CHAPTER VII.

FACTORS AND ANALYSIS.

86. The **factors** of a number (see Art. 49) have already been defined as the numbers which, multiplied together, produce the given number.

Thus, the factors of 187 are 11 and 17; of 60 are 3, 4, and 5.

87. Illustration of the Value of a Knowledge of the Factors of Numbers.—If it is required to determine the value of

$$\frac{252 \times 240}{54 \times 35}$$

a knowledge of the factors of the given numbers enables us greatly to abbreviate the work. For, since dividing the dividend and divisor by the same number leaves the value of the quotient unchanged, we may divide 252 and 54 by the number 9, which is a factor of both, and proceed in like manner till all the factors common to both divisor and dividend are removed. Thus,

$$\frac{\overset{\overset{4}{\cancel{28}}}{\cancel{252}} \times \overset{\overset{8}{\cancel{48}}}{\cancel{240}}}{\underset{\cancel{6}}{\cancel{54}} \times \underset{\cancel{7}}{\cancel{35}}} = 4 \times 8 = 32, \textit{ Result.}$$

Similarly, an indicated quotient of two large numbers may often be reduced to a simple form by means of a knowledge of the factors of the numbers.

Thus, if we have $\frac{235}{376}$, and know that 235 = 47 × 5, and 376 = 47 × 8, we have

$$\frac{235}{376} = \frac{47 \times 5}{47 \times 8} = \frac{5}{8}.$$

These illustrations show the importance of as thorough a knowledge as possible of the factors of numbers and of the processes of determining them.

88. Prime and Composite Numbers.—A **prime number,** or **prime**, is a number which is not divisible by any number except itself and unity.

Thus, 2, 3, 17, 47, etc., are prime numbers.

A **composite number** is a number which can be divided by one or more numbers besides itself and unity.

Thus, 4, 6, 18, etc., are composite numbers.

89. Even and Odd Numbers.—As a rule, the most important possible factor of a number is 2.

An **even number** is a number exactly divisible by 2; as 2, 4, 6, etc.

An **odd number** is one not exactly divisible by 2; as 1, 3, 5, etc.

90. Powers and Exponents.—A power is the product of two or more identical factors. Thus, since

$$125 = 5 \times 5 \times 5$$

125 is said to be the *third* power of 5.

A **second power** is called a **square**, thus 25 is the square of 5; a **third power** is called a **cube**, thus 125 is the cube of 5.

An **exponent** is a small figure written above and to the right of a number to indicate how many times the number is taken as a factor (and to save the labor of writing out all the identical factors).

Thus, since $729 = 3 \times 3 \times 3 \times 3 \times 3 \times 3$, we may write $729 = 3^6$, the 6 being the exponent of 3.

Similarly, since $400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5$, we may write $400 = 2^4 \times 5^2$.

91. Short Methods of Determining Whether 2, 3, 4, 5, 8, 9, are Factors of a Given Number.—To determine whether a given number contains another number as a factor, the direct method is to divide the given number by the supposed factor. This work, however, may be greatly abbreviated in the case of 2, 3, 4, 5, 8, 9, and some other numbers. Thus, in order to determine whether 2 is a factor of a number, it is not necessary to divide the entire number by 2, but only the last or right-hand digit. Hence, we substitute the less labor of dividing the last figure of a number for the greater labor of dividing the entire number.

The following abbreviated methods of determining the factors are of especial importance:

A number is divisible,

1. By 2, if its last or right-hand figure is divisible by 2.

Thus, 765918 is divisible by 2 since 8 is divisible by 2.

2. By 4, if the number expressed by its last two digits is divisible by 4.

Thus, 742368 is divisible by 4 since 68 is.

3. By 8, if the number expressed by the last three digits is divisible by 8.

Thus, 659256 is divisible by 8 since 256 is.

4. By 3, if the sum of the digits composing the number is divisible by 3.

Thus, 8721561 is divisible by 3 since the sum of the digits of the number is 30, which is a number divisible by 3.

5. By 9, if the sum of the digits composing the number is divisible by 9.

Thus, 87219 is divisible by 9 since the sum of the digits is 27, which is a number divisible by 9.

6. By 5, when the last digit in the number is either 5 or 0.

7. By 6, when the number is divisible by both 2 and 3.

The reasons for the first three of the above tests of divisibility are similar. For, in the first case, any number may be regarded as made up of a number of tens and an additional group of units. Since 10 is divisible

by 2, any number of tens is also divisible by 2; hence, if the last figure is also divisible by 2, the entire number is divisible by 2.

Similarly, in the second case, any number may be regarded as made up of a number of hundreds and an additional number composed of two digits. Since 100 is divisible by 4, any number of hundreds is divisible by 4; hence, if the additional number expressed by the last two digits is divisible by 4, the entire number is divisible by 4.

Similarly, the test for divisibility by 8 depends on the fact that 8 is an exact divisor of 1000. Let the student make a formal statement of the reasoning involved.

The reason for the test of the divisibility of a number by 3 is as follows: Any number larger than 10 may be separated into two parts, viz.: a part which is a multiple of 9 (and hence divisible by 3); and a second part which is equal to the sum of the digits of the number. Hence, if the latter part is divisible by 3, the entire number is. Thus, to test 852 as to its divisibility by 3,

$$\begin{aligned} \text{Since } 800 &= 8\,(99 + 1) = 8 \times 99 + 8 \\ \text{and } 50 &= 5\,(9 + 1) = 5 \times 9 + 5 \\ \text{and } 2 &= 2 \\ 852 &= (8 \times 99 + 5 \times 9) + (8 + 5 + 2). \end{aligned}$$

Since the multiples of 9 are divisible by 3, the divisibility of the entire number by 3 depends on whether the sum of the digits 8 + 5 + 2 is divisible by 3.

Let the student state in like manner the reason for the abbreviated method of determining whether a number is divisible by 9.

Let the student also state the reason for the test of the divisibility of a number by 5.

The student should observe that the positional system of notation adopted in representing numbers makes possible these abbreviated tests of divisibility. Let the pupil determine which of them could be applied to numbers expressed in the Roman notation.

92. Prime Factors of a Number.—To determine the prime factors of a number, it is sufficient to *divide the given number by a prime factor (it is generally best to divide first by the smallest prime factor), then divide the quotient obtained by another prime factor, and so on till a quotient is obtained which is itself prime.*

Ex. Separate 2040 into its prime factors.

OPERATION.

$$\begin{array}{r} 2\underline{)2040} \\ 2\underline{)1020} \\ 2\underline{)510} \\ 3\underline{)255} \\ 5\underline{)85} \\ 17 \end{array} \qquad \therefore 2040 = 2^3 \times 3 \times 5 \times 17, \textit{ Factors.}$$

EXERCISE 21.

Find the prime factors of:

1. 15, 18, 20, 25, 27, 28, 32, 36, 40, 42, 48.
2. 50, 56, 60, 64, 72, 80, 84, 88, 92, 98, 105.
3. 108, 112, 124, 128, 136, 148, 150, 165.
4. 224, 396, 480, 600, 842, 873, 919, 960.
5. 1315, 1599, 3003, 2145, 3696, 4081, 12121.

Tell by inspection whether or not each of the following numbers is divisible by 2, 3, 4, 5, 6, 8, 9, 10, or 20.

6. 120, 130, 140, 156, 171, 217, 240, 498.
7. 3428, 7653, 9345, 76532, 97605, 123456.
8. 98010, 152460, 216216, 445038, 876543210.

Determine whether the following numbers are prime or composite:

9. 81, 83, 87, 93, 111, 201, 271, 343, 427.
10. 319, 507, 533, 851, 917, 1189, 1927.
11. Which are the more numerous, odd numbers or even numbers? Prime numbers or odd numbers? Are all prime numbers odd?
12. Obtain tests for the divisibility of a large number by 12. By 15. By 18. By 36. By 40. By 32.
13. Ascertain what the sieve of Eratosthenes is, and by its use form a list of prime numbers from 1 to 500.
14. By the aid of this table determine whether 839 is a prime number. Is it necessary to divide by *all* of the prime numbers less than 1333 to determine whether it is prime? Which may be omitted?

93. Cancellation.—It has been shown that dividing both divisor and dividend by the same number does not change the quotient. So much labor is saved by this means that the process is frequently used, and it is convenient to give it a special name.

Cancellation is the operation of striking out a factor common to both divisor and dividend.

Ex. 1. Compute $\frac{50 \times 18 \times 84}{12 \times 3 \times 75}$ by cancellation.

OPERATION.

$$\frac{\overset{2}{\cancel{50}} \times \overset{\overset{2}{\cancel{6}}}{\cancel{18}} \times \overset{7}{\cancel{84}}}{\cancel{12} \times \cancel{3} \times \underset{\cancel{3}}{\cancel{75}}} = 28, \textit{ Quotient.}$$

EXPLANATION.

50 and 75 have the common factor, 25, which may be canceled, giving 2 in the place of 50, and 3 in the place of 75. 12 will divide 12 and 84, giving 1 and 7. 3 will divide 3 and 18, giving 1 and 6. 3 again will divide 3 and 6, giving 1 and 2. Hence, the quotient is $2 \times 2 \times 7$, or 28.

The quotient is not changed in its denomination by cancellation, but this is not the case with the *remainder*, if there be one. To obtain the true remainder, it is necessary to multiply the remainder after the cancellation, by all the factors cancelled out.

Ex. By cancellation $\frac{\overset{\cancel{4}}{\cancel{36}} \times \overset{14}{\cancel{42}} \times \overset{2}{\cancel{6}}}{\underset{\cancel{3}}{\cancel{12}} \times \cancel{9} \times \underset{5}{\cancel{15}}} = \frac{28}{5} = 5,$

with apparent remainder of 3. But the true remainder is obtained by multiplying 3 by $12 \times 9 \times 3$ (the factors cancelled out), giving 972, the remainder which should have been obtained if the division had been performed without any cancellation.

It is left as an exercise for the pupil to discover the reason of this process.

EXERCISE 22.

Reduce:

1. $\frac{6 \times 10}{20}$.

2. $\frac{24}{3 \times 2}$.

3. $\frac{8 \times 15}{20 \times 3}$.

4. $\frac{24 \times 25}{75}$.

5. $\frac{56 \times 27}{9 \times 42}$.

6. $\frac{65 \times 84}{26 \times 25}$.

7. $\frac{144 \times 125}{100 \times 12}$.

8. $\frac{88 \times 105}{20 \times 77}$.

9. $\frac{216 \times 343}{98 \times 42}$.

10. $\frac{5 \times 6 \times 20}{4 \times 15 \times 2}$.

11. $\frac{7 \times 9 \times 16}{8 \times 14 \times 3}$.

12. $\frac{17 \times 20 \times 27}{9 \times 34 \times 5}$.

13. $\frac{38 \times 40 \times 55}{44 \times 19 \times 50}$.

14. $\frac{48 \times 68 \times 77}{84 \times 187}$.

15. $\frac{60 \times 75 \times 81}{45 \times 50 \times 54}$.

16. $\frac{90 \times 96 \times 98}{168 \times 630}$.

17. $\frac{205 \times 666 \times 18}{81 \times 185 \times 84}$.

18. $\frac{360 \times 405 \times 7}{75 \times 504}$.

19. $\frac{384 \times 162 \times 275}{216 \times 200 \times 99}$.

20. Divide $16 \times 18 \times 24 \times 30$ by $45 \times 32 \times 72$.
21. Divide $60 \times 70 \times 85 \times 96$ by $42 \times 125 \times 64$.
22. Divide $128 \times 132 \times 150$ by $275 \times 48 \times 84$.
23. Divide $345 \times 396 \times 425$ by $187 \times 276 \times 375$.

Ascertain the value and the true remainder in each:

24. $\frac{15 \times 28 \times 96}{77 \times 40}$.

25. $\frac{18 \times 25 \times 126}{45 \times 28 \times 35}$.

26. $\frac{48 \times 50 \times 51}{60 \times 34 \times 16}$.

27. $\frac{58 \times 57 \times 56}{21 \times 24 \times 87}$.

28. $\frac{63 \times 64 \times 198}{42 \times 99 \times 72}$.

29. $\frac{95 \times 96 \times 98}{343 \times 38 \times 36}$.

ANALYSIS.

94. Units Used for Computation Purposes.—Besides units in general use, such as \$1, 1 yard, etc., certain special units are often employed in solving particular examples, simply as an aid in computation.

95. Analysis is the solution of problems by the aid of special units devised to aid in the computation.

Ordinarily we have given in the problem the value of the unit when taken a given number of times. The process of analysis consists (1) in determining the value of the unit taken once, and (2) the value of the unit when taken a required number of times. These two steps are called reasoning *to the unit* and *from the unit.*

Ex. 1. If 6 horses cost \$420, what will 15 horses cost?

ANALYSIS.—The unit considered in this problem is the cost of 1 horse. Thus,

$$\$420 = \text{cost of 6 horses (6 units).}$$

$$\frac{\$420}{6} = \text{cost of 1 horse (1 unit).}$$

$$\frac{\$420 \times 15}{6} = \text{cost of 15 horses (15 units).}$$

$$\frac{\overset{70}{\cancel{\$420}} \times 15}{\cancel{6}} = \$1050, \text{ cost of 15 horses.}$$

It should be noticed that ordinarily it is of advantage merely to indicate the division which gives the value of the single unit, and not to obtain the quotient itself by actual division (thus we write $\frac{\$420}{6}$ and not \$70 as the cost of 1 horse), in order to take advantage of possible cancellations in the final computation.

Though the above statement of the analysis is all that the student need write down as the solution of the problem, he should be able to give clearly and exactly the reasoning used. Thus, in the above example, if \$420 is the cost of 6 horses, 1 horse will cost as many dollars as 6 is contained times in \$420, or $\frac{\$420}{6}$; and if 1 horse costs $\frac{\$420}{6}$, 15 horses will cost 15 times $\frac{\$420}{6}$ or \$1050.

Ex. 2. If 9 books cost \$20, what will 54 books cost?

The unit is the cost of 1 book.

$\$20 =$ cost of 9 books.

$\frac{\$20}{9} =$ cost of 1 book.

$\frac{\$20 \times 54}{9} =$ cost of 54 books.

$\frac{\$20 \times \overset{6}{\cancel{54}}}{\cancel{9}} = \$120$, *Result.*

Two or more steps are often necessary in obtaining the value of the computation unit (*i. e.*, in reasoning to the unit) and also at times in reasoning from the unit. Thus:

Ex. 3. A workman received \$21 for 15 days' work of 7 hours each. How many dollars will he receive for 17 days' work of 10 hours each?

The unit which controls the computation is the number of dollars received for 1 hour's work.

Hence,

$7 =$ No. hours in 1 day's work.

$15 \times 7 =$ No. hours in 15 days' work.

$\$21 =$ wages for 15×7 hours' work.

$\frac{\$21}{15 \times 7} =$ wages for 1 hour's work (the unit).

$\frac{\$21 \times 10}{15 \times 7} =$ wages for 10 hours' work.

$\frac{\$21 \times 10 \times 17}{15 \times 7} =$ wages for 17 days' work of 10 hours each.

$$\frac{\overset{3}{\cancel{\$21}} \times \overset{2}{\cancel{10}} \times 17}{\underset{3}{\cancel{15}} \times \cancel{7}} = \$34, \textit{ Result.}$$

After some practice in working similar examples, the above statement may be conveniently abbreviated as follows:

$\frac{\$21}{15 \times 7} =$ wages for 1 hour's work.

$\frac{\$21 \times 10 \times 17}{15 \times 7} =$ wages for 10×17 hours' work.

The unit of computation in a problem is often given explicitly, it being required to determine the number of times the unit is used.

Ex. 4. How many pounds of sugar at 6 cents a pound can be obtained in exchange for 10 dozen eggs at 21 cents a dozen?

The cost of 1 pound of sugar is the unit which controls the computation.

21 cents $=$ value 1 dozen eggs.

21 cents $\times$ 10 $=$ value 10 dozen eggs.

6 cents $=$ value 1 pound sugar.

$\frac{21 \text{ cents} \times 10}{6 \text{ cents}} =$ No. pounds sugar at 6 cents a pound which can be obtained for 21×10 cents.

$$\frac{\overset{7}{\cancel{21}} \times \overset{5}{\cancel{10}}}{\underset{2}{\cancel{6}}} = 35, \textit{ No. of pounds.}$$

Ex. 5. A milkman has 20 cows, each of which gives 8 quarts of milk daily. He sells the milk for 6 cents a quart. How many pieces of cloth, each containing 40 yards and costing 15 cents a yard, can he obtain for the milk of 10 days?

6 cents $\times 8 \times 20 \times 10 =$ value of milk for 10 days.

15 cents $\times 40 =$ cost of 1 piece of cloth.

$\frac{6 \text{ cents} \times 8 \times 20 \times 10}{15 \text{ cents} \times 40} =$ No. of pieces of cloth received for 6 cents $\times 8 \times 20 \times 10$.

$$\frac{\overset{2}{\cancel{6}} \times 8 \times \overset{5}{\cancel{20}} \times \cancel{10}}{\underset{3}{\cancel{15}} \times \underset{4}{\cancel{40}}} = 16, \textit{ No. of pieces.}$$

Ex. 6. 12 men working 8 hours a day do a piece of work in 15 days. How many days will it take 8 men working 10 hours a day?

The computation unit is the work done by one man in 1 hour; then

$12 \times 8 \times 15 =$ No. units work done by 12 men in 15 days of 8 hours each.

$8 \times 10 =$ No. units work done by 8 men in 1 day of 10 hours.

$\frac{12 \times 8 \times 15}{8 \times 10} =$ No. days it will take 8 men working 10 hours a day to do $12 \times 8 \times 15$ hours' work for 1 man.

$$\frac{\overset{6}{\cancel{12}} \times \cancel{8} \times \overset{3}{\cancel{15}}}{\cancel{8} \times \underset{2}{\cancel{10}}} = 18, \textit{ No. of days.}$$

EXERCISE 23.

1. If 6 stamps cost 30 cents, what will 14 stamps cost? 45 stamps?

2. If 9 pads cost 72 cents, what will 7 pads cost?

3. If 12 pounds of candy cost 216 cents, how much will 21 pounds cost?

4. When $415 will buy 5 acres of land, how many dollars are 17 acres worth?

5. What will 25 cattle cost when 7 cattle are worth $161?

6. If a bar of iron 12 feet long weighs 192 pounds, how much will a similar bar 19 feet long weigh?

7. If a stock of 45 chairs is worth $279, what is another stock of 55 similar chairs worth?

8. If a class of 74 men weigh together 11766 pounds, about what will a similar class of 111 men weigh?

9. A family of 7 drink 17 quarts of water each day. How many quarts will a town of 14000 people drink?

10. If 16 people use $20 worth of meat each week, how much will 50 people use in a year of 52 weeks?

11. A workman receives $6 for working 5 days of 9 hours each. How many dollars should he receive for the labor of 15 days of 10 hours each?

12. If a laborer receives $25 for the work of 15 days of 7 hours, how many dollars will be paid him after 28 days' work of 9 hours each?

13. If $264 are paid 11 men for the labor of 16 days, each 10 hours, how much should be paid 40 men for 25 days' labor, each day of 6 hours?

14. It cost $290 to print and bind 75 books, of 812 pages each. What will be the cost of printing and binding 81 books, of 560 pages each, at the same rate?

15. If a force of 63 men can do a certain task in 7 days, of 8 hours each, how many men, working 6 hours a day, will be needed to do a similar task in 12 days?

16. For the construction of a certain wall, 8 rods long, 36 men were required, working 10 hours each, of 24 days. How many days, of 8 hours each, will it take 55 men to construct a like wall, 22 rods long?

17. On the erection of a wall, 75 feet long, 6 feet wide, and 8 feet high, 30 men worked 17 days, of 8 hours each. How long a wall, 4 feet wide and 7 feet high, can a force of 34 men build in 40 days, of 7 hours each?

18. How many pounds of rice, at 8 cents a pound, can be bought for 12 pounds of butter, at 20 cents a pound?

19. A merchant exchanges 45 yards of cloth, worth $2 a yard, for silk, worth $5 a yard. How many yards of silk does he receive?

20. 12 casks of vinegar, each containing 16 gallons, and worth 10 cents a gallon, are given in exchange for potatoes, worth 60 cents a bushel. How many bushels of potatoes are received?

21. How many firkins of butter, each containing 50 pounds, worth 23 cents a pound, will be returned for 115 bales of hay, at 90 cents a bale?

22. A farmer sells the wool from 60 sheep, at 13 cents a pound, each fleece weighing 4 pounds. How many rolls of matting, at 52 cents a yard, can he buy with the money, if each roll contains 15 yards?

23. Each line of a book, of 150 pages, contains 12 words, and there are 30 lines on a page. If the printing costs 3 cents a word, how many bales of paper, each containing 10 bundles, of 20 quires each, and worth 18 cents a quire, can be bought with the proceeds of printing the book?

CHAPTER VIII.

GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE.

GREATEST COMMON DIVISOR.

96. Common Factors.—In order to simplify work as much as possible (as for instance in cancellation), we need to know not only *a* number that will divide each of two given numbers, but also the *largest* number that will divide both of them, and to have methods of determining this number which will cover all cases.

Thus, in order to simplify the indicated quotient $\frac{1128}{705}$, we need to know that the largest number that will divide both dividend and divisor is 141, or, if we do not know it, to have some way of determining it.

97. The **greatest common divisor** (or G. C. D.) of two or more numbers is the largest number that will divide them all.

It is also sometimes called the highest common factor, or H. C. F.

Thus, $12 is the G. C. D. of $24, $36, $60.

The G. C. D. may also be described as the largest unit group which can be used to measure all of a set of numbers.

When two numbers have no common factor except unity, the numbers are said to be *prime to each other;* thus, 8 and 15 are prime to each other, though neither of them is itself a prime number.

98. Aids in Finding the G. C. D.—It is helpful in finding the G. C. D. of two or more numbers to understand that

First, if a number be a factor of each of two or more numbers, it must be a factor of their G. C. D.

Thus, since 5 is a factor of 30, 75, and 90, it must be a factor of their G. C. D.

This principle enables us to separate the process of finding the G. C. D. of two or more numbers into several often comparatively simple steps of finding the prime factors common to all of the numbers, and a last step of taking the product of these common prime factors.

Second, if a number be a factor of two numbers, it must be a factor of the sum or difference of any multiples of these numbers.

This principle is illustrated if, for instance, we tie toothpicks into bundles, of 12 each, and have 8 bundles in one heap and 5 bundles in another heap. 12 toothpicks will evidently be a divisor of each entire heap, or of the sum of the two heaps, or of their differences, or of the sum or difference of any multiples of such heaps (since such a sum or difference will be composed entirely of bundles containing 12 each).

Thus, again, if 9 is a factor of both 333 and 855, it is a factor of their sum or difference, or of $855 - 2 \times 333$; that is, of the number 189, smaller than either of the original numbers.

This principle enables us to simplify the work of finding the G. C. D. of two large numbers, by using smaller and smaller numbers, obtained by successive subtractions of multiples of a smaller number from a larger. See Art. 101.

99. I. Short Division Method of Finding the G. C. D.—If the numbers whose G. C. D. is sought, be small, the most convenient method of proceeding is to *arrange the given numbers in a row, and divide by any number that will divide all the given numbers; similarly divide the quotients obtained till there is no number which will divide all the quotients; the product of all the divisors will be the G. C. D.*

Ex. Find the G. C. D. of 84, 126, 210.

OPERATION.

```
2)84, 126, 210
3)42,  63, 105
7)14,  21,  35
   2,   3,   5
```

Hence, $2 \times 3 \times 7 = 42 =$ G. C. D.

If the pupil be already thoroughly acquainted with the prime factors of the numbers whose G. C. D. is sought, it is sometimes convenient to separate each of the given numbers into its prime factors, and multiply together the prime factors that are common to all the numbers.

Ex. Find the G. C. D. of 24, 72, 120.

OPERATION.

$24 = 2^3 \times 3$

$72 = 2^3 \times 3^2$

$120 = 2^3 \times 3 \times 5$

$\therefore 2^3 \times 3 = 24 = \text{G. C. D.}$

EXERCISE 24.

Find the G. C. D. of the following groups of numbers:

1. 18, 42.
2. 24, 60.
3. 60, 105.
4. 90, 198.
5. 112, 256.
6. 168, 273.
7. 216, 384.
8. 630, 924.
9. 30, 42, 72.
10. 32, 56, 88.
11. 36, 84, 180.
12. 180, 144, 198.
13. 168, 288, 216.
14. 135, 378, 324.
15. 210, 300, 450.
16. 192, 288, 416.
17. 144, 252, 400.
18. 432, 540, 918.
19. 24, 40, 56, 104.
20. 42, 98, 112, 140.
21. 40, 70, 90, 150.
22. 72, 120, 144, 264.
23. 108, 162, 198, 270.
24. 288, 480, 720, 336.

100. II. **Long Division Method of Finding G. C. D.**— When the numbers whose G. C. D. is sought are large, it is best to proceed by the method indicated in a general way in Art. 98. For, by the aid of this principle, it can be shown that the G. C. D. of any two numbers, taken as *divisor* and *dividend*, is the same as the G. C. D. of the *divisor* and *remainder*.

For since, denoting the quotient by *m*,

we have, *Divisor*) *Dividend*(*m*
m × *Divisor*,
Remainder,

and hence *Remainder* = *Dividend* − *m* × *Divisor*,

and also, *Dividend* = *m* × *Divisor* + *Remainder*.

Now, every number that will divide both divisor and dividend exactly must also divide the remainder exactly (by Art. 98, Second Principle, since the

remainder is the difference between the dividend and a multiple of the divisor). Hence, all the common factors of both divisor and dividend are also common factors of the divisor and remainder.

Conversely, whatever number will divide the divisor and remainder exactly is also a factor of the dividend (by Art. 98, since the dividend is the sum of the remainder and a multiple of the divisor). Hence, all the common factors of the divisor and remainder are also common factors of the divisor and dividend.

Hence, every common factor of the one pair of numbers is a common factor of the other pair also; hence, the two pairs have the same G. C. D. Hence, we may substitute the smaller pair, the divisor and remainder, for the larger pair, the divisor and dividend, and by successive uses of this principle, finally determine the G. C. D.

Ex. Find the G. C. D. of 841, 1740.

OPERATION.

```
841)1740(2
    1682
      58)841(14
         58
         261
         232
          29)58(2
             58
G. C. D. = 29
```

EXPLANATION.

Dividing 841 into 1740, we obtain 2 for a quotient and 58 for a remainder. But by the principle proved above, the G. C. D. of 841 and 58 is the same as the G. C. D. of 841, 1740. Proceeding in like manner, the G. C. D. of 29 and 58 is the same as the G. C. D. of the original pair of numbers, 841 and 1740.

By the use of symbols, the proof given above that the G. C. D. of the divisor and remainder is the same as the G. C. D. of the divisor and dividend, may be put in an abbreviated form thus:

Denote the smaller of two numbers (the divisor) by A, the larger (the dividend) by B, the quotient by m, the remainder by R. We have

```
A) B (m
   m A        or     R = B − m × A
   R          also   B = R + m × A
```

Then every factor of the numbers A and B is also a factor of R (by Art. 98, since $R = B - m \times A$); and, hence, is a factor of the pair of numbers A and R.

Conversely, every factor of the pair of numbers A and R is also a factor of B (Art. 98, since $B = R + m \times A$), and, hence, is a factor of the pair of numbers A and B.

Hence, every factor of the one pair of numbers is a factor of the other pair $\therefore$ the G. C. D. of A and B = G. C. D. of A and R.

Hence, to find the G. C. D. of two numbers, *divide the less number into the greater, the remainder into the divisor, and thus continue until there is no remainder; the last divisor will be the G. C. D. of the two original numbers.*

To find the G. C. D. of three or more large numbers, first find the G. C. D. of two of the numbers by the above method, then obtain the G. C. D. of this result and a third number, and so on till all the numbers have been used. The last G. C. D. obtained is the G. C. D. of all the original numbers.

EXERCISE 25.

Find the G. C. D. of:

1. 55, 75.
2. 68, 92.
3. 90, 138.
4. 96, 152.
5. 126, 153.
6. 189, 261.
7. 176, 275.
8. 252, 480.
9. 187, 510.
10. 182, 533.
11. 390, 675.
12. 882, 903.
13. 918, 675.
14. 1457, 899.
15. 2736, 4389.
16. 182, 196, 357.
17. 209, 198, 473.
18. 272, 400, 816.
19. 782, 969, 1156.
20. 216, 360, 280.
21. 270, 315, 735.
22. 546, 455, 702.
23. 1584, 2772, 3276.
24. 2088, 2349, 3016.
25. 3330, 2035, 3663.

LEAST COMMON MULTIPLE.

101. A **common multiple** of two or more numbers is a number which is exactly divisible by all of them.

Thus, $600 is a common multiple of $15, $20, and $30.

The **least common multiple** (or L. C. M.) of two or more numbers is the least number which is divisible by them all.

Thus, $60 is the L. C. M. of $15, $20, $30.

The most useful application of the L. C. M. is in determining the least common multiple of the denominators of a set of fractions. This enables us to determine the largest unit which will measure each of a set of fractional quantities, just as the G. C. D. enables us to determine the largest unit which will measure a set of integral quantities.

102. I. Short Division Method of Determining the L. C. M. of Several Numbers.—If the numbers whose L. C. M. is desired, are small, the most convenient method of proceeding in order to determine their L. C. M. is to *arrange the given numbers in a row; divide by any prime factor that will divide at least two of them, bringing down each undivided number along with the quotients; continue the process till the quotients are all prime to each other; the L. C. M. will be the product of all the divisors and final quotients.*

If any one of the numbers is contained exactly in (∴ is a factor of) any other of the given numbers, it may be struck out. For, in finding the L. C. M. of the larger number, we find that of the smaller number also.

Ex. 1. Find the L. C. M. of 12, 21, 30, 36, 63, 70.

OPERATION.

2)12, 21, 30, 36, 63, 70
3)15, 18, 63, 35
3)5, 6, 21, 35
2, 7, 35

$2 \times 3 \times 3 \times 2 \times 35 = 1260$, L. C. M.

EXPLANATION.—12 is contained in 36, and 21 in 63; hence, they are struck out, and the L. C. M. of the remaining numbers is found. (Similarly 5 and 7 are struck out in the course of the process.) By dividing by the prime factors 2, 3, 3, and multiplying them and the final quotients together, each prime factor will occur in the final product the highest number of times it occurs in any one number; hence, the product thus obtained will be the L. C. M.

If the prime factors of the given numbers are well known, it is sometimes more convenient to separate each of the given numbers into its prime factors, and take the product of all the different factors, using each factor the greatest number of times it occurs in any single number.

Ex. 2. Find the L. C. M. of 48, 72, 120.

$$48 = 2^4 \times 3.$$
$$72 = 2^3 \times 3^2.$$
$$120 = 2^3 \times 3 \times 5.$$
$$\therefore \text{L. C. M.} = 2^4 \times 3^2 \times 5 = 720.$$

EXERCISE 26.

Find the L. C. M. of:

1. 8, 12.	*7.* 6, 10, 15.	*13.* 105, 168, 120.
2. 9, 15.	*8.* 12, 16, 36.	*14.* 280, 144, 210.
3. 10, 25.	*9.* 18, 45, 70.	*15.* 180, 189, 315.
4. 18, 30.	*10.* 24, 45, 40.	*16.* 210, 231, 330.
5. 24, 42.	*11.* 32, 81, 72.	*17.* 182, 286, 308.
6. 28, 70.	*12.* 48, 84, 210.	*18.* 385, 420, 660.

19. 12, 20, 36, 54.	*25.* 20, 24, 25, 36, 45.
20. 22, 44, 88, 108.	*26.* 30, 36, 35, 56, 80.
21. 12, 24, 63, 84.	*27.* 33, 42, 66, 70, 84.
22. 8, 15, 18, 120.	*28.* 45, 54, 65, 91, 63.
23. 10, 25, 40, 75.	*29.* 63, 75, 81, 98, 105.
24. 7, 16, 77, 132.	*30.* 85, 102, 105, 110, 165.

103. II. Long Division Method of Determining the L. C. M. of two or more Numbers.—When it is required to find the L. C. M. of two large numbers, which cannot be readily factored, it is best to proceed by first finding their G. C. D. by the long division method.

Ex. Find the L. C. M. of 841 and 1740.

We first find the G. C. D. of the numbers, thus:

841)1740(2
1682
58)841(14
58
261
232
29)58(2
58

∴ G. C. D. = 29

Dividing each of the given numbers by their G. C. D., we have

29)841(29
58
261
261

∴ 841 = 29 × 29

29)1740(60
174
0

∴ 1740 = 29 × 60

Hence, to find the L. C. M., we may proceed as in Art. 102:

29)29 × 29, 29 × 60
29, 60

∴ L. C. M. = 29 × 29 × 60
= 29 × 1740
= 50460.

Hence, in general, to find the L. C. M. of two large numbers, *find first the G. C. D. of the given numbers; divide one of the given numbers by the G. C. D. and multiply the quotient by the other number; the product will be the L. C. M. of the two numbers.*

To find the L. C. M. of three or more large numbers, first find the L. C. M. of two of the given numbers, then the L. C. M. of this result and another of the given numbers, and so on till all of the given numbers have been used. The last L. C. M. obtained is the L. C. M. of all the given numbers.

EXERCISE 27.

Find the L. C. M. of:

1. 264, 319.	*5.* 450, 648.	*9.* 456, 684, 720.
2. 320, 408.	*6.* 832, 650.	*10.* 280, 448, 640.
3. 506, 308.	*7.* 252, 329, 357.	*11.* 396, 495, 660.
4. 390, 525.	*8.* 288, 405, 477.	*12.* 945, 810, 1260.

EXERCISE 28.

1. Find the G. C. D. of 72, 96, 132.
2. Find the L. C. M. of 60, 75, 90.
3. Find the G. C. D. of 672 and 526.
4. Find the L. C. M. of 12, 15, 25, 28, 35.
5. Find the G. C. D. of 782, 867, 969.
6. Find the L. C. M. of 1066 and 962.

Find the G. C. D. and the L. C. M. of:

7. 18, 42, 54, 96.	*10.* 1008, 1365.
8. 24, 40, 120, 160.	*11.* 195, 510, 468.
9. 84, 210, 378.	*12.* 406, 945, 980.

13. What is the greatest width of carpet that will exactly fit three rooms of widths 15 feet, 24 feet, and 33 feet respectively?

14. A merchant having 54 yards of one kind of cloth, 84 yards of another, and 132 yards of a third, wishes to cut them into patterns of equal length. What is the greatest possible length of each pattern?

15. With a 4-quart, a 5-quart, and a 6-quart vessel, what is the size of the smallest can which may be filled exactly by each?

16. Find the length of the shortest line that can be measured exactly by rods of lengths 6 feet, 8 feet, 10 feet, and 12 feet.

17. What is the length of the longest rod which will exactly measure 209 feet, 242 feet, and 341 feet?

18. A farm produces 442 bushels of oats, 728 bushels of corn, and 585 bushels of wheat. The grain is removed in equal cases and all are full. What is the greatest capacity of each case, provided there is no mixing of the grains?

19. How can the L. C. M. of two numbers, which are prime to each other, be found? Of two prime numbers?

20. How many common multiples may 2 or more numbers have?

21. Find the difference between the G. C. D. of 480 and 520, and the L. C. M. of 5, 6, 15, 20.

CHAPTER IX.

COMMON FRACTIONS.

104. Derived Units.—A certain unit, as one pound, having been chosen for the purpose of weighing objects in general, it is often convenient to obtain from this primary unit (one pound) other derived units to be used for weighing special classes of objects. Thus, one ton (or 2000 pounds) is used in weighing objects of small value in proportion to their bulk, such as hay, coal, etc., and one ounce is used in weighing objects of great value in proportion to their bulk, as spices, gold, drugs, etc. Similarly, from any primary unit, derived units may be obtained adapted to special uses.

When the derived unit is an exact part of the primary unit, it is termed a *fractional unit* or *fraction*.

105. Fractional Units.—Thus, for measuring long distances, the *mile* is the convenient unit; but for many purposes, as, for instance, in running races in athletic games, it is convenient to divide the unit into 4 equal parts, and call one of them one-fourth of a mile; similarly we form other fractional units from the miles, as one-eighth, one-half, one-sixteenth of a mile, etc. These are all fractional units or fractions, and are expressed by writing the number of parts into which the given unit is divided under the figure 1. Thus, $\frac{1}{8}$ of a mile means one-eighth of a mile.

If a given fractional unit be taken any number of times, the result is still a fraction, and is denoted by writing the number of times the unit is taken above the line instead of 1. Thus, "$\frac{5}{8}$ mile" is an abbreviation for "5 units" of the value $\frac{1}{8}$ mile.

Sometimes a fractional unit receives a special name, as when one-twelfth of a foot is called an "inch"; or it may be made into a physical object, as