CHAPTER V.

DIVISION.

59. Illustration.—A man having $18 is staying at a hotel at a cost of $3 a day. How many days can he stay?

Since he spends $3 a day, the number of days he can stay can be determined by subtracting $3 from $18, then $3 from the remainder, and so on in succession till the $18 are exhausted, and then counting the number of times $3 has been subtracted. The number of subtractions will be the number of days required.

If, however, the student is familiar with the multiplication table, it is much easier to recall that the number which multiplies 3 and makes 18 is 6. We thus substitute the less labor of memory for the greater labor of repeated subtraction and the still greater labor of counting off dollars in groups of 3 each from 18, and counting the number of groups counted off.

60. Definitions.—The process of determining how many times one number may be subtracted from another—that is, is contained in another—by a brief method (as by the aid of the multiplication table above), is termed *division.*

The **dividend** is the number from which the successive subtractions are made.

The **divisor** is the number successively subtracted.

The **quotient** is the number of times the divisor is subtracted. Thus, in the example in Art. 59, $18 is the dividend, $3 is the divisor, and 6 is the quotient.

Since the short way of dividing is to recall that multiplier which, applied to a given number, will produce another given number, we may say that—

Division is the process by which, one factor and the product being given, the other factor is determined. The given factor is the *divisor*, the given product is the *dividend*, and the required factor is the *quotient.*

61. Exact and Inexact Division.—If we divide $18 by 3, we find that, after subtracting six times, no dollars remain. When there are no units left over from a division, the division is said to be **exact.**

If we attempt to divide $18 by $5, we find that, after subtracting $5 three times, $3 are left.

When units are left over from a division, the division is said to be **inexact.** The units left over are called the **remainder.**

62. Relation of Quantities in Division.—Since the two factors of a number multiplied together equal the number, it follows that—

1. In exact division, *dividend* = *divisor* × *quotient.*

2. In inexact division, *dividend* = *divisor* × *quotient* + *remainder.* Or, using symbols, and denoting the dividend by D, divisor by d, quotient by Q, remainder by R,

$$D = d \times Q + R.$$

Thus, $\$18 = \$5 \times 3 + \$3.$

63. Symbols for Division.—The ordinary sign of division is $\div$, which reads "divided by." Placed between two numbers, it means that the first number is to be divided by the second. Thus, $36 \div 9$ means that 36 is to be divided by 9. Division may also be indicated by a horizontal line, with the dividend above and divisor below, as $\frac{36}{9}$; or by a curved line, with the divisor on the left, and the dividend on the right, as 9)36.

64. Table for Division.—Just as the multiplication of all large numbers is performed by means of a few simple primary products, which are learned once for all at the outset, so the division of all large numbers may be resolved into a few simple primary divisions, to be learned once for all, and used repeatedly afterward. These simple primary divisions

are formed by taking each product in the multiplication table, and determining one of its factors when the other is given.

Thus, since $9 \times 6 = 54$, we may write $9)\overline{54}$, $6)\overline{54}$;

Or, $54 \div 9 = 6$; $54 \div 6 = 9$.

Let the pupil form a division table in this way and thoroughly master it.

EXERCISE 13.

ORAL OR DRILL EXERCISE.

1. Tell immediately the quotient in each case:

15 ÷ 3.	22 ÷ 11.	36 ÷ 6.	72 ÷ 9.
27 ÷ 9.	24 ÷ 6.	18 ÷ 6.	15 ÷ 5.
24 ÷ 4.	25 ÷ 5.	42 ÷ 7.	84 ÷ 12.
35 ÷ 5.	90 ÷ 10.	36 ÷ 4.	77 ÷ 11.
21 ÷ 7.	84 ÷ 7.	96 ÷ 8.	48 ÷ 6.
36 ÷ 12.	72 ÷ 12.	81 ÷ 9.	32 ÷ 4.
28 ÷ 4.	49 ÷ 7.	54 ÷ 6.	60 ÷ 5.
56 ÷ 7.	30 ÷ 5.	63 ÷ 7.	132 ÷ 12.
63 ÷ 9.	45 ÷ 9.	16 ÷ 8.	88 ÷ 11.
48 ÷ 12.	48 ÷ 8.	20 ÷ 5.	121 ÷ 11.
64 ÷ 8.	66 ÷ 11.	40 ÷ 8.	108 ÷ 12.
42 ÷ 6.	72 ÷ 8.	60 ÷ 10.	56 ÷ 8.
3)36	7)49	9)36	11)132
5)45	8)72	4)44	12)84
7)63	6)42	8)24	7)35
12)96	5)55	9)108	12)60
11)99	8)32	6)72	6)30

2. If a man earns $6 a day, how many days must he work to earn $36? $48? $72?

3. How many weeks in 56 days? In 49 days? In 77 days?

4. How many sponges at 8 cents apiece can be bought with 64 cents? With 72 cents? With 96 cents?

5. There are 4 quarts in a gallon; how many gallons in 20 quarts? In 40 quarts? In 48 quarts?

6. What is the remainder in each case following?

25 ÷ 4.	38 ÷ 8.	80 ÷ 9.	63 ÷ 6.
29 ÷ 9.	43 ÷ 5.	90 ÷ 11.	100 ÷ 12.
30 ÷ 7.	47 ÷ 6.	70 ÷ 8.	105 ÷ 11.
35 ÷ 4.	68 ÷ 9.	75 ÷ 7.	115 ÷ 12.

7. What is the price of each orange when 9 oranges cost 45 cents? When they cost 72 cents?

8. If there are 9 square feet in a square yard, how many square yards in 108 square feet? In 90 square feet?

9. If there are 48 roods in 12 acres, how many roods in 1 acre?

10. When 9 quarts of milk cost 63 cents, what is the price of 1 quart?

11. If I save $132 in 12 months, how much must I average each month?

12. In 8 days a digger opened 72 yards of ditch. How many yards could he open in 1 day?

13. If a wheelman rides 96 miles in 12 hours, how far does he ride in 1 hour? In 7 hours?

65. Abstract and Concrete Numbers in Division.—Abstract number is of the first importance in division, since it enables us to form a division table good for all kinds of particular concrete quantity. But, as in multiplication, so in division, certain limitations must be observed in the use of abstract and concrete numbers. Since we can divide a number of units by a number of like units or by a number of groups (but not by a number of unlike units), it follows that

1. If the dividend is an abstract number, the divisor must be an abstract number also, and hence the quotient too will be abstract.

2. If the dividend is a particular kind of concrete number and the divisor also is concrete, it must be a concrete number of the same kind with the dividend. Thus, we can divide $18 by $3, or 18 marbles by 3 marbles, but not $18 by 3 marbles.

In this case the *quotient is abstract number.*

3. If the dividend is concrete number, the divisor may be abstract number. Thus, $18 may be divided by 6; that is, divided into 6 equal parts.

In this case the *quotient is a concrete number of the same kind with the dividend.*

This kind of division is sometimes called *partition*, since it consists essentially of dividing a given number into a number of equal groups or parts.

In the preceding oral exercise, let the pupil point out which examples are cases of partition.

We now proceed to show how, by the use of the division table, any number, however large, may be divided by a smaller number.

66. I. Short Division.—When the divisor is a single digit, the process is called *short division*. The dividend may in this case be resolved into small partial dividends mentally, the divisions performed mentally, and the figures of the quotient set down at once.

Ex. 1. Divide \$9452 into 4 equal parts.

OPERATION.

4)\$9452

\$2363, *Quotient.*

EXPLANATION.

We set down the divisor at the left of the dividend, and divide it into the different orders of units of which the dividend is composed, beginning with the highest and dividing each separately. 4 is contained in 9 (thousands) 2 (thousands) times, with a remainder of 1 (thousand). Setting down the 2 beneath the thousands and combining the 1 (thousand) with 4 (hundreds), we have 14 (hundreds) as the next partial dividend. 4 is contained in 14 (hundreds) 3 (hundreds) times, with a remainder of 2 (hundreds). Setting down the 3 (in the hundreds place) and combining the 2 (hundreds) remainder with the 5 (tens), we have 25 (tens) as the next partial dividend. 4 is contained in 25 (tens) 6 (tens) times, with a remainder of 1 (ten). Setting down the 6 (tens) in the quotient, the last partial dividend is 12 (units). Into this, 4 is contained 3 times. Hence, the division is exact, and the entire quotient is \$2363.

Ex. 2. Divide \$31559 by 7.

OPERATION.

Divisor 7)\$31559, *Dividend,*

\$4508, *Quotient*, with a *Remainder* of \$3.

EXPLANATION.—Since 7 is not contained in 3 (ten thousands), we treat 3 as a remainder, and divide 7 into 31 (thousands) as the first partial dividend, obtaining the quotient 4 (thousands), with 3 (thousands) as a remainder, and proceed as in Ex. 1. Similarly when 7 is not contained in 5 tens, we set down 0 in the tens place in the quotient, and take 59 as the next partial dividend. 7 is contained in 59 (units) 8 times, with a remainder of 3 (units). Hence the division is inexact, and the quotient is \$4508, with a remainder of \$3. The remainder is sometimes written over the divisor, with a line between, and set down as a (fractional) part of the quotient. The entire quotient in this case would then be \$4508$\frac{3}{7}$.

67. Other Cases of Short Division.—The method of short division can be employed when the divisor is 11 or 12, or a larger number if the student is familiar with the table of the products of such numbers by the nine digits.

Ex. 1. Divide 42084 by 12.

OPERATION.

12)42084

3507, *Quotient.*

Again, if the divisor consist of a single digit followed by two or more zeroes, the division is best performed as a short division.

Ex. 2. Divide \$897563 by 300.

Since the divisor is an exact number of hundreds, that part of the dividend which is less than \$100 (viz., \$63) may be set aside as a part of the remainder, and the number of hundreds in the dividend divided by the number of hundreds in the divisor; hence, we have

300)\$897563

\$2991, *Quotient* + \$263, *Remainder.*

68. Verification.—To test the accuracy of the work done in division, we may multiply the divisor and quotient together, and add the remainder, if there be any, to the product. If the result equals the dividend and there have been no compensating errors in the two processes, the work is correct (see Art. 62).

Thus, to test the accuracy of the work in the last example, we have

$$\begin{array}{r} \$2991 \\ \underline{\quad\quad 300} \\ \$897300 \\ \underline{\quad\quad \$263} \\ \$897563. \end{array}$$

This result equals the original dividend; hence, the work in the division is (probably) correct.

EXERCISE 14.

Divide rapidly and orally:

1. 60 by 3.	*11.* 57 by 3.	*21.* 240 by 4; by 8.
2. 84 by 4.	*12.* 58 by 2.	*22.* 360 by 9; by 6.
3. 96 by 3.	*13.* 64 by 4.	*23.* 630 by 7; by 10.
4. 48 by 2.	*14.* 65 by 5.	*24.* 720 by 12; by 9.
5. 69 by 3.	*15.* 68 by 4.	*25.* 560 by 8; by 4.
6. 93 by 3.	*16.* 72 by 3.	*26.* 840 by 7; by 12.
7. 86 by 2.	*17.* 75 by 5.	*27.* 960 by 8; by 6.
8. 56 by 4.	*18.* 78 by 6.	*28.* 880 by 11; by 4.
9. 42 by 3.	*19.* 81 by 3.	*29.* 420 by 6; by 3.
10. 54 by 2.	*20.* 94 by 2.	*30.* 600 by 5; by 10.

Copy and divide the following:

31. 3)426 inches.	*45.* 7)19999	*59.* 12)494820
32. 2)590 feet.	*46.* 7)456939	*60.* 12)688608
33. 3)714 days.	*47.* 8)301376	*61.* 3)1213526
34. 2)592 men.	*48.* 8)123456	*62.* 2)16181590
35. 3)477 tons.	*49.* 8)579752	*63.* 3)1524291
36. 2)738 men.	*50.* 8)703144	*64.* 4)3231616
37. 4)344 days.	*51.* 9)555579	*65.* 4)8352396
38. 3)825 pecks.	*52.* 9)745083	*66.* 5)9541035
39. 4)628 yards.	*53.* 9)430056	*67.* 5)47521015
40. 5)645 feet.	*54.* 9)388971	*68.* 5)3800275
41. 4)30248	*55.* 9)8035083	*69.* 6)3046824
42. 5)16785	*56.* 9)559431	*70.* 7)5666612
43. 6)43974	*57.* 11)345631	*71.* 7)19633328
44. 7)66668	*58.* 11)499037	*72.* 8)96111264

73. Divide 7488, 56703, 1341117, and 627144 by 3.

74. Divide 5208, 617102, 9031758, and 12345678 by 6.

75. Divide 175345, 420275, 753105, and 5123045 by 5.

76. Divide 302208, 4251612, 25801278, and 10002018 by 6.

77. Divide 30028, 1756320, 75690376, and 5004108 by 4.

78. Divide 66965, 3201305, 3710420, and 9876540 by 5.

79. Divide 33334, 7080906, 1230463, and 20230721 by 7.

80. Divide 520352, 2705608, 3391576, and 70001208 by 8.

81. Divide 23742, 1012347, 50123458, and 70123456 by 9.

82. 567886 by 11.	*84.* 76488 by 12.	*86.* 930156 by 12.
83. 587180 by 11.	*85.* 65472 by 12.	*87.* 5750412 by 12.

88. Divide 35816, 44781, 4075973, and 10170688 by 11.

89. Divide 28284, 609756, 888384, and 46818072 by 12.

90. A man divided $7434 equally among 6 children. How many dollars did each receive?

91. There are 7 days in a week. How many weeks in 2002 days?

92. At $5 a barrel, how many barrels of flour can be bought for $3940?

93. A boy receives $9 a week. How many weeks must he work in order that he may receive $7020?

94. There are 12 inches in a foot. How many feet in 63360 inches?

95. If there are 4 pecks in a bushel, how many bushels in 16300 pecks?

96. Change 35091 square feet to square yards, if there are 9 square feet in one square yard.

97. Multiply 504 by 231, and divide the product by 9.

98. Divide 75320 by 8, and multiply the quotient by 76.

99. Multiply 204 by 917, and divide the product by 12. Then divide this quotient by 7.

100. Find the product of $225 \times 716 \times 135$, and divide it by 5. Divide the quotient by 12. Divide this quotient by 9.

101. If the product is 43744, and the multiplier is 8, find the multiplicand.

Find the divisor if the:

102. Dividend is 3078 and the quotient is 9.
103. Dividend is 176072 and the quotient is 8.
104. Dividend is 463815 and the quotient is 11.

Find the quotient and remainder in each case:

105. 8)31415	*112.* 20)98750	*119.* 300)976580
106. 9)21763	*113.* 30)67670	*120.* 400)95670
107. 7)1416838	*114.* 40)841370	*121.* 500)1796300
108. 6)360517	*115.* 80)35670	*122.* 600)92370
109. 5)312059	*116.* 90)123456	*123.* 700)516784
110. 10)16758	*117.* 100)26750	*124.* 800)56793
111. 11)176531	*118.* 200)976350	*125.* 900)387650

Find the dividend if the:

126. Divisor is 7, quotient is 43, and remainder is 5.
127. Divisor is 12, quotient is 327, and remainder 10.
128. Divisor is 23, quotient is 76, and remainder 18.
129. Quotient is 416, divisor 207, and remainder 194.
130. Quotient is 356, remainder 401, and divisor 510.

69. II. Long Division.—When the divisor is so large (larger than 12) that the products of it by the nine digits cannot be retained mentally and subtracted, it is necessary in the division to set down in succession the partial dividends and the successive subtrahends and remainders. When division is performed in this way, it is called *long division*.

Ex. Divide 8746 by 37.

Operation.	Verification.
Dividend.	
Divisor 37)8746(236, *Quotient.*	236
74	37
134	1652
111	708
236	8732
222	14
14, *Remainder.*	8746, (= *Dividend.*)

Explanation.—We determine the first partial dividend by beginning at the left and taking the smallest number of digits that will contain the divisor; 37 is not contained in 8, but is contained in 87 (hundreds), 2 (hundreds) times, with a remainder of 13 (hundreds). Setting down the 2 (hundreds) to the right as part of the quotient, and combining the 13 (hundreds) remainder with the 4 (tens), we have 134 (tens) as the next partial dividend. 37 is contained in 134 (tens), 3 (tens) times, with a remainder of 23 (tens). Setting down the 3 (tens) as part of the quotient, and combining the 23 (tens) remainder with the 6 (units), we have 236 (units) as the last partial dividend. 37 is contained in 236 (units), 6 (units) times, with a remainder of 14 (units). Hence, the entire quotient is 236, with a remainder of 14.

It is now left as an exercise for the pupil to write out a *general rule for long division.*

In regard to long division, it is to be remarked that

1. If the work is properly done, the remainder is in all cases less than the divisor. If in the course of the work a remainder is obtained equal to or greater than the divisor, it shows that the last figure of the quotient is too small.

2. If at any time the divisor multiplied by the last figure gives a product greater than the partial dividend under which it is placed, it shows that too large a number has been taken for the last figure of the quotient.

3. If at any time the partial dividend is less than the divisor, and hence will not contain it, a zero is to be set down in the quotient, another figure of the dividend brought down, and the work continued as before.

Ex. Divide 216912714 by 71873.

71873)216912714(3018, *Quotient.*
215619
129371
71873
574984
574984

EXERCISE 15.

Divide

1. 322 inches by 14.	*4.* 728 days by 26.
2. 540 feet by 15.	*5.* 1457 years by 31.
3. 391 men by 23.	*6.* 1204 feet by 43.

7. 2530 tons by 55.
8. 2989 min. by 61.
9. 6300 hours by 75.
10. 5312 pages by 83.
11. 7990 by 94.
12. 2668 by 23.
13. 3166 by 42.
14. 9968 by 56.
15. 17152 by 67.
16. 24198 by 74.
17. 29602 by 82.
18. 36210 by 85.
19. 51428 by 92.
20. 71808 by 96.

21. 7380 by 36.
22. 13815 by 45.
23. 21268 by 52.
24. 32448 by 64.
25. 47479 by 79.
26. 25800 by 25.
27. 77666 by 38.
28. 185288 by 46.
29. 302328 by 57.
30. 595765 by 85.
31. 1968 by 123.
32. 2430 by 135.
33. 6048 by 108.
34. 8695 by 235.
35. 9222 by 318.

36. 31832 by 184.
37. 400189 by 187.
38. 70305 by 215.
39. 106953 by 231.
40. 64250 by 125.
41. 120239 by 193.
42. 182688 by 346.
43. 113057 by 207.
44. 442458 by 523.
45. 164984 by 328.
46. 434995 by 719.
47. 719037 by 891.
48. 444024 by 881.
49. 377289 by 927.
50. 306327 by 503.
51. 802392 by 998.
52. 155484 by 126.
53. 721644 by 231.
54. 1722855 by 331.
55. 2135328 by 354.
56. 2752461 by 423.
57. 3019272 by 429.
58. 4125954 by 509.
59. 5488416 by 608.

60. 5582225 and 7430600 by 1325.
61. 8010555 and 17266376 by 2461.
62. 13841372 and 29745036 by 3812.
63. 21801270 and 25961535 by 4115.
64. 43673280 and 54034432 by 6208.

65. 27300 by 350.
66. 70680 by 760.
67. 37800 by 1400.
68. 170100 by 2700.
69. 1663200 by 5400.
70. 9010000 by 17000.

Find the quotient and remainder in each case:

71. $96731 \div 309$.
72. $59178 \div 421$.
73. $96733 \div 1209$.
74. $188576 \div 2761$.
75. $9980736 \div 2047$.
76. $13896789 \div 8308$.

77. If there are 24 hours in 1 day, how many days are there in 2136 hours? In 3168 hours?

78. How many beeves at $35 each can a dealer buy with $2380? With $6650?

79. There are 144 square inches in 1 square foot. How many square feet are there in 11952 square inches?

80. How many barrels of flour in a load containing 40180 pounds, a barrel of flour weighing 196 pounds?

81. The distance from the earth to the sun is 93000000 miles. How many days would it take a cyclist to ride that distance, traveling 124 miles a day? How many years, if there are 365 days in a year?

82. How many days would it take a locomotive to run that far, at the rate of 744 miles a day?

83. How long would light require, to come from the sun to earth, at the rate of 186000 miles a second? What is the name of your quotient?

84. At $117 a share, how many shares can be purchased with $843336?

85. What is the price of 1 acre of land, when 371 acres cost $87185?

86. There are 792 inches in a chain, and 63360 inches in a mile. How many chains in a mile?

87. The product is 350102, and the multiplier is 386. Find the multiplicand.

88. The quotient is 567, and the dividend is 2442069. Find the divisor.

89. How many yards of cloth at $3 a yard must be returned for 7 barrels of flour at $6 a barrel?

90. How many tons of hay worth $18 a ton must be given in exchange for 24 loads of coal at $9 a load?

91. At $36 an ox, how many oxen will be required, to pay for a farm of 104 acres, worth $81 an acre?

92. The 425 families of a village agree to bear equally the expense of paving 612 rods of street, costing $25 a rod. How many dollars will each family contribute?

Divide:

93. 5168254 by 1898.
94. 26638950 by 4314.
95. 69908524 by 7543.
96. 9111878 by 3954.
97. 32393290 by 4978.
98. 68115312 by 7165.
99. 37318800 by 3405.
100. 17829888 by 3072.
101. 83555703 by 8701.
102. 25360303 by 6329.
103. 32658915 by 5435.
104. 73728456 by 9208.
105. 280967545 by 6435.
106. 422582680 by 5785.
107. 60088326832 by 76048.
108. 3380002995264 by 67008.
109. 803000835205 by 200705.

Find the value of each of the following:

110. $7 \times 6 + 63 \div 9$.
111. $9 + 40 \times 2 - 72 \div 3$.
112. $80 + 121 \div 11 - 90$.
113. $196 \div 14 + 324 \div 18$.
114. $8 \times 7 - 18 \div 3 + 45 \div 15 - 105 \div 5$.
115. $256 \div 16 - 7 \times 2 + 11 \times 14 - 90 \div 9$.
116. $1728 \div 12 - 169 \div 13 - 289 \div 17 - 361 \div 19$.

70. Factorial Division.—If a divisor can be separated into two or more small factors, the work of division can often be diminished by dividing the dividend by the factors of the divisor in succession, instead of dividing by the entire divisor at once.

Ex. 1. Divide 11060 by 35.

The factors of 35 are 7 and 5.

Hence, to divide 11060 by 35, we divide first by 7, and then divide the quotient obtained by 5, thus:

7)11060
5)1580
316, *Quotient.*

By dividing by 7 we separate 11060 into 1580 groups of 7 each, and by dividing by 5 we separate the 1580 groups of 7 each into 316 groups of 5×7, or 35 each.

The method of determining the remainder in factorial division is best shown by an example.

Ex. 2. Divide 6083 by 84 by the factorial method.

OPERATION.

3)6083
4)2027 groups of 3 units, with a remainder of 2 units.
7)506 groups of 3×4 units, with a remainder of 3 groups of 3 units.
72 groups of $3 \times 4 \times 7$ units, with a remainder of 2 groups of 3×4 units.

Hence, the quotient is 72, and the entire remainder consists of three parts, viz., 2 units, and 3×3 or 9 units, and $2 \times 3 \times 4$ or 24 units; or $2 + 9 + 24$, 35.

Hence, to find the entire remainder in factorial division,

Multiply each remainder by the divisor which produced the given remainder and by all preceding divisors, and add the results.

EXERCISE 16.

By the factorial method, divide:

1. 888 by 24.
2. 1736 by 28.
3. 3570 by 42.
4. 4230 by 45.
5. 7074 by 54.
6. 15744 by 64.
7. 13794 by 66.
8. 25632 by 72.
9. 31350 by 75.
10. 41426 by 77.
11. 50787 by 81.
12. 68712 by 84.
13. 38304 by 96.
14. 41846 by 98.
15. 49665 by 105.
16. 79596 by 108.
17. 91056 by 112.
18. 124272 by 144.

Find the quotient and the remainder in each, by the factorial method:

19. 355 by 36.
20. 927 by 48.
21. 791 by 64.
22. 999 by 56.
23. 975 by 121.
24. 3446 by 135.
25. 4289 by 144.
26. 4873 by 150.
27. 5327 by 216.
28. 582 by 243.
29. 6899 by 324.
30. 7163 by 385.
31. 8563 by 420.
32. 8873 by 540.
33. 9997 by 756.

71. The Value of Division, even in a single problem, is realized on comparing the labor employed in division with what the labor would be if the same problem were solved by repeated subtraction. Take, for example, the following problem:

A railroad contractor has $400000, and is paying out $3276 a day; he desires to know how many days his money will last. To determine the number of days by successive subtractions of $3276 would require the setting down and subtracting of this number more than 120 times. To obtain the required number of days in this way would require at least 50 times as much labor as by the method of long division.

The value of division is still further realized when some adequate notion is formed of the extent of its possible application. Division, like multiplication, has this wide application owing to the extended use of groups uniform in the number of units which they contain. Thus it is an advantage to have the barrels employed for a given purpose, as to measure oil, contain the same number of gallons; to sell all the yards in a given piece of cloth at the same price per yard, etc.

Where uniform groups of units are not explicitly given, they may often be formed by the proper analysis, and the solution of the given problem effected by division.

Ex. 1. In a given election where 10896 votes were cast, the successful candidate had a majority of 324. How many votes did each candidate receive?

We may regard the number of votes received by the defeated candidate as the primary or unit group of the problem. The successful candidate received this unit group and 324 votes besides. Hence, the total number of votes cast, 10896, equals twice the unit group and 324 votes besides. Hence, subtracting 324 votes from 10896 votes, we have 10572 votes as equal to twice the unit group, and the unit group may be obtained by dividing 10572 by 2. Hence, 10572 ÷ 2, or 5286, is the number of votes received by defeated candidate; and 5286 + 324, or 5610, is the number of votes received by successful candidate.

Ex. 2. A man dying left an estate of $84000, of which his wife was to receive a certain part, his daughter half as much as his wife, and his niece half as much as his daughter. How many dollars did each inherit?

If we regard the number of dollars received by his niece as the unit, then his daughter received twice this unit, and his wife four times it. Hence, the number of dollars bequeathed must equal 7 times the unit taken. If 7 times the unit equals $84000, the unit group itself may be obtained by dividing $84000 by 7, giving $12000 as a quotient. From this the value of each share is readily determined.

EXERCISE 17.

1. In an election in which 22795 votes were cast for 2 candidates, the successful candidate received a majority of 461 votes. How many votes were cast for each?

2. John has twice as many marbles as William, and William has three times as many as George; all have 130 marbles. How many has each?

3. A boy has four times as many examples as his sister, and both together have 100. How many has each?

4. Divide 90 into two parts so that one is twice as large as the other. One is 5 times as large as the other.

5. How could the examples of Exercise 15 have been done other than by division? How else could you do Example 21 there?

6. How could you do Example 60 but by division? Which is more simple? Could you do Example 81 the other way?

7. Without multiplication or division, how could the 92d example be done?

8. There are 10 times as many boys in a certain school as there are teachers, and 3 times as many girls as boys; altogether there are 492. How many teachers, boys, and girls separately?

9. Separate the number 132 into two parts such that one is 11 times the other. Into two other parts such that one is 21 times the other.

10. A has twice as many acres as B, B has twice as many as C, C has three times as many as D, and all have 1100 acres. How many has each?

11. In a certain election 56329 votes were cast for 2 candidates, the successful candidate receiving a majority of 1071 votes. How many did each receive?

CHAPTER VI.*

ABBREVIATED PROCESSES.

72. Fundamental Arithmetical Operations and their Abbreviations.—The four operations, addition, subtraction, multiplication, division, are called the fundamental operations of arithmetic, since all subsequent arithmetical work consists of these in various combinations. Hence, it is important that every possible method of abbreviating these fundamental processes be investigated and mastered. They are themselves, it is to be remembered, abbreviations of more tedious work, as of the counting and grouping of units; yet, in many cases, the four operations can be still further abbreviated. The abbreviations used diminish not only the amount of work, but also the likelihood of error.

ABBREVIATED MULTIPLICATION.

73. Multiplication by 25.—Since $25 = 100 \div 4$, to multiply a number by 25,

Annex two zeroes to the multiplicand (i. e., multiply by 100) and divide by 4.

Ex. Multiply \$8769 by 25.

4)\$876900

\$219225, *Product.*

The multiplication, when performed in this way, calls for less mental effort and the making of fewer figures. It can be readily mastered and made a purely mental process.

74. Multiplication by an Aliquot Part of 100.—Of like nature is the multiplication by any other aliquot or exact

*In teaching young or backward pupils, the teacher should omit all of Chapter VI, except Arts. 73, 74, 79, 80, 84.

part of 100. This calls for a slight knowledge of fractions, but should be mentioned in this connection.

Thus,	$12\frac{1}{2} = \frac{1}{8}$ of 100.	$50 = \frac{1}{2}$ of 100.
	$16\frac{2}{3} = \frac{1}{6}$ of 100.	$62\frac{1}{2} = \frac{5}{8}$ of 100.
	$25 = \frac{1}{4}$ of 100.	$66\frac{2}{3} = \frac{2}{3}$ of 100.
	$33\frac{1}{3} = \frac{1}{3}$ of 100.	$75 = \frac{3}{4}$ of 100.
	$37\frac{1}{2} = \frac{3}{8}$ of 100.	$87\frac{1}{2} = \frac{7}{8}$ of 100.

Hence, for example, to multiply a number by $33\frac{1}{3}$, *annex two zeroes to the multiplicand and divide by* 3. Let the student state and illustrate a rule for multiplying by each of the other aliquot parts of 100. Examples illustrating this kind of multiplication will be found in Exercise 18.

75. When the Multiplier is a Series of 9's except the last Digit.—In this case the multiplication may be abbreviated into the process of *first multiplying by* 1 *followed by as many zeroes as there are figures in the multiplier (i. e., annexing this many zeroes to the multiplicand), and then deducting the product of the multiplicand by the excess of the multiplier used over the given multiplier.*

Ex. Multiply 13721685 by 99998.

Since $99998 = 100000 - 2$, we first multiply 13721685 by 100000, and then deduct 13721685×2 from the product so obtained. Thus we have

1372168500000

27443370

1372141056630, *Product.*

If the student will now perform the multiplication directly, he can estimate the labor saved by the abbreviated process.

76. Multiplication in which the Partial Products are not set Down, but are Added Mentally.—When multiplying by a number of but two or three digits, it is often of advantage to form in immediate succession all the partial products of the same order and add them mentally, setting down only the last figure of each result in its proper place in the product, and carrying the other figure mentally.

Ex. 1. Multiply 47 by 63.

Operation.	Explanation.
47 63 2961, *Product.*	We have the following partial products: 7 (units) × 3 (units) = 21 (units). Set down 1 (unit) and carry 2 (tens); 3 (units) × 4 (tens) + 6 (tens) × 7 (units) + 2 (tens) carried = 56 (tens). Set down 6 (tens) and carry 5 (hundreds); 6 (tens) × 4 (tens) + 5 (hundreds) carried = 29 (hundreds), which we set down.

Ex. 2. Multiply 587 by 346.

Operation.

587
346
203102, *Product.*

The partial products, with figures carried, may be grouped as follows:
(Units) $7 \times 6 = 42$ units.
(Tens) $4 + 8 \times 6 + 4 \times 7 = 4 + 48 + 28 = 80$ tens.
(Hundreds) $8 + 6 \times 5 + 4 \times 8 + 3 \times 7 = 91$ hundreds.
(Thousands) $9 + 4 \times 5 + 3 \times 8 = 53$ thousands.
(Tens of thousands) $5 + 3 \times 5 = 20$ tens of thousands.

77. Abbreviations due to Limitations of Accuracy in Measuring Quantities to be Multiplied.—As has been explained in Art. 19, each digit of a number has a value depending on the place which it occupies in the number (as well as the absolute value of the digit). Thus the figure 6, when in the thousands place, has a hundred times the value it has when in the tens place. It is important that the student use every means to keep in mind this difference. For convenience in printing and writing, the digits of a number in the units, tens, etc., places are made of uniform size; but it is useful, in thinking of them, to regard them as of different sizes proportional to the positional values. Thus, while looking at the number 586, let the pupil think of the 5 as a digit ten times as large as the 8, and the 6 as a digit only one-tenth as large as the 8, thus ☞ 5_{8_6}.

Similarly in the number 7652138, if 7 be given a size in proportion to its positional value as compared with the number, it would be a figure 8 feet long, while the 8 in comparison with the 2 would vanish into invisibility.

If numbers be thought of in this way, we easily realize that the digits composing a number are of less and less importance as we go to the right, and in certain cases their value vanishes into insignificance, and computations with them may be limited accordingly.

78. Computations Based on Measurements.—No measurement is accurate beyond the sixth or seventh figure; this is owing to the limitations of our eyesight and sense of touch-perception, and to the ultimate imperfections in all our instruments of measurement.

Thus, a mile (63360 inches) can be measured only to within $\frac{1}{10}$ inch of its true length; an inch can be measured only to within a millionth part of itself, etc. So great a degree of accuracy, however, can be obtained only by applying every possible refinement of accuracy. Ordinary measuring, such, for instance, as that done by a carpenter, is accurate only to the second or third figure, that is, to within $\frac{1}{100}$ or $\frac{1}{1000}$ part. Hence,

Computations based on measurements cannot be accurate beyond the fifth or sixth place of figures, and in ordinary work not beyond the third figure.

All numerical work, therefore, which does not affect the accuracy of the result within the required limits, may be omitted.

Ex. Multiply 3274 by 4125 so that the result shall be accurate to three places.

We carry out the work to five places, so that on adding the partial products the figure in the third place may be accurate, thus,

Operation.	Explanation.
3274 4125 13096 327 64 15 13502, thousands.	We first multiply by 4, the digit of highest denomination in the multiplier. Since this gives a partial product containing five figures, as required, before multiplying by 1, the next digit in the multiplier, we strike out the units figure 4 in the multiplicand and then multiply. Similarly we strike out one figure in the multiplicand before multiplying by each successive figure in the multiplier. The number of figures struck out fixes the denomination of the product.

13500000 is the product correct to the third place.

EXERCISE 18.

Perform the following multiplications by the shortest method:

1. 356×25.	*10.* 584×125.	*19.* 77×37.
2. $477 \times 33\frac{1}{3}$.	*11.* 648×375.	*20.* 296×98.
3. $968 \times 12\frac{1}{2}$.	*12.* 743×750.	*21.* 447×96.
4. 777×25.	*13.* 26×29.	*22.* 738×98.
5. $324 \times 37\frac{1}{2}$.	*14.* 32×36.	*23.* 315×998.
6. $586 \times 62\frac{1}{2}$.	*15.* 42×37.	*24.* 716×997.
7. $420 \times 66\frac{2}{3}$.	*16.* 53×61.	*25.* 1763×999.
8. $676 \times 87\frac{1}{2}$.	*17.* 78×43.	*26.* 2385×996.
9. 736×75.	*18.* 85×76.	*27.* 3485×997.

Find approximately the following products:

28. 231×463.	*32.* 4671×3085.
29. 378×512.	*33.* 7681×4321.
30. 918×761.	*34.* 8506×7533.
31. 4231×3248.	*35.* 4761×3289.

ABBREVIATED DIVISION.

79. Division by 25.—The result obtained by dividing a group of units by 25 is the same as that obtained by dividing four times as large a group by 100. The latter process usually involves less labor. Hence, to divide a number by 25,

Multiply the given number by 4 and divide by 100.

Ex. Divide 6237 by 25.

OPERATION.

$$\begin{array}{r} 6237 \\ \hline 4 \\ \hline 100)\overline{24948} \\ \hline 249\frac{48}{100}, \text{ or } 249\frac{12}{25}. \end{array}$$

Hence, the quotient is 249, with a remainder 12.

80. Division by an Aliquot Part of 100 may be abbreviated in a manner similar to that used in abbreviating the multiplication by such numbers.

Ex. Divide 89676 by $33\frac{1}{3}$.

$$\begin{array}{r} 89676 \\ \hline 3 \\ \hline 100)\overline{269028} \\ \hline 2690\frac{28}{100} \text{ or } 2690\frac{9\frac{1}{3}}{33\frac{1}{3}}; \end{array}$$

that is, the quotient is 2690, while the remainder is $9\frac{1}{3}$. Let the student formulate rules for dividing a number by aliquot parts of 100.

For examples, see Exercise 19.

81. When the divisor is a series of 9's except the last digit.

Ex. Divide 7865923 by 98.

The abbreviated process consists essentially in dividing by 100, and then forming remainders which are divided in succession.

OPERATION.

Quotients.	*Remainders.*
78659	23
1573	18
31	46
	62
1	49
	2
80264	51

EXPLANATION.

Dividing 7865923 by 100, we obtain 78659 as a quotient and 23 as a remainder. 98 is contained the same number of times as 100, with a further remainder of 2 for every time 100 is contained; that is, a further remainder of 157318. Dividing 157318 by 100, and continuing the process as above, we obtain a series of quotients, 78659, 1573, 31 (which are to be added together), and a series of remainders, 23, 18, 46, 62. Adding the remainders, we obtain 149; dividing this by 100, and then by 98, we obtain a further quotient, 1, and a final remainder, 51. Adding the quotients, the final quotient for the divisor 98 is 80264, with remainder, 51.

82. Abbreviation by Setting Down only the Successive Partial Dividends.—Students with especial aptitude for numerical work may learn to abbreviate long division by performing mentally the subtractions required, and setting down only the partial dividends.

Ex. Divide 460686 by 71.

OPERATION.

71)460686(6488, *Quotient.*
346
628
606
38, *Remainder.*

EXPLANATION.

In ordinary long division we would multiply 71 by 6 (the first figure of the quotient), and set down the product, 426, under 460, and subtract. In the abbreviated process we multiply each figure of 71 by 6, and subtract figure by figure, setting down only the remainder. Thus, $6 \times 1 = 6$, and 6 from 10 leaves 4; set down 4. 6×7 is 42, and 42 from 45 leaves 3, which we set down. Annexing 6, we have 346 as the next partial dividend. We proceed in like manner till the division is completed.

83. Division of Numbers Determined by Measurements.—If one or both of the numbers employed in a division be determined by measurement, the result of the division cannot be accurate beyond the fifth or sixth figure. Hence, all numerical work which does not affect the accuracy of the result beyond the fifth or sixth place may be omitted. This is done by beginning at a certain stage of the work to strike off the final digit of the divisor, instead of bringing down a new figure from the dividend and annexing it to the partial dividend.

Ex. Divide 872372500 by 15273, so that the quotient shall be accurate to 4 figures.

To be accurate to 4 places, we must also determine the fifth figure. If we obtain two figures of the quotient in the ordinary way, and then begin to strike off the final digits of the divisor, it will insure that two digits of the divisor will remain at the close of the process.

OPERATION.

15273)872372500(57118
76365
108722
106911
1811
1527
284
153
131
122

Hence, the required quotient is 57120.

It is to be observed that striking out the last figure on the right of the divisor, the last figure struck out is in each case multiplied and the product carried, if it is more than 5; thus, in forming the last subtrahend, $8 \times 15 = 120$, but since 2 has just been struck out, and $8 \times 2 = 16$, the last subtrahend is $120 + 2$, or 122.

EXERCISE 19.

Divide most briefly:

1. 76125 by 25.	*14.* 57650 by 875.	*27.* 76327 by 92.
2. 89375 by 25.	*15.* 48765 by 125.	*28.* 56345 by 95.
3. 88900 by $33\frac{1}{3}$.	*16.* 94670 by 875.	*29.* 87632 by 132.
4. 63850 by $33\frac{1}{3}$.	*17.* 1598 by 47.	*30.* 179630 by 241.
5. 64350 by 75.	*18.* 3216 by 67.	*31.* 31086 by 99.
6. 47250 by $37\frac{1}{2}$.	*19.* 4088 by 73.	*32.* 42751 by 98.
7. 679500 by $62\frac{1}{2}$.	*20.* 5747 by 79.	*33.* 87630 by 98.
8. 460600 by $87\frac{1}{2}$.	*21.* 3888 by 43.	*34.* 88561 by 97.
9. 385500 by $37\frac{1}{2}$.	*22.* 7163 by 59.	*35.* 92176 by 97.
10. 832300 by $87\frac{1}{2}$.	*23.* 58951 by 61.	*36.* 98753 by 98.
11. 56780 by 125.	*24.* 38765 by 83.	*37.* 147632 by 998.
12. 35716 by 375.	*25.* 17973 by 85.	*38.* 187632 by 999.
13. 87632 by 625.	*26.* 47381 by 87.	*39.* 256319 by 998.

Perform the following divisions accurately to 4 figures:

40. 1112223334 by 3567.	*42.* 5634217689 by 16328.
41. 3216789567 by 7238.	*43.* 87632765176 by 29045.

COMBINATIONS OF OPERATIONS.

84. Order of Operations.—When several operations are combined in a single process, it is often possible to perform these operations in different orders; by performing them in one order rather than another, much labor is often saved.

Ex. 1. Find the difference between the products

$$3217 \times 85 \text{ and } 3217 \times 83.$$

FIRST COMPUTATION.

3217 3217
85 83
16085 9651
25736 25736
273445 267011
267011
6434, *Difference.*

SECOND COMPUTATION.

$85 - 83 = 2$
3217
2
6434, *Difference.*

The pupil will notice that the second computation requires less than one fourth the labor of the first computation, and that in the second computation as compared with the first the likelihood of error is diminished even more.

Ex. 2. Compute $\frac{156 \times 892}{78}$.

First Computation.	Second Computation.
156	78)156(2
892	156
312	
1404	892 × 2 = 1784, *Result.*
1248	
78)139152(1784, *Quotient.*	

In general, it is evident that if a, b, c, n be symbols denoting any numbers,

1. $a \times b \times c = a \times (b \times c) = (a \times b) \times c$.
2. $(a + b)\, c = ac + bc$.
3. $(a - b)\, c = ac - bc$.
4. $\frac{a \times b}{c} = \frac{a}{c} \times b$.
5. $\frac{a \div b}{c} = \frac{a}{c} \div \frac{b}{c}$.
6. $\frac{a}{b} = \frac{a \times n}{b \times n}$.
7. $\frac{a}{b} = \frac{a \div n}{b \div n}$.

The symbol, ( ), is called the **parenthesis.** When two or more numbers are inclosed in a parenthesis, it means that they are to be treated as a whole and subjected to the same operation.

Let the student state the above principles in general language. For instance, 7 may be expressed thus,

If the divisor and dividend be divided by the same number, the quotient is not changed.

EXERCISE 20.

1. I bought 460 acres at $26 an acre, and sold them at $37 an acre. Required my total gain.

2. During June a butcher took in $37 on each of 25 days. During July he took in $43 on each of 25 days. What were his total receipts? What was the July excess?

3. Monday, 3126 people attended the theatre; and Tuesday, 4719 people attended. If the tickets were $2 each, what were the total receipts? The Tuesday excess?

4. A train leaves New York for Chicago, 900 miles away, at the rate of 30 miles an hour, and at the same time another leaves Chicago at 45 miles an hour. How long before they meet? How far from Chicago do they meet? How long between the arrivals of the trains at destinations?

5. The excursion fare between two places is $17. What does the R. R. Co. receive from the passengers in a train of 8 cars, each carrying 63 passengers?

6. From the sum of 1436, 2785, 43697, 5638, take the product of 341 and 75.

7. Divide the difference between 3467519 and 5321963 by the quotient of 38836 divided by 73.

8. Multiply the sum of 2076, 35941, 763, 9876, and 21638, by the difference between the greatest two of them.

9. When the divisor is 409, the quotient 1703, and the remainder 245, what is the dividend?

10. When the minuend is 57632051, and the remainder 14678932, what is the subtrahend?

11. When the dividend is 100, the quotient 8, and the remainder 4, what is the divisor?

12. When the dividend is 2606526, the quotient 1478, and the remainder 812, what is the divisor?

13. An officer in distributing an appropriation of $697000 among the counties of a State, gave each county $33190, and had $10 remaining. How many counties were there?

85. Logarithms.—By the use of a series of auxiliary numbers called logarithms, numerical work may be further abbreviated. By the use of logarithms, multiplication is converted into addition, division into subtraction, etc. The consideration of these methods, however, comes later.