144.5136.
16. 53.4072.
17. 41.888.
18. 468.7267.
19. 551.0366.
20. 12.732.
21. 119.36.
22. 18.207.
23. 33.04.
24. 24 ft.
25. 205 rd.
26. 81 ft.
27. 40 ft.
28. 96 ft.
29. 26.7036 in.
30. 24881 mi.
31. 381.72 ft.

Exercise 159.

1. 280 sq. ft.
2. 852 sq. ft.
3. 200 sq. yd.
4. 35 A.
5. 39 sq. in.
6. 135 sq. in.
7. $163\frac{7}{11}$ sq. rd.
8. $337\frac{1}{2}$ sq. yd.
9. 340 A.
10. 113.0976 sq. in.
11. 78.54 sq. rd.
12. 509.29 sq. ft.
13. 36 sq. in.
14. 90 sq. ft.
15. 180 sq. yd.
16. 306 sq. rd.
17. 756 sq. mi.
18. 10.82 sq. in.
19. 2771.2 sq. rd.
20. 414.691 sq. ft.
21. 7.245 A.
22. $79200.
23. $4908.75.
24. (a) 1392 sq. ft.
    (b) 30 ft.
    (c) 11784 sq. ft.

Exercise 160.

1. 120 sq. in.
2. 48 sq. ft.
3. 150 sq. in.
4. $997\frac{1}{2}$ sq. ft.
5. 132 sq. ft.
6. 131.95 sq. ft.
7. 1005.31 sq. ft.
8. 2764.608 sq. yd.
9. 4.7124 sq. ft.
10. 1507.968 sq. in.
11. 113.0976 sq. ft.
12. 1134.1176 sq. in.
13. $291.60.
14. $55.85.
15. $395.84.
16. 148.925 sq. ft.

Exercise 161.

1. 120 cu. ft.
2. 720 cu. ft.
3. $2338\frac{1}{3}$ cu. ft.
4. 1728 cu. in. ; 2197 cu. in.
5. 300 cu. ft.
6. $186\frac{2}{3}$ cu. yd.
7. 17.6715 cu. ft.
8. 530.145 cu. in.
9. 523.6 cu. in. ; 1436.758 cu. ft.
10. 696.9116 cu. in.
11. 366 cu. ft.
12. 929.914 cu. ft.
13. 47001.6 gal.
14. 17.4097 cu. in.
15. 30.4896 cu. in.
16. 163.405 cu. ft. ; 153.938 sq. ft.
17. 240000 drops.
18. 8.45 in. ; 12.32 in.
19. 3.94 ft.
20. 378.675 bu.
21. 207.84 cu. ft. ; 40.037 ft. ; 360.333 sq. ft.

Exercise 162.

1. 194.4 sq. in.
2. $21\frac{1}{3}$ in.
3. 759.5 sq. ft.
4. 153.94 sq. in.
5. 275 sq. ft.
6. 175.5 cu. in.
7. 2143.75 cu. ft.
8. 4 : 9 ; 27 : 125.
9. 7 : 9 ; 49 : 81 ; 343 : 729.
10. Four times as much; eight times as much.
11. 4 in.
12. $6\frac{12}{13}$ ft.
13. 1012.5 yd. ; 37.5 yd.
14. $1000.
15. 1 : 121 ; 1 : 1331.
16. 10.09 rd.
17. 1 : 2 ; 1 : $\sqrt[3]{4}$.
18. $\$7\frac{1}{3}$.
19. $\$2\frac{7}{9}$.
20. 194.74 lb.

Exercise 168.

1. = 73240 dm. = 732.4 Dm. = 732400 cm.
2. = .3608 Kg. = 36080 cg. = 3608 dg.
3. = 7124500 sq. cm. = .071245 sq. Hm. = .071245 Ha.
4. = 50321.7 ml. = 5.03217 Dl. = .0503217 Kl.
5. = .0055171 Ha. = .55171 a. = .55171 sq. Dm.
6. = 25000 cu. cm. = 25 l. = .25 dst.
7. = 53 dst. = 5300 cu. dm. = 5.3 st.
8. = 1234.5 Dm. = 123450 dm. = 12345000 mm.
9. = 3.2671 Dg. = .032671 Kg. = 32671 mg.
10. = 10673 ml. = 1.0673 Dl. = .010673 Kl.
11. = .0000083 cu. m. = .0083 l. = 8300 cu. mm.
12. = .4671 cu. m. = 467100 cu. cm. = 4671 dl.
13. = 50070000 mg. = 500700 dg. = 50.07 Kg.
14. = 3.755 Ha. = 37550 ca. = 375.5 sq. Dm.
15. = 4000 a. = 400000 sq. m. = 400000 ca. = 4000 sq. Dm.
16. = 34575 cm. = 3.4575 Hm. = .34575 Km. = 345750 mm.
17. = 86.32 sq. m. = .008632 Ha. = .8632 sq. Dm.
18. = 38500 cl. = .385 cu. m. = .000000000385 cu. Km.
19. 75 Kg. ; 1000 Kg. ; 1.7 Kg. ; 3.5 Kg. ; 11000 Kg.
20. 75 l. ; 5000 l. ; .03 l. ; .587 l. ; 1.3875 l.
21. = 4230 dl. = 423000 cu. cm. = 423 cu. m. 423000 g. = 423 Kg.

Exercise 169.

1. 96.05 m.
2. 16.2 a.
3. 50.6090003 cu. m.
4. 1458.6061 g.
5. 3242.405 l.
6. 8071.97 a.
7. 1906.233 m.
8. 7178.22661 g.
9. 30.917389128 cu. m.
10. 11356.212 l.
11. 380.827 l.
12. 226385 sq. m.

13. 79.927 m.
14. 41.691 l.
15. 899.943 sq. m.
16. 2534.43 g.
17. 7547.362 m.
18. 7284.654 l.
19. 57307.924375 l.
20. 175.2 m.
21. 256227.3 g.
22. 133.817 m.
23. 163000 g.
24. 1154.56 Ha.
25. 64.592 Hl. = 6.4592 cu. m.
26. .755 sq. m.
27. 642600 sq. m.
28. 2.7 sq. m.
29. 21.12 sq. m.
30. .2.
31. 9000.
32. 2000000000.
33. .000000154.
34. 3409.013 m.
35. 1626.13 a.
36. 1156.01612 g.
37. 1600 m. ; 2153.43+ m.
38. 31.212 cu. m.
39. 120 l.
40. 955080 Kg.
41. 36 Kl. ; 36000 Kg.
42. $\frac{5}{6}$ m.
43. 74.88 Kl.
44. 666 Hl. = 66.6 st.
45. $2238.60.
46. 1011.84 cu. m. = 1011.84 st. = 1011840 l. ; 1011840 Kg. ; 10118.4 Hl.
47. 8 l.
48. $\frac{1}{4}$ m.
49. 56700 Kg.
50. 1436.4 st. = 1436400 l. = 1436.4 Kl.
51. 1436400 Kg.
52. 2585520 Kg.
53. $34473.60.
54. 7.736 a.
55. 11.28 m.
56. $1\frac{1}{4}$ %.
57. 85.1 % ; 42.55 Kg. ; 1050 cu. cm.
58. .00025 cu. m.
59. 1.346 m.

Exercise 170.

[These reductions can be only approximate.]

1. 8.045 Km.
2. 4.58 Hl.
3. 22.66 Ha.
4. 170.33 l.
5. 72.48 st.
6. 9.938 cu. m.
7. 27.22 Kg.
8. 141.88 bu.
9. 497.12 rd.
10. 98.425 ft.
11. 8.83 cd.
12. 10.5 A.
13. 33.296 gal.
14. 36.89 pk.
15. 2.205 cwt.
16. 1135.04 bu.
17. 255.38 bu.
18. 3.621 Hm.
19. 11340 g.
20. 176.58 cu. ft.
21. 375.59 sq. rd.
22. 7257.6 Kg.
23. .7946 pk.
24. 6.653 cu. in.
25. 430.56 sq. ft.
26. 1135.56 dl.
27. 478.4 sq. yd.
28. .63 in.
29. .1544 pt.
30. 2034.54 cl.
31. 3.44 Km.
32. 1.707 Hl.
33. 13.72 l.
34. 16.44 Ha.
35. 3420.14 Kg.
36. 18 A. 97 sq. rd. 1.8 sq. yd.
37. 51 bu. 3 pk. 7.5 qt.
38. 8 T. 5 cwt. 96 lb. 3.7 oz.
39. 23 gal. 1 pt. .2 gi.
40. 15 mi. 22 rd. 1.06 ft.
41. 37.48 lb.
42. 30.864 T.
43. 6.614 T.
44. 3785.2 Kg.

45. 191125 Hg.
46. 16.4 g.
47. 11772 cu. yd.
48. 8884.5 T.
49. 12.6 in.
50. 3.39 dm.
51. $32.25.
52. 322.7 Hl.
53. 1.186 A.
54. 2048.07 lb.
55. 62.46 lb.
56. 22.35 m.
58. 119.234 Kg.
59. 1616628 Kg.
60. 8040.2 rev.
62. 199.5 l. ; 403.9 lb.
63. 11.416 qt. (milk) ; 9.81 qt. (corn).
64. $106.84.
65. $162.52 ; 50+ %.
66. 9116.6 gal.
67. $3736.15.
68. Gained ; nearly 24%.
69. 7.457 mi.
70. 38.016 Km. ; 23.623 mi.
71. $84.56.
72. 10.43 l.
73. 4.584 Ha.
74. (a) 145800 l.
(b) 160.72 T.
(c) 38527.65 gal.
(d) 4137.22 bu.
(e) 145.8 st.
(f) 1.674 a.
(g) 12.68 rd.

Exercise 171.

1. 330 kg. = 727.52 lb.
2. .0018 cu. m. = .06357 cu. ft.
3. 298.94 T.
4. 1.38 cu. ft.
5. 1190.4 m. tons.
6. 15901.86 Kg.
7. 47 cu. dm. = 2868.034 cu. in.
8. 42.86 cu. dm.
9. 15648.25 lb.
10. 6.7 (nearly).
11. 8.88.
12. .3 (nearly).
13. .75.
14. 1.3.
15. 3.78.
16. .98.
17. 10.48.
18. .0013.
19. (a) W't = sp. gr. × w't same volume water.
(b) Sp. gr. = w't ÷ w't same volume water.
(c) W't same volume water = w't ÷ sp. gr.
Volume water can be found if its weight is known.
20. 9 cu. m.
21. 8.283 l.
22. .758.
23. 851.7 g.
24. 25.632 g.
25. 2886+ coins.
26. 7.6+ l.
27. 8680.6 lb.
28. 1.236 Kg.
29. 4588.5 Kg.
30. .0019; 1.9 Kg.
31. .0516 a.
32. 28700 Kg.
33. $\frac{1}{160}$ %.
34. 9.756 dm.
35. 3.5 kg. ; $42\frac{6}{7}$ %.
36. 316.128 cu. in.
37. 235.35 Kg.
38. 4.7 m.
39. 66.679 Kg.; 7200 Kg.
40. 2.5 dm.

Exercise 172.

1. 9.309 gal.
2. 11.5.
3. 2 yr. 169 da. 6 hr. $43\frac{1}{3}$ sec.
4. 57.
5. 88 sq. rd. 26 sq. yd. 8 sq. ft.
6. 3.6.
7. $2\frac{42}{125}$.
8. 14975.
9. $$7.86\frac{2}{3}$.
10. $32.85.
11. $$273\frac{1}{4}$.
12. $30\frac{12}{13}$ %.
13. 43 ; 78788.

14. 20 books.
15. $3\frac{3}{5}$ da.
16. $\frac{66}{72}$; $\frac{21}{72}$; $\frac{64}{72}$; $\frac{56}{72}$; $\frac{225}{72}$; $\frac{27}{72}$; $\frac{32}{72}$; $\frac{64}{72}$.
17. Increased, $\frac{27}{400}$.
18. $\frac{19}{30}$.
19. $7\frac{3}{5}$ bu.
20. $4.
21. $17\frac{1}{2}$ da.
22. 55396 min.
23. $120.
24. $64\frac{11}{25}$.
25. $1\frac{19}{31}$.
26. 42.
27. 6 T. 7.5 cwt.
28. $286.
29. 1.008.
30. 5 mi. 55 rd. 2 yd. 2 ft. 6.96 in.
31. The former.
32. 1984 bu.
33. $5\frac{1}{7}$ da.
34. $532\frac{52}{119}$ times.
35. $18492.
36. $17.10\frac{1}{2}$%.
37. $114\frac{3}{7}$ gal.
38. 12.276 bu.
39. 507064 in.
40. $167.41.
42. 5%.
43. 8 hr. 2 min. 22 sec.
44. $13\frac{1}{3}$%.
45. $5280.
46. $130.87.
47. 31116 rails.
48. $936.
49. Most, Saturday; least, Thursday.
50. A, $\frac{5}{21}$; B, $\frac{4}{21}$; C, $\frac{10}{63}$; D, $\frac{8}{63}$.
51. $\$69\frac{1}{3}$.
52. $1\frac{5}{8}$.
53. 17 mi. 616 ft.
54. $39\frac{11}{32}$ acres; $\$4871\frac{19}{30}$.
55. $\$1185\frac{5}{37}$.
56. $\frac{115}{616}$.
57. 13 feet.
58. $196\frac{1}{3}$.
59. $\frac{5}{16}$.
60. $40800; $57120; $45696.
61. 21582 gr.
62. The former.
63. 6 mi. 123 rd. 2 yd. 2 ft. 4 in.
64. $\$43\frac{767}{1260}$.
65. 62 cents (nearly).
66. 47.
67. $19\frac{1}{2}$ yd.
68. 3 mi. 90 rd. 1 yd. 1 ft. 10 in.
69. $5\frac{7}{12}$.
70. $1\frac{1}{5}$ T.; 9 T.
71. $274.68.
72. 67.
73. $12715.50.
74. 10143620 sec.
75. 21.909 rd.
76. 12.9+ in.; 6.14 in. (nearly).
77. $666\frac{2}{3}$ lb.
78. Gained; $7.50; $6\frac{2}{3}$%.
79. B, 30 da.; C, 10 da.; together, $7\frac{1}{2}$ da.

14. 20 books.
15. $3\frac{3}{5}$ da.
16. $\frac{66}{72}$; $\frac{71}{72}$; $\frac{64}{72}$; $\frac{56}{72}$; $\frac{225}{72}$; $\frac{27}{72}$; $\frac{33}{72}$; $\frac{94}{72}$.
17. Increased, $\frac{27}{400}$.
18. $\frac{13}{30}$.
19. $7\frac{3}{5}$ bu.
20. $4.
21. $17\frac{1}{2}$ da.
22. 55396 min.
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57. 13 feet.
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66. 47.
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78. Gained; $7.50; $6\frac{2}{3}$%.
79. B, 30 da.; C, 10 da.; together, $7\frac{1}{2}$ da.

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THE EDUCATIONAL FORUM, *The Auditorium, Chicago, Illinois.*—The subject of algebra has in this book (Durell and Robbins Grammar School Algebra) been simplified, and the practical reason for each step is given in such a plain common-sense way that algebra is made far more attractive than by any previous text-book....This method is extremely practical, and adds materially to the interest of the pupil.

WILLIAM J. BOONE, *President of the College of Idaho, Caldwell, Idaho.*—The Durell and Robbins School Algebra presents the subject in the liveliest, clearest, and most forceful manner. I am acquainted with about two dozen texts on elementary algebra; but I consider Durell and Robbins the best.

PROF. G. H. DOSCH, *Department of Mathematics, Central Pennsylvania College.*—We are using the Durell and Robbins School Algebra Complete. We are delighted with the results. It is an excellent work, clear, thorough, and up to date.

square

13 = 169
14 = 196
15 = 225
16 = 256
17 = 289
18 = 324
19 = 361
20 = 400
21 = 441
22 = 484
23 = 529
24 = 576
25 = 625

memorise

2 = 8
3 = 27
4 = 64
5 = 125
6 = 216
7 = 343
8 = 512
9 - 729
10 - 1000
11 = 1331
12 = 1728