ADVANCED ARITHMETIC

LYMAN

WITHOUT ANSWERS

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ADVANCED ARITHMETIC

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PREFACE

THE need felt for an advanced text-book in arithmetic that shall develop fundamental principles and at the same time include the essentials of commercial practice is responsible for the appearance of this book. The author believes that mental training is an important feature in the study of arithmetic, but that the study need lose none of this training by the introduction of practical business methods. Consequently, throughout the work, the aim has been not only to develop the principles of the subject, both by means of demonstrations and exercises, but also to employ such methods and short processes as are used in the best commercial practice, and to exclude cumbersome methods and useless material.

The book is intended for pupils who have completed the grammar school work in arithmetic, and contains abundant material for a review and an advanced course.

The exercises have been selected largely from actual business transactions. A few have been taken from standard foreign works.

Brief historical notes are occasionally inserted in the hope that they will be of interest and value.

The author is indebted to several friends, who, after careful reading of manuscript, or proof sheets, or both, have offered valuable suggestions.

E. A. LYMAN.

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ADVANCED ARITHMETIC

NOTATION AND NUMERATION

1. Our remote ancestors doubtless did their counting by the aid of the ten fingers. Hence, in numeration it became natural to divide numbers into *groups of tens.* This accounts for the almost universal adoption of the *decimal scale* of notation.

2. It is uncertain what the first number symbols were. They were, probably, fingers held up, groups of pebbles, notches on a stick, etc. Quite early, however, groups of strokes I, II, III, IIII, ⋯, were used to represent numbers.

3. The earliest written symbols of the Babylonians were *cuneiform* or *wedge-shaped* symbols. The vertical wedge (𒁹) was used to represent unity, the horizontal wedge (𒀸) to represent ten, and the two together (𒁹𒀸) to represent one hundred. Other numbers were formed from these symbols by writing them adjacent to each other. Thus,

𒁹𒁹𒁹 $= 1 + 1 + 1 = 3,$

𒀸𒀸𒁹𒁹 $= 10 + 10 + 1 + 1 = 22,$

𒀸𒁹𒀸 $= 10 \times 100 = 1000,$

𒁹𒁹𒁹𒁹𒁹𒁹𒀸𒀸𒁹𒁹 $= 5 \times 100 + 10 + 2 = 512.$

To form numbers less than 100 the symbols were placed adjacent to each other and the numbers they represented were added. To form numbers greater than 100 the symbols representing the number of hundreds were placed at the left of the symbol for one hundred and used as a multiplier.

4. The Egyptians used *hieroglyphics*, pictures of objects, or animals that in some way suggested the idea of the number they wished to represent. Thus, one was represented by a vertical staff (ı), ten by a symbol shaped like a horseshoe (∩), one hundred by a short spiral (℮), one hundred thousand by the picture of a frog, and one million by the picture of a man with outstretched hands in the attitude of astonishment. They placed the symbols adjacent to each other and added their values to form other numbers. Thus, ℮∩ı $= 100 + 10 + 1 = 111$. The Egyptians had other symbols also.

5. The Greeks used the *letters* of their *alphabet* for number symbols, and to form other numbers combined their symbols much as the Babylonians did their wedge-shaped symbols.

6. The Romans used *letters* for number symbols, as follows:

1	5	10	50	100	500	1000
I	V	X	L	C	D	M

Numbers are represented by combinations of these symbols according to the following principles:

(1) The *repetition* of a symbol *repeats* the value of the number represented by that symbol; as, III = 3, XX = 20.

(2) The value of a number is *diminished* by placing a symbol of less value *before* one of greater value; as,

IV = 4, XL = 40, XC = 90. The less number is subtracted from the greater.

(3) The value of a number is *increased* by placing a symbol of less value *after* one of greater value, as XI = 11, CX = 110. The less number is added to the greater number.

(4) The value of a number is *multiplied* by 1000 by placing a bar over it, as $\overline{\text{C}} = 100{,}000$, $\overline{\text{X}} = 10{,}000$.

7. Among the ancients we do not find the characteristic features of the Arabic, or Hindu system where each symbol has two values, its *intrinsic value* and its *local value*, *i.e.* the value due to the position it occupies. Thus, in the number 513 the intrinsic value of the symbol 5 is *five*, its local value is *five hundred*. Written in Roman notation 513 = DXIII. In the Roman notation each symbol has its intrinsic value only.

8. The ancients lacked also the symbol for *zero*, or the absence of quantity. The introduction of this symbol made place value possible.

9. With such cumbersome symbols of notation the ancients found arithmetical computation very difficult. Indeed, their symbols were of little use except to record numbers. The Roman symbols are still used to number the chapters of books, on clock faces, etc.

10. The Arabs brought the present system, including the symbol for zero and place value, to Europe soon after the conquest of Spain. This is the reason that the numerals used to-day are called the *Arabic* numerals. The Arabs, however, did not invent the system. They received it and its figures from the Hindus.

11. The origin of each of the number symbols 4, 5, 6, 7, 9, and probably 8 is, according to Ball, the initial letter of the corresponding numeral word in the Indo-Bactrian alphabet in use in the north of India about 150 B.C. 2 and 3 were formed by two and three parallel strokes written cursively, and 1 by a single stroke. Just when the zero was introduced is uncertain, but it probably appeared about the close of the fifth century A.D. The Arabs called the sign 0, sifr (sifra = empty). This became the English *cipher* (Cajori, "History of Elementary Mathematics").

12. The Hindu system of notation is capable of unlimited extension, but it is rarely necessary to use numbers greater than billions.

13. In the development of any series of number symbols into a complete system, it is necessary to select some number to serve as a base. In the Arabic, or Hindu system *ten* is used as a base; *i.e.* numbers are written up to 10, then to 20, then to 30, and so on. In this system 9 digits and 0 are necessary. If five is selected as the base, but 4 digits and 0 are necessary. If twelve is selected, 11 digits and 0 are necessary.

The following table shows the relations of numbers in the scales of 10, 5, and 12. (*t* and *e* are taken to represent ten and eleven in the scale of 12.)

BASE															
10	1	2	3	4	5	6	7	8	9	10	11	12	21	48	100
5	1	2	3	4	10	11	12	13	14	20	21	22	41	143	400
12	1	2	3	4	5	6	7	8	9	*t*	*e*	10	19	40	84

Ex. 1. Reduce 431_5 to the decimal scale.

Note. 431_5 means 431 in the scale of 5.

Solution.

$$\begin{aligned} &4 \text{ represents } 4 \times 5 \times 5 = 100 \\ &3 \text{ represents } 3 \times 5 \quad\;\; = \;\;15 \\ &1 \text{ represents } 1 \qquad\quad\; = \;\;\;1 \\ \hline &\therefore 431_5 \qquad\qquad\; = 116_{10} \end{aligned}$$

Ex. 2. Reduce 4632_{10} to the scale of 8.

Solution.

$$\begin{array}{r|l} 8 & 4632 \\ \hline & 579 \quad 0 = 579 \text{ units of the second order and none of the first order.} \\ \hline & 72 \quad 3 = \;\;72 \text{ units of the third order and 3 of the second order.} \\ \hline & 9 \quad 0 = \;\;\;9 \text{ units of the fourth order and none of the third order.} \\ \hline & 1 \quad 1 = \;\;\;1 \text{ unit of the fifth order and 1 of the fourth order.} \end{array}$$

$$\therefore 4632_{10} = 11030_8.$$

EXERCISE 1

1. What number symbols are needed for the scale of 2? of 8? of 6? of 11? Write 12 and 20 in the scale of 2.

2. Reduce 234_5 and 546_7 to the decimal scale.

3. Reduce 7649_{10} to the scale of 4.

4. Compare the local values of the two 9's in 78,940,-590,634. What is the use of the zero? Why is the number grouped into periods of three figures each? Read it.

5. If 4 is annexed to the right of 376, how is the value of each of the digits 3, 7, 6 affected? if 4 is annexed to the left? if 4 is inserted between 3 and 7?

6. What is the local value of each figure in 76,345? What would be the local value of the next figure to the right of 5? of the next figure to the right of this?

7. For what purpose is the decimal point used?

8. Read 100.004 and 0.104; 0.0002; 0.0125 and 100.0025.

ADDITION

14. If the arrangement is left to the computer, numbers to be added should be written in columns with units of like order under one another.

15. In adding a column of given numbers, the computer should think of results and not of the numbers.

He should not say three and two are five and one are six and four are ten and nine are nineteen, but simply five, six, ten, *nineteen*, writing down the 9 as he names the last number. The remaining columns should be added as follows: three, seven, nine, fifteen, *seventeen*, writing down the 7; nine, fifteen, seventeen, twenty-four, *twenty-seven*, writing down the 7; nine, *eighteen*, writing down the 18. Time in looking for errors may be saved by writing the numbers to be carried underneath the sum as in the exercise.

```
  329
  764
  221
 9642
 7823
18779
  211
```

16. Checks. If the columns of figures have been added **upward**, check by adding **downward**. If the two results agree, the work is probably correct.

Another good check for adding, often used by accountants, is to add beginning with the left-hand column.

Thus, the sum of the thousands is 16 thousands, of the hundreds 26 hundreds, of the tens 16 tens, and of the units 19 units.

```
16000   or   16
 2600        26
  160        16
   19         19
18779      18779
```

EXERCISE 2

1. What is meant by the order of a digit? Define *addend*, *sum*.

2. Why should digits of like order be placed in the same column? State the general principle involved.

3. Why should the columns be added from right to left? Could the columns be added from left to right and a correct result be secured? What is the advantage in beginning at the right?

4. In the above exercise, why is 1 added ("carried") to the second column? 1 to the third column? 2 to the fourth column?

17. Accuracy and **rapidity** in computing should be required from the first. Accuracy can be attained by *acquiring the habit of always checking results.* Rapidity comes with *much practice.*

18. The 45 simple combinations formed by adding consecutively each of the numbers less than 10 to itself and to every other number less than 10 should be practiced till the student can announce the sum at sight. These combinations should be arranged for practice in irregular order similar to the following:

1	2	2	5	9	8	1	4	5	7	6	4	2	3	4
1	2	3	1	3	8	9	7	6	8	6	9	4	6	4
6	6	1	1	3	4	9	5	2	7	1	2	5	3	5
9	7	2	8	7	5	9	8	7	9	6	8	5	4	7
5	4	2	2	6	4	3	1	2	1	8	3	1	3	7
9	8	5	9	8	6	5	3	6	7	9	3	4	8	7

19. Rapid counting by ones, twos, threes, etc., up to nines is very helpful in securing both accuracy and rapidity.

Ex. Begin with 4 and add 6's till the result equals 100. Add rapidly, and say simply 4, 10, 16, 22, · · ·, 94, 100.

20. It is helpful also to know combinations, or groups that form certain numbers. Thus,

1	2	3	4	5
9	8	7	6	5

and

8	7	6	6	5	5	4
1	2	2	3	4	3	3
1	1	2	1	1	2	3

, etc., are groups that form 10, and

9	9	9	9	8	8	8	7
9	8	7	6	8	7	6	7
2	3	4	5	4	5	6	6

are groups that form 20.

21. Such groups should be carefully studied and practiced until the student readily recognizes them in his work. He should also familiarize himself with other groups. The nine-groups and the eleven-groups are easy to add, since adding nine to any number diminishes the units' figure by one, and adding eleven increases the units' and the tens' digits each by one.

EXERCISE 3

1. Begin with 8 and add 7's till the result is 50.
2. Begin with 3 and add 8's till the result is 67.

Form the following sums till the result exceeds 100:

3. Begin with 3 and add 7's.
4. Begin with 7 and add 8's.
5. Begin with 5 and add 9's.
6. Begin with 8 and add 5's.
7. Begin with 5 and add 6's.
8. Begin with 6 and add 3's.

Add the following columns, beginning at the bottom, and check the results by adding downward. Form such groups as are convenient and add them as a single number. In the first two exercises groups are indicated.

9.	10.	11.	12.	13.	14.	15.	16.
6	7	5	25.4	2122	275	5427	47.683
3	9	4	76.1	7642	267	6742	72.125
1	1	1	34.59	8321	979	8374	94.467
8	4	6	43.33	9789	231	9763	53.2124
5	5	8	67.27	2432	486	2134	91.576
2	2	4	81.2	5765	523	5666	14.421
9	4	2	28.3	1297	752	3249	32.144
6	7	9	32.99	6423	648	1678	67.6797
5	2	7	16.25	1678	486	2432	19.045
3	8	4	53.11	3212	529	5469	54.091
9	7	3	91.5	7679	926	8761	86.2459
8	4	2	85.4	2144	842	2332	27.654
2	4	1	74.1	1576	236	5467	98.346
1	5	5	22.22	4467	574	1023	84.6211

In commercial operations it is sometimes convenient to add numbers written in a line across the page. If totals are required at the right-hand side of the page, add from left to right and *check* by adding from right to left.

Add:

17. 23, 42, 31, 76, 94, 11, 13, 27, 83, 62, 93.
18. 728, 936, 342, 529, 638, 577, 123, 328, 654.
19. 1421, 2752, 7846, 5526, 3425, 1166, 7531, 8642.
20. 46, 72, 88, 44, 39, 37, 93, 46, 64, 73, 47.
21. 1728, 3567, 2468, 5432, 4567, 2143, 9876, 6789.

Find the sum of the following numbers by adding the columns and then adding the results horizontally. Check by adding the rows horizontally and then adding the columns of results.

22.	7642	6241	5331	
	3124	4724	8246	
	9372	3623	2793	
				51096
23.	793	864	927	
	531	642	876	
	927	426	459	
24.	7942	8349	2275	3673
	9527	2136	3411	4212
	6524	7641	5675	7987
	3171	1234	2892	6425

25.	26	72	126	467	354	987	54	86
	13	34	45	56	67	67	87	43
	98	87	765	453	342	465	783	5
	21	5	43	350	9	11	321	24
	8	25	196	961	649	378	452	36
	77	66	555	444	888	999	111	222

Exercises for further practice in addition can be readily supplied by the teacher. The student should be drilled till he can add accurately and rapidly. Accuracy, however, should never be sacrificed to attain rapidity.

Expert accountants, by systems of grouping and much practice, acquire facility in adding two or even three columns of figures at a time. Elaborate calculating machines have also been invented, and are much used in banks and counting offices. By means of these machines, columns of numbers can be tabulated and the sum printed by simply turning a lever.

SUBTRACTION

22. In **subtraction** it is important that the student should be able to see at once what number added to the smaller of two numbers of one figure each will produce the larger. Thus, if the difference between 5 and 9 is desired, the student should at once think of 4, the number which added to 5 produces 9.

23. Again, if the second number is the smaller, as in 7 from 5, the student should think of 8, the number which added to 7 produces 15, the next number greater than 7 which ends in 5.

24. The complete process of subtraction is shown in the following exercise:

8534	7 and 7 are 14, carry 1. **(Why carry 1?)**
5627	3 and 0 are 3.
2907	6 and 9 are 15, carry 1.
	6 and 2 are 8.

25. The student should think "What number added to 5627 will produce 8534?" After a little practice, it is unnecessary to say more than 7 and 7, 3 and 0, 6 and 9, 6 and 2, writing down the underscored digit just as it is named.

26. Check. To check, add the remainder and the subtrahend upward, since in working the exercise the numbers were added downward.

27. The above method of subtraction is important not only because it can be performed rapidly, but because it is very useful in long division. It is also the method of "making change" used in stores.

28. There are two other methods of subtraction in common use. The processes are shown in the following exercises:

$$\text{(1)}\quad \begin{array}{l} 643 = 600 + 40 + 3 = 500 + 130 + 13 \\ \underline{456 = 400 + 50 + 6 = 400 + \ \ 50 + \ \ 6} \\ 187 = \qquad\qquad\qquad\quad 100 + \ \ 80 + \ \ 7 \end{array}$$

6 from 13, 7; 5 from 13, 8; 4 from 5, 1.

$$\text{(2)}\quad \begin{array}{l} 643 = 600 + 40 + 3,\ 600 + 140 + 13 \\ \underline{456 = 400 + 50 + 6,\ 500 + \ \ 60 + \ \ 6} \\ 187 = \qquad\qquad\qquad 100 + \ \ 80 + \ \ 7 \end{array}$$

6 from 13, 7; 6 from 14, 8; 5 from 6, 1.

EXERCISE 4

1. Define the terms *subtrahend*, *minuend*, *difference*.

2. How should the terms be arranged in subtraction? Where do we begin to subtract? Why?

3. Is the difference affected by adding the same number to both subtrahend and minuend? Is this principle used in either (1) or (2)?

4. If a digit in the minuend is less than a digit of the corresponding order in the subtrahend, explain how the subtraction is performed in both (1) and (2).

29. Arithmetical Complement. The arithmetical complement of a number is the difference between the number and the next higher power of 10. Thus, the arithmetical complement of 642 is 358, since $358 + 642 = 1000$. The arithmetical complement of 0.34 is 0.66, since $0.66 + 0.34 = 1$.

EXERCISE 5

1. Name rapidly the complements of the following numbers: 75, 64, 32, 12, 90, 33, 25, 0.25, 0.16, 125, 500, 5000, 1250, 625.

2. Name the amount of change a clerk must return if he receives a five-dollar bill in payment of each of the following amounts: $1.25, $3.75, $2.34, $3.67, $0.25, $0.88, $4.91, $1.85.

3. Name the amount of change returned if the clerk receives a ten-dollar bill in payment of each of the following amounts: $7.34, $3.42, $9.67, $5.25, $2.67, $6.45, $4.87, $0.68, $3.34.

Determine in each of the following exercises what number added to the smaller number will produce the larger. The student will notice that in some cases the subtrahend is placed over the minuend. It is often convenient in business to perform work in this way.

4.	5.	6.	7.	8.	9.
9	36	75	246	8937	5280
5	42	31	167	9325	3455

10.	11.	12.	13.
7621	2339	9654327	4680215
6042	5267	6098715	9753142

14. Show that to subtract 73854 from 100000 it is necessary only to take 4 from 10 and each of the remaining digits from 9.

15. Subtract 76495 from 100000, and 397.82 from 1000, as in Ex. 14.

16. Show that to subtract 3642 from 5623 is the same as to add the arithmetical complement of 3642 and subtract 10000 from the sum.

17. From 8757 take the sum of 1236, 2273 and 3346.

8757	
1236	6, 9, 15 and 2; 17.
2273	5, 12, 15 and 0; 15.
3346	4, 6, 8 and 9; 17.
	4, 6, 7 and 1; 8.
1902	

18. From 53479 take the sum of 23, 1876 and 41253.

19. From 7654 take the sum of 3121, 126 and 2349.

20. From 764295 take the sum of 45635, 67843, 125960 and 213075.

21. A clerk receives a twenty-dollar bill in payment of the following items: $2.25, $11.50, $0.13, $0.75. How much change does he return?

22. Find the value of $2674 + 1782 - 1236 + 8420 - 4536$ by adding the proper arithmetical complements and subtracting the proper powers of ten.

30. To find the balance of an account.

Dr. FIRST NATIONAL BANK, YPSILANTI, *in acct. with* JOHN SMITH *Cr.*

1904					1904				
Aug. 3	Balance	1	486	87	Aug. 4	By check		500	00
Aug. 22	To deposit		290	00	Aug. 10	By check		57	30
Sept. 30	To deposit		198	75	Sept. 1	By check		235	75
Oct. 24	To deposit		773	40	Sept. 21	By check		11	80
Nov. 20	To deposit		110		Oct. 15	By check		97	30
					Nov. 3	By check	1	000	00
					Nov. 25	Balance		956	87
		2	859	02			2	859	02
Nov. 25	Balance		956	87					

The preceding form represents the account of John Smith with the First National Bank from Aug. 3 till Nov. 25. The items at the left of the central dividing line are the amounts that the bank owes Mr. Smith. This side is called the **debit** side of the account. The items at the right represent the amounts withdrawn by Mr. Smith. This side is called the **credit** side of the account. The difference between the sums of the credits and the debits is called the **balance** of the account.

It is evident that the debit side of the above account is greater than the credit side. Therefore, to balance the account, add the debit side first, and then subtract the sum of the credit side from the result, as in Ex. 17 above. The difference will be the balance, or the amount left in the bank to the credit of Mr. Smith. The work can be checked by adding the balance to the credit column. The result should equal the sum of the debit column.

EXERCISE 6

Find the balance of each of the following accounts:

1.

Dr.		*Cr.*	
234	50	246	84
798	34	125	00
500	00	450	00
212	60	55	30
351	00	97	30
100	00	60	00
75	00		
198	30		

2.

Dr.		*Cr.*	
1250	00	527	30
888	80	131	60
210	60	927	50
1100	00	500	00
2681	50	975	25
69	00	659	75
10	46	50	00
100	00		

3.

Dr.		*Cr.*	
235	67	564	90
1000	00	75	00
750	25	34	68
104	69	100	00
566	66	1200	00
195	75	275	80
302	00	625	30
259	00		

4. On May 1 R. F. Joy had a balance of $1376.24 to his account in the bank. He deposited on May 1, $189; June 27, $166; July 28, $75; Aug. 5, $190.60; Aug. 10, $192.22. He withdrew by check the following amounts: June 1, $153; June 10, $300; July 3, $25; July 27, $575.50. What was his balance Aug. 15?

MULTIPLICATION

31. The multiplication table should be so well known that the factors will at once suggest the product. Thus, 7×6, or 6×7, should at once suggest 42.

32. The student should also be able to see at once what number added to the product of two numbers will produce a given number. Thus, the number added to 4×9 to produce 41 is $\underline{5}$, or 4×9 and $\underline{5}$ are 41.

It is a common practice in multiplication to write the multiplier first as $2 \times \$5 = \$10$. In this case the sign ($\times$) is read "times." If the multiplier is written after the multiplicand, as in $\$5 \times 2 = \$10$, the sign ($\times$) is read "multiplied by." The multiplier is always an abstract quantity (Why?), but the multiplicand may be either abstract or concrete.

33. The following examples show the complete process of multiplication:

Ex. 1. Multiply 2743 by 356.

Solution. In multiplying one number by another it is not necessary to begin with the units' digit of the multiplier. We may begin with either the units' digit or the digit of the highest order. In fact, it is frequently of decided advantage to begin with the digit of highest order, especially in multiplying decimals; but care should be taken in placing the right-hand figure of the first partial product. Since 3 hundred times 3 units = 9 hundred, the 9 must be put in the third or hundreds' place, etc.

2743	2743
356	356
16458	8229
13715	13715
8229	16458
976508	976508

Ex. 2. Multiply 3.1416 by 26.34.

Solution. In beginning the multiplication we see that $26 \times 0.0006 = 0.012$. Hence the 2 is written in the thousandths' place. The work is then completed as indicated in the annexed example. It will readily be seen that the rest follows after pointing off the first partial product correctly.

The advantage of beginning with the digit of the highest order is seen in approximations (see p. 59), where considerable work is thereby saved.

3.1416
26.34
62.832
18.8496
.94248
.125664
82.749744

34. Check. Multiplication may be checked by using the multiplicand as the multiplier and performing the multiplication again. However, the check by "casting out the nines" (p. 41), is more convenient.

EXERCISE 7

1. Define *multiplier*, *multiplicand*, *product*.

2. Explain why multiplication is but an abridged method of addition.

3. Can the multiplier ever be a concrete number? Explain.

4. How should the terms be arranged in multiplication? Does it make any difference in what order we multiply by the digits of the multiplier? Might we begin to multiply with the 5 in *Ex.* 1 and with the 6 in *Ex.* 2?

5. How is the order of the right-hand figure of each partial product determined?

6. How does the presence of a zero in the multiplier affect the work?

7. In multiplying 3.1416 by 26.34, can we tell at once how many integral places there will be in the product? Can we tell the number of decimal places?

8. How many decimal places will there be in each of the following products: 21.34×5.9? 98.65×76.43? 321.1×987.543? 1.438×42.345?

35. The following short methods are useful:

1. *To multiply any number by* 5, 25, $16\frac{2}{3}$, $33\frac{1}{3}$, 125.

Since $5 = \frac{10}{2}$, to annex a cipher and divide by 2 is the same as to multiply by 5. The student in a similar manner should explain short processes of multiplying by 25, $16\frac{2}{3}$, $33\frac{1}{3}$, 125.

2. *To multiply any number by* 9.

Since $9 = 10 - 1$, it is sufficient to annex a cipher to the number and subtract the original number.

Ex. Multiply 432 by 9.

$$\begin{array}{r} 432 \times 10 = 4320 \\ 432 \times \ \ 1 = \ \ 432 \\ \hline 432 \times \ \ 9 = 3888 \end{array}$$

3. *To multiply any number by* 11.

Since $11 = 10 + 1$, it is sufficient to annex a cipher to the number and add the original number.

Ex. Multiply 237 by 11.

$$\begin{array}{r} 237 \times 10 = 2370 \\ 237 \times \ \ 1 = \ \ 237 \\ \hline 237 \times 11 = 2607 \end{array}$$

This result can readily be obtained by writing down the right-hand figure first and then the sums of the first and second figures, the second and third, etc., and finally the left-hand figure.

4. *To multiply any number by a number differing but little from some power of* 10.

Annex as many ciphers to the number as there are ciphers in the next higher power of 10, and subtract the product of the number multiplied by the complement of the multiplier.

Ex. Multiply 335 by 996. $996 = 1000 - 4$.

$$\begin{array}{r} 335 \times 1000 = 335000 \\ 335 \times \ \ \ \ 4 = \ \ \ 1340 \\ \hline 335 \times \ \ 996 = 333660 \end{array} \qquad \text{In practice written} \quad \begin{array}{r} 335 \\ 1340 \\ \hline 333660 \end{array}$$

5. *To multiply any number by a number of two figures ending with* 1.

Multiply by the tens' figure of the multiplier, writing this product under the number one place to the left.

Ex. Multiply 245 by 71.

$$\begin{array}{r} 245 \times \ \ 1 = \ \ \ 245 \\ 245 \times 70 = 17150 \\ \hline 245 \times 71 = 17395 \end{array}$$

6. *To multiply any number by a number between twelve and twenty.*

Multiply by the units' figure of the multiplier, writing the product under the number one place to the right.

Ex. Multiply 427 by 13.

$$\begin{array}{r} 427 \times 10 = 4270 \\ 427 \times \ \ 3 = 1281 \\ \hline 427 \times 13 = 5551 \end{array}$$

7. *To square a number ending in* 5.

$35^2 = 3 \times 400 + 25$, $45^2 = 4 \times 500 + 25$, $55^2 = 5 \times 600 + 25$, etc.

8. *To multiply by a number when the multiplier contains digits which are factors of other parts of the multiplier.*

Ex. Multiply 25631 by 74221.

Since 7 is a factor of 42 and 21, multiply by 7, placing the first figure in the partial product under 7. (Why?) Then multiply this product by 6 ($42 = 6 \times 7$), placing the first figure under 2 in hundreds' place. (Why?) Then multiply the first partial product by 3 ($21 = 3 \times 7$), placing the first figure under 1. (Why?) The sum of these partial products will be the product of the numbers.

$$\begin{array}{r} 25631 \\ 74221 \\ \hline 179417 \\ 1076502 \\ 538251 \\ \hline 1902358451 \end{array}$$

EXERCISE 8

Name rapidly the products of the successive pairs of digits in each of the following numbers:

1. 75849374657.
2. 265374867598.
3. 67452367885.
4. 98765432345.

5. In each of the following groups of digits add rapidly to the product of the first two the sum of all that follow: 567, 432, 7654, 3456, 9753, 3579, 8642, 2468, 7896, 5436, 3467.

6. Multiply 1264 by 125; by $12\frac{1}{2}$; by $1\frac{1}{4}$.

7. Multiply 76.26 by $16\frac{2}{3}$; by $33\frac{1}{3}$.

8. Multiply 2348 by 25; by $2\frac{1}{2}$; by 50; by 0.5.

9. Multiply 645 by 9; by 11; by 17; by 41.

10. Multiply 8963 by 848.

11. Multiply 37439 by 4832.

12. Show that to multiply a number by 625 is the same as to multiply by 10000 and divide by 16.

13. Subtract 5×12631 from 87642.

The work should be done as follows:

	5×1 and 7, 12.
87642	5×3 and 1 and 8, 24.
12631	5×6 and 2 and 4, 36.
24487	5×2 and 3 and 4, 17.
	5×1 and 1 and 2, 8.

14. Subtract 3×2462 from 9126.

15. Subtract 6×42641 from 768345.

16. Subtract 2×86473 from 291872.

When the same number is to be used as a multiplier several times, work may be saved by forming a table of its multiples. Thus,

$5764 \times 784 =$		1	784
3136	(4)	2	1568
4704	(6)	3	2352
5488	(7)	4	3136
3920	(5)	5	3920
4518976		6	4704
		7	5488
		8	6272
		9	7056

The partial products in each case are taken from the table.

17. Use the above table and multiply 5764, 74591, 84327, 23145, each by 784.

18. Form a table of multiples of 6387, and use it to find the product of 7482, 3.1416, 742896, 342312, 67564534, 897867, 65768798, 56024.85, each by 6387.

19. Multiply 2785 by 9998, and 1728 by 997.

20. Multiply 78436 by 25×125.

21. Multiply 32.622 by 0.0125.

22. Multiply 486.72 by 0.25×0.25.

23. Multiply 320.4 by 5×1.25.

24. Multiply 5763 by $16\frac{2}{3} \times 33\frac{1}{3}$.

DIVISION

36. In **division** the student should be able to see at once how many times a given digit is contained in any number of two digits with the remainder. Thus, 7 is contained in 46, 6 times with a remainder 4. The student should think simply 6̲ and 4 over.

Ex. 6)354279
      59046 remainder 3.

The whole mental process should be 5̲ and 5, 9̲ and 0, 0̲ and 2, 4̲ and 3, 6̲ and 3.

Two interpretations arise from considering division as the inverse of multiplication.

Thus, since $4 \times \$6 = \$24$.

(1) $\$24 \div 4 = \$6$, separation into groups. $24 has been separated into 4 equal groups.

(2) $\$24 \div \$6 = 4$, involving the idea of measuring, or being *contained in*.

$6 is contained in $24, 4 times.

37. The following examples show the complete process of long division.

```
      346
4541)1571186
     13623
      20888
      18164
       27246
       27246
```

It assists in determining the order of the digits in the quotient to write them in their proper places above the dividend.

38. The work in long division may be very much abridged by omitting the partial products and writing down the remainders only. These remainders are obtained by the method used in Ex. 13, p. 26.

Ex. Divide 764.23 by 2.132.

The work will be simplified by multiplying both numbers by 1000 to avoid decimals. The first remainder, 1246, is obtained as follows:

```
       358
2132)764230     3 × 2, 6 and 6, 12.
      12463     3 × 3, 9 and 1, 10 and 4, 14.
       18030    3 × 1, 3 and 1, 4 and 2, 6.
         974    3 × 2, 6 and 1, 7.
```

Then bring down 3 and proceed as before to form the other remainders.

39. Check. Division may be checked by multiplying the quotient by the divisor, the product plus the remainder should equal the dividend. The check by "casting out the nines" (p. 42) may be used.

EXERCISE 9

1. Define *dividend, divisor, quotient, remainder.*

2. Explain the two interpretations arising from considering division as the inverse of multiplication. $5 \times \$10 = \$50$. Give the two interpretations as applied to this example.

3. How is the order of the right-hand figure in each partial product determined?

4. Explain why the sum of the partial products plus the remainder, if any, must equal the dividend if the work is correct.

5. Explain why the quotient is not affected by multiplying both dividend and divisor by the same number.

40. If the same number is used as a divisor several times, or if the dividend contains a large number of places, work may be saved by forming a table of multiples of the divisor. Thus:

Ex. Divide 786342 by 4147.

1	4147	189
2	8294	4147)786342
3	12441	37164
4	16588	39882
5	20735	2559 remainder.
6	24882	
7	29029	
8	33176	
9	37323	

EXERCISE 10

1. Divide 987262, 49789 and 314125 each by 4147.
2. Divide 896423, 76425, 9737894 each by 5280.
3. Divide 44.2778 by 63.342.

Find the value of:

4. $32.36 \div 8.9$.
5. $1.25 \div 0.5$ and $12.5 \div 0.05$.
6. $144 \div 1.2$ and $14.4 \div 12$.
7. $625 \div 25$ and $62.5 \div 2.5$.
8. $1125 \div 50$ and $11.25 \div 9.5$.
9. $5280 \div 12.5$ and $580 \div 125$.
10. $750 \div 2.5 \div 0.5$.

41. In addition to the checks on the fundamental processes given above, it is well when possible *to form the habit of estimating results before beginning the solution of a*

problem. Thus, in multiplying $19\frac{1}{2}$ by $12\frac{1}{2}$ it is evident that the result will be about $12 \times 20 = 240$.

In using this check the student should form a rough estimate of the result, then solve the problem and compare results. A large error will be at once detected.

EXERCISE 11

Solve the following, first giving approximate answers, then the correct result:

1. Multiply $15.3 \times 3\frac{48}{49}$ (about 15×4).
2. Divide 594 by $\frac{99}{302}$ (about $594 \div \frac{1}{3}$).
3. Divide 32.041 by 0.499 (about $32.041 \div \frac{1}{2}$).
4. How much will 21 horses cost at $145 each?
5. Multiply 30.421 by 20.516.
6. At $12\frac{1}{2}$ ct. a dozen, how much will $6\frac{1}{2}$ doz. eggs cost?
7. At $37\frac{1}{2}$ ct. a pound, how much will 11 lb. of coffee cost?
8. How many bushels of potatoes can be bought for $5.25 at 35 ct. a bushel?
9. At $1.12½ a barrel, how many barrels of salt can be bought for $22.50?
10. How far will a train travel in $12\frac{1}{2}$ hr. at the rate of 45 mi. an hour?
11. How much will $8\frac{3}{4}$ T. of coal cost at $7.25 a ton?
12. The net cost of printing a certain book is 49 ct. a copy. How much will an edition of 2500 cost?
13. At the rate of 40 mi. an hour, how long will it take a train to run 285 mi.?

FACTORS AND MULTIPLES

42. A **factor** or **divisor** of a number is any integral number that will exactly divide it.

43. A number that is divisible by 2 is called an **even** number, and one that is not divisible by 2 an **odd** number.

Thus, 24 and 58 are even numbers, while 17 and 83 are odd numbers.

44. A number that has no factors except itself and unity is called a **prime** number.

Thus, 1, 2, 3, 5, 7, etc., are prime numbers.

45. Write down all of the odd numbers less than 100 and greater than 3. Beginning with 3 reject every third number; beginning with 5 reject every fifth number; beginning with 7 reject every seventh number. The numbers remaining will be all of the prime numbers between 3 and 100. (Why?)

46. This method of distinguishing prime numbers is called the **Sieve of Eratosthenes**, from the name of its inventor, Eratosthenes (276–196 B.C.). He wrote the numbers on a parchment and cut out the composite numbers, thus forming a sieve.

47. A number that has other factors besides itself and unity is called a **composite** number.

48. Numbers are said to be **prime to each other** when no number greater than 1 will exactly divide each of them.

Are numbers that are prime to each other necessarily prime numbers?

49. An integral number that will exactly divide two or more numbers is called a **common divisor, or a common factor** of these numbers.

Thus, 2 and 3 are common divisors of 12 and 18.

50. The greatest common factor of two or more numbers is called the **greatest common divisor (g. c. d.)** of the numbers.

Thus, 6 is the g. c. d. of 12 and 18.

51. A **common multiple** of two or more numbers is a number that is exactly divisible by each of them.

Thus, 12, 18, 24, and 48 are common multiples of 3 and 6, while 12 is the **least common multiple** (l. c. m.) of 3 and 6.

52. It is of considerable importance in certain arithmetical operations, particularly in cancellation, to be able readily to detect small factors of numbers. In proving the tests of divisibility by such factors, the two following principles are important.

1. *A factor of a number is a factor of any of its multiples.*

Proof. Every multiple of a number contains that number an exact number of times; therefore, it contains every factor of the number.

Thus, 5 is a factor of 25, and hence of 3×25, or 75.

2. *A factor of any two numbers is a factor of the sum or difference of any two multiples of the numbers.*

Proof. Any factor of two numbers is a factor of any of their multiples by Principle 1. Therefore, as each multiple is made up of parts each equal to the given factor, their sum or difference will be made up of parts equal to the given factor, or will be a multiple of the given factor.

Thus, 3 is a factor of 12 and of 15, and hence of $5 \times 12 + 2 \times 15$, or 90. 3 is also a factor of $5 \times 12 - 2 \times 15$, or 30.

53. Tests of Divisibility. 1. *Any number is divisible by* 2 *if the number represented by its last right-hand digit is divisible by* 2.

Proof. Any number may be considered as made up of as many 10's as are represented by the number exclusive of its last digit plus the last digit. Then, since 10 is divisible by 2, the first part, which is a multiple of 10, is divisible by 2. Therefore, if the second part, or the number represented by the last digit, is divisible by 2, the whole number is.

Thus, $634 = 63 \times 10 + 4$ is divisible by 2 since 4 is.

2. *Any number is divisible by* 4 *if the number represented by the last two digits is divisible by* 4.

Proof. Any number may be considered as made up of as many 100's as are represented by the number exclusive of its last two digits plus the number represented by the last two digits. Then, since 100 is divisible by 4, the first part, which is a multiple of 100, is divisible by 4. Therefore, if the number represented by the last two digits is divisible by 4, the whole number is.

Thus, $85648 = 856 \times 100 + 48$ is divisible by 4 since 48 is.

3. *Any number is divisible by* 5 *if the last digit is* 0 *or* 5.

The proof, which is similar to the proof of 1, is left for the student.

Note. 0 is divisible by any number, and the quotient is always 0.

4. *Any number is divisible by* 8 *if the number represented by its last three digits is divisible by* 8.

The proof is left for the student.

5. *Any number is divisible by* 9 *if the sum of its digits is divisible by* 9.

Proof. Since $10 = 9 + 1$, any number of 10's = the same number of 9's + the same number of units; since $100 = 99 + 1$, any number of 100's = the same number of 99's + the same number of units; since $1000 = 999 + 1$, any number of 1000's = the same number of 999's + the same number of units; etc. Therefore, any number is made up of a multiple of 9 + the sum of its digits, and hence is divisible by 9 if the sum of its digits is divisible by 9.

Thus, $7362 = 7 \times 1000 + 3 \times 100 + 6 \times 10 + 2$

$$= 7(999 + 1) + 3(99 + 1) + 6(9 + 1) + 2$$

$$= 7 \times 999 + 3 \times 99 + 6 \times 9 + 7 + 3 + 6 + 2$$

= a multiple of 9 + the sum of the digits.

Therefore, the number is divisible by 9 since $7 + 3 + 6 + 2 = 18$ is divisible by 9.

6. *Any number is divisible by* 3 *if the sum of its digits is divisible by* 3.

The proof, which is similar to the proof of Principle 5, is left for the student.

7. *Any even number is divisible by* 6 *if the sum of its digits is divisible by* 3.

The proof is left for the student.

8. *Any number is divisible by* 11 *if the difference between the sums of the odd and even orders of digits, counting from units, is divisible by* 11.

Proof. Since $10 = 11 - 1$, any number of 10's = the same number of 11's − the same number of units; since $100 = 99 + 1$, any number of 100's = the same number of 99's + the same number of units; since $1000 = 1001 - 1$, any number of 1000's = the same number of 1001's − the same number of units; etc. Therefore, any number is made up of a multiple of 11 + the sum of the digits of odd order − the sum of the digits of even order, and hence is divisible by 11 if the sum of the digits of odd order − the sum of the digits of even order is divisible by 11.

Thus, $753346 = 7\times100000+5\times10000+3\times1000+3\times100+4\times10+6$

$$=7(100001-1)+5(9999+1)+3(1001-1)+3(99+1)+4(11-1)+6$$

$$=7\times100001+5\times9999+3\times1001+3\times99+4\times11-7+5-3+3-4+6$$

= a multiple of 11 + the sum of the digits of odd order − the sum of the digits of even order.

Therefore, the number is divisible by 11 since $5+3+6-(7+3+4)=0$ is divisible by 11.

9. *The test for divisibility by 7 is too complicated to be useful.*

EXERCISE 12

1. Write three numbers of at least four figures each that are divisible by 4.

2. Write three numbers of six figures each that are divisible by 9.

3. Is 352362257 divisible by 11? by 3?

4. Without actual division, determine what numbers less than 19 (except 7, 13, 14, 17) will divide 586080.

5. Explain short methods of division by 5, 25, $16\frac{2}{3}$, $33\frac{1}{3}$, 125.

6. Divide 3710 by 5; by 25; by 125; by $12\frac{1}{2}$.

7. Divide 2530 by 0.5; by 0.025; by 1.25.

8. Prove that to divide by 625 is the same as to multiply by 16 and divide by 10000.

9. State and prove a test for divisibility by 12; by 15; by 18.

10. If 7647 is divided by 2 or 5, how will the remainder differ from the remainder arising from dividing 7 by 2 or 5? Explain.

11. If 26727 is divided by 4 or 25, how will the remainder differ from the remainder arising from dividing 27 by 4 or 25? Explain.

12. Explain how you can find the remainder arising from dividing 26727 by 8 or 125 in the shortest possible way.

54. Relative Weight of Symbols of Operation. In the use of the symbols of operation (+, −, ×, ÷), it is important that the student should know that the numbers connected by the signs × and ÷ must first be operated upon and then those connected by + and −; for the signs of multiplication and division connect factors, while the signs of addition and subtraction connect terms. Factors must be combined into simple terms before the terms can be added or subtracted.

Thus, $5+2\times3-15\div5+4=12$, the terms 2×3 and $15\div5$ being simplified before they are combined by addition and subtraction.

55. The ancients had no convenient symbols of operation. Addition was generally indicated by placing the numbers to be added adjacent to each other. Other operations were written out in words. The symbols + and − were probably first used by Widman in his arithmetic published in Leipzig in 1489. He used them to mark excess or deficiency, but they soon came into use as symbols of operation. × as a symbol of multiplication was used by Oughtred in 1631. The dot (·) for multiplication was used by Harriot in 1631. The Arabs indicated division in the form of a fraction quite early. ÷ as a symbol of division was used by Rahn in his algebra in 1659. Robert Recorde introduced the symbol = for equality in 1557. : was used to indicate division by Leibnitz and Clairaut. In 1631 Harriot used > and < for greater than and less than. Rudolff used √ to denote square root in 1526.

56. Greatest Common Divisor. In many cases the g. c. d. of two or more numbers may readily be found by factoring, as in the following example:

Ex. Find the g. c. d. of 3795, 7095, 30030.

$$3795 = 3 \times 5 \times 11 \times 23,$$
$$7095 = 3 \times 5 \times 11 \times 43,$$
$$30030 = 2 \times 3 \times 5 \times 11 \times 91,$$

and since the g. c. d. is the product of all of the prime factors that are common to the three numbers, it is $3 \times 5 \times 11 = 165$.

57. Euclid, a famous Greek geometer, who lived about 300 B.C., gave the method of finding the g. c. d. by division. This method is useful if the prime factors of the numbers cannot be readily found.

Ex. Find the g. c. d. of 377 and 1479.

		1479
377	3	1131
348	1	348
29	12	348

The g. c. d. cannot be greater than 377, and since 377 is not a factor of 1479, it is not the g. c. d. of the two numbers.

Divide 1479 by 377. Then, since the g. c. d. is a common factor of 377 and 1479, it is a factor of $1479 - 3 \times 377$, or 348 (Principle 2, p. 33).

Therefore, the g. c. d. is not greater than 348. If 348 is a factor of 377 and 1479, it is the g. c. d. sought.

But 348 is not a factor of 377. Therefore, it is not the g. c. d. sought.

Divide 377 by 348. Then, since the g. c. d. is a factor of 377 and 348, it is a factor of $377 - 348$, or 29 (Principle 2, p. 33).

Therefore, the g. c. d. is not greater than 29, and if 29 is a factor of 348, 377, and 1479, it is the g. c. d. sought. (Why?)

29 is a factor of 348. Therefore, it is a factor of 377 and of 1479. (Why?)

Therefore, 29 is the g. c. d. sought.

58. Least Common Multiple. In many cases the l. c. m. of two or more numbers may readily be found by factoring, as in the following example.

Ex. Find the l. c. m. of 414, 408, 3330.

$$414 = 2 \times 3 \times 3 \times 23,$$
$$408 = 2 \times 2 \times 2 \times 3 \times 17,$$
$$3330 = 2 \times 3 \times 3 \times 5 \times 37.$$

The l. c. m. must contain all of the prime factors of 414, 408, 3330, and each factor must occur as often in the l. c. m. as in any one of the numbers. Thus, 3 must occur twice in the l. c. m., 2 must occur three times, and 23, 17, 5, 37 must each occur once.

Therefore, the l. c. m. $= 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 23 \times 17 \times 37 = 5208120$.

59. When the numbers cannot readily be factored, the g. c. d. may be used in finding the l. c. m.

Since the g. c. d. of two numbers contains all of the factors that are common to the numbers, if the numbers are divided by the g. c. d., the quotients will contain all the factors that are not common. The l. c. m. is therefore the product of the quotients and the g. c. d. of the numbers.

Ex. Find the l. c. m. of 14482 and 32721.

The g. c. d. of 14482 and 32721 is 13.
$14482 \div 13 = 1114$. ∴ the l. c. m. of the two numbers is

$$1114 \times 32721 = 36451194.$$

EXERCISE 13

1. Find the l. c. m. and g. c. d. of 384, 2112, 2496.
2. Find the l. c. m. of 3, 5, 9, 12, 14, 16, 96, 128.
3. Find the g. c. d. and l. c. m. of 1836, 1482, 1938, 8398, 11704, 101080, 138945.
4. Prove that the product of the g. c. d. and l. c. m. of two numbers is equal to the product of the numbers.
5. What is the length of the longest tape measure that can be used to measure exactly two distances of 2916 ft. and 3582 ft. respectively?
6. Find the number of miles in the radius of the earth, having given that it is the least number that is divisible by 2, 3, 4, 5, 6, 8, 9, 10, 11, 12.

CASTING OUT NINES

60. The check on arithmetical operations by casting out the nines was used by the Arabs. It is a very useful check, but fails to detect such errors as the addition of 9, the interchange of digits, and all errors not affecting the sum of the digits. (Why?)

The remainder arising from dividing any number by 9 is the same as that arising from dividing the sum of its digits by 9.

Thus, the remainder arising by dividing 75234 by 9 is 3, the same as arises by dividing $7+5+2+3+4$ by 9.

The student should adapt the proof of Principle 5, p. 34, to this statement.

61. The most convenient method is to add the digits, dropping or "casting out" the 9 as often as the sum amounts to that number.

Thus, to determine the remainder arising from dividing 645738 by 9, say 10 (reject 9), 1, 6, 13 (reject 9), 4, 7, 15 (reject 9), 6. Therefore, 6 is the remainder. After a little practice the student will easily group the 9's. In the above, 6 and 3, 4 and 5, could be dropped, and the excess in 7 and 8 is seen to be 6 at once.

62. Check on Addition by casting out the 9's.

Ex. Add 56342, 64723, 57849, 23454 and check the work by casting out the 9's.

Since each number is a multiple of 9 plus some remainder, the numbers can be written as indicated in the annexed solution.

$$\begin{aligned}
56342 &= 9 \times 6260 + 2 \text{ rem.}\\
64723 &= 9 \times 7191 + 4 \text{ rem.}\\
57849 &= 9 \times 6427 + 6 \text{ rem.}\\
23454 &= 9 \times 2606 + 0 \text{ rem.}\\
\hline
202368 &= 9 \times 22484 + 12 \text{ rem.}
\end{aligned}$$

But $12 = 9 + 3$.

$$\therefore 202368 = 9 \times 22484 + 9 + 3$$
$$= 9 \times 22485 + 3.$$

Thus, the excess of 9's is 3 and the excess in the sum of the excesses, 2, 4, 6, and 0, is 3, therefore the work is probably correct.

63. The proof may be made general by writing the numbers in the form $9x + r$. This can be done since all numbers are multiples of 9 plus a remainder. Hence, by expressing the numbers in this form and adding we have for the sum a multiple of 9 plus the sum of the remainders. Therefore, *the excess of the 9's in the sum is equal to the excess in the sum of the excesses.*

$$\begin{aligned}
&9x + r\\
&9x' + r'\\
&9x'' + r''\\
&\text{——}\\
&\text{——}\\
\hline
&9(x + x' + x'' + \cdots)\\
&+ r + r' + r'' + \cdots)
\end{aligned}$$

64. Check on Multiplication by casting out the 9's.

Since any two numbers may be written in the form $9x + r$ and $9x' + r'$, multiplying $9x + r$ by $9x' + r'$, we have $81xx' + 9(x'r + xr') + rr'$. From this it is evident that the excess of 9's in the product arises from the excess in rr'. Therefore, *the excess of 9's in any product is equal to the excess in the product found by multiplying the excesses of the factors together.*

$$\begin{aligned}
&9x + r\\
&\underline{9x' + r'}\\
&9xr' + rr'\\
&\underline{81xx' + 9x'r}\\
&81xx' + 9(x'r + xr') + rr'
\end{aligned}$$

Ex. Multiply 3764 by 456 and check by casting out the 9's.

$$3764 \times 456 = 1716384.$$

The excess of 9's in 3764 is 2; the excess in 456 is 6; the excess in the product of the excesses is 3 ($2 \times 6 = 12$; $12 - 9 = 3$); the excess in 1716384, the product of the numbers, is 3. Therefore, the work is probably correct.

65. Check on Division by casting out the 9's.

Division being the inverse of multiplication, the dividend is equal to the product of the divisor and quotient plus the remainder. Therefore, *the excess of 9's in the dividend is equal to the excess of 9's in the remainder plus the excess in the product found by multiplying the excess of 9's in the divisor by the excess of 9's in the quotient.*

Ex. Divide 74563 by 428 and check by casting out the 9's.

$$74563 \div 428 = 174 + \tfrac{91}{428},$$

or

$$74563 = 174 \times 428 + 91$$

The excess of 9's in 74563 is 7; in 174, 3; in 428, 5; in 91, 1. Since 7, the excess of 9's in 74563 = the excess in 3 × 5 + 1, or 16, which is the product of the excesses in 174 and 428 plus the excess in 91, the work is probably correct.

EXERCISE 14

1. State and prove the check on subtraction by casting out the 9's.

2. Determine without adding whether 89770 is the sum of 37634 and 52146.

3. Add 74632, 41236, 897321 and 124762, and check by casting out the 9's.

4. Multiply 76428 by 5937, and check by casting out the 9's.

5. Determine without multiplying whether 2718895 is the product of 3785 and 721.

6. Show by casting out 9's that 18149 divided by $56 = 324\frac{5}{56}$.

7. Show that results may also be checked by casting out 3's; by casting out 11's.

8. Is 734657 divisible by 9? by 3? by 11?

9. Perform the following operations and check: 91728 × 762; 849631 ÷ 2463; 17 × 3.1416; 78.54 ÷ 3.1416.

10. Does the proof for casting out the 9's hold as well for 4, 6, 8, etc.? May we check by casting out the 8's? Explain.

MISCELLANEOUS EXERCISE 15

1. What is the principle by which the ten symbols, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, are used to represent any number?

2. Why is the value of a number unaltered by annexing zeros to the right of a decimal?

3. How is the value of each of the digits in the number 326 affected by annexing a number, as 4, to the right of it? to the left of it?

4. How is the value of each of the digits of 7642 affected if 5 is inserted between 6 and 4?

5. Write 4 numbers of 4 places each that are divisible by (*a*) 4, (*b*) 2 and 5, (*c*) 6, (*d*) 8, (*e*) 9, (*f*) 11, (*g*) 16, (*h*) 12, (*i*) 15, (*j*) 18, (*k*) 3, (*l*) 50, (*m*) 125, (*n*), both 6 and 9, (*o*) both 8 and 3, (*p*) both 30 and 20.

6. Determine the prime factors of the following numbers: (*a*) 3426, (*b*) 8912, (*c*) 6600, (*d*) 6534, (*e*) 136125, (*f*) 330330, (*g*) 570240.

7. Mr. Long's cash balance in the bank on Feb. 20 is $765.75. He deposits, Feb. 21, $150; Feb. 25, $350.25; Feb. 26, $97.50; and withdraws, Feb. 23, $200; Feb. 24, $123.40 and $112.50; Feb. 28, $321.75. What is his balance March 1?

8. Form a table of multiples of the multiplier and multiply 7642, 93856, 24245, 6420246, each by 463.

9. Form a table of multiples of the divisor and use it in dividing 86420, 97531, 876123, 64208, each by 765.

10. Use a short method to multiply 8426 by $16\frac{2}{3}$; by $33\frac{1}{3}$; by 945; by 432.

11. Find, without dividing, the remainder when 374265 is divided by 3; by 9; by 11.

12. Evaluate $45 + 32 \times 25 - 800 \div 125 + 180 \times 33\frac{1}{3}$.

13. Determine by casting out the 9's whether the following are correct: (*a*) $786 \times 648 = 509328$; (*b*) $24486 \div 192 = 127 + 102$ rem.; (*c*) $415372 \div 267 = 1555 + 187$ rem.; (*d*) $16734 \times 3081 = 52557454$.

14. Perform each of the operations indicated in Ex. 13.

15. Subtract from 784236 the sum of 7834, 5286, 23462 and 345679.

16. What are the arithmetical complements of 12000, 1728, 3.429, 86, 0.1, 125?

17. Light travels at the rate of 186000 mi. per second. Find the distance of the sun from the earth if it takes a ray of light from the sun 8 min. 2 sec. to reach the earth.

18. A cannon is 2 mi. distant from an observer. How long after it is fired does it take the sound to reach the observer if sound travels 1090 ft. per second?

19. Replace the zeros in the number 760530091 by digits so that the number will be divisible by both 9 and 11.

20. Show that every even number may be written in the form $2n$ and every odd number in the form $2n+1$ where n represents any integer.

21. Show that the product of two consecutive numbers must be even and the sum odd.

22. Show that all numbers under and including 15 are factors of 360360.

23. Find, without dividing, the remainder after 364257 has been divided by 3; by 9; by 11.

24. Evaluate $10 + 144 \times 25 - 2130 \div 15 + 5 \times 3$.

25. Write 4 numbers of 5 places each that are divisible by both 9 and 11.

26. Write 4 numbers of 6 places each that are divisible by both 3 and 6.

27. Write 4 numbers of 4 places each that are divisible by 4, 5, 6.

28. Evaluate $47 \times 68 + 68 \times 53$.

29. Evaluate $346 \times 396.84 - 146 \times 396.84$.

30. Evaluate $27 \times 3.1416 - 41 \times 3.1416 + 49 \times 3.1416 + 65 \times 3.1416$.

31. If lemons are 20 ct. a dozen and oranges are 25 ct., how many oranges are worth as much as $12\frac{1}{2}$ doz. lemons?

32. A farmer received 6 lb. of coffee in exchange for 9 doz. eggs at $12\frac{1}{2}$ ct. a dozen. How much was the coffee worth per pound?

33. Two piles of the same kind of shot weigh respectively 1081 lb. and 598 lb. What is the greatest possible weight of each shot?

FRACTIONS

66. Historically the **fraction** is very old. A manuscript on arithmetic, entitled "Directions for obtaining a Knowledge of All Dark Things," written by Ahmes, an Egyptian priest, about 1700 B.C., begins with fractions. In this manuscript all fractions are reduced to fractions with unity as the numerator. Thus, the first exercise is $\frac{2}{9} = \frac{1}{6} + \frac{1}{18}$.

67. While the Egyptians reduced all fractions to those with constant numerators, the Babylonians used them with a constant denominator of 60. Only the numerator was written, with a special mark to denote the denominator. This method of writing fractions lacked only the symbol for zero and the substitution of the base 10 for 60 to become the modern decimal fraction. Sexagesimal fractions are still used in the measurement of angles and time.

68. The Romans used duodecimal fractions exclusively. They had special names and symbols for $\frac{1}{12}, \frac{2}{12}, \cdots, \frac{11}{12}, \frac{1}{24}, \frac{1}{48}$, etc. To the Romans, fractions were concrete things. They never advanced beyond expressing them in terms of **uncia** ($\frac{1}{12}$), **silicus** ($\frac{1}{4}$ uncia), **scrupulum** ($\frac{1}{24}$ uncia), etc., all subdivisions of the **as**, a copper coin weighing one pound.

69. The sexagesimal and duodecimal fractions prepared the way for the decimal fraction, which appeared in the latter part of the sixteenth century. In 1585 Simon Stevin of Bruges published a work in which he used the notations 7 4′ 6″ 5‴ 9⁗, or 7⓪ 4① 6② 5③ 9④ for 7.4659. During the early stage of its development the decimal fraction was written in various other forms, among which are found the following: $7\ \overset{\text{I}}{4}\ \overset{\text{II}}{6}\ \overset{\text{III}}{5}\ \overset{\text{IV}}{9}$, $7\ \overset{1}{4}\ \overset{2}{6}\ \overset{3}{5}\ \overset{4}{9}$, 7|4659, 7|$\underline{4659}$, 7.4659. The decimal point was first used, in 1612, by Pitiscus in his trigonometrical tables; but the decimal fraction was not generally used before the beginning of the eighteenth century.

70. The **primary conception** of a fraction is *one or several of the equal parts of a unit.* Thus, the fraction $\frac{4}{5}$ indicates that 4 of the 5 equal parts of a unit are taken.

71. The term *naming* the number of parts into which the unit is divided is called the **denominator.** The term *numbering* the parts is called the **numerator.**

72. A **proper fraction** is less than unity; an **improper fraction** is equal to or greater than unity.

73. A number consisting of an integer and a fraction is called a **mixed number.**

74. Our conception of a fraction must, however, be enlarged as we proceed, and be made to include such expressions as $\frac{2.5}{3.25}$, $\frac{3}{-7}$, 3.14159 $\cdots$, $\frac{\frac{3}{4}}{\frac{5}{7}}$, etc. The more **general conception** of a fraction is that it is *an indicated operation in division* where the numerator represents the dividend and the denominator the divisor.

75. Decimal fractions are included in the above definition, as 0.5 means $\frac{5}{10}$ or $\frac{1}{2}$. Using the decimal point (0.5) is simply another way of writing $\frac{5}{10}$.

76. General Principles. *Multiplying the numerator or dividing the denominator of a fraction by a number multiplies the fraction by that number.*

Let $\frac{5}{7}$ be any fraction where 7 represents the number of parts into which unity is divided, and 5 the number of these parts taken.

(1) $\frac{3 \times 5}{7} = 3 \times \frac{5}{7}$, since there are three times as many of the 7 parts of unity as before.

(2) $\frac{5}{7 \div 3} = 3 \times \frac{5}{7}$, since dividing the denominator by 3 divides by 3 the number of equal parts into which unity is divided and therefore the fraction is 3 times as large as before.

77. *Dividing the numerator or multiplying the denominator of a fraction by a number divides the fraction by that number.*

(1) $\frac{7 \div 4}{9} = \frac{7}{9} \div 4$, since dividing the numerator by 4 divides by 4 the number of parts taken without changing the value of the parts.

(2) $\frac{7}{9 \times 4} = \frac{7}{9} \div 4$, since multiplying the denominator by 4 multiplies by 4 the number of parts into which unity is divided and therefore the fraction is $\frac{1}{4}$ as large as before.

78. *Multiplying or dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction.*

(1) $\frac{5 \times 2}{5 \times 3} = \frac{2}{3}$. Multiplying both numerator and denominator by 5 both multiplies and divides the value of the fraction by 5. The value of the fraction therefore remains unchanged.

(2) $\frac{2 \div 5}{3 \div 5} = \frac{2}{3}$. Dividing both numerator and denominator by 5 both divides and multiplies the value of the fraction by 5. The value of the fraction therefore remains unchanged.

79. *A mixed number may be reduced to an improper fraction and an improper fraction may be reduced to a mixed number or an integer.*

Thus, $5\frac{3}{4} = \frac{5 \times 4 + 3}{4}$. Since $5 \times 4 =$ the number of 4ths in 5 and $5 \times 4 + 3 =$ the number of 4ths in $5\frac{3}{4}$, $\therefore\ 5\frac{3}{4} = \frac{23}{4}$.

Reversing the process,

$$\frac{23}{4} = 23 \div 4 = 5 + \frac{3}{4} = 5\frac{3}{4}.$$

80. *When the numerator and denominator of a fraction are prime to each other, the fraction is said to be in its lowest terms.*

Ex. Express $\frac{42}{70}$ in its lowest terms.

$$\frac{42}{70} = \frac{3 \times 14}{5 \times 14} = \frac{3}{5}.$$

81. *Two or more fractions may be reduced to equivalent fractions having a common denominator.*

Ex. 1. Reduce $\frac{3}{4}$, $\frac{5}{9}$, $\frac{1}{12}$, to equivalent fractions having a common denominator.

The l. c. m. of 4, 9, 12, is 36.

$$\frac{3}{4} = \frac{3 \times 9}{4 \times 9} = \frac{27}{36}.$$

$$\frac{5}{9} = \frac{5 \times 4}{9 \times 4} = \frac{20}{36}.$$

$$\frac{1}{12} = \frac{1 \times 3}{12 \times 3} = \frac{3}{36}.$$

$\therefore\ \frac{27}{36}, \frac{20}{36}, \frac{3}{36}$ are fractions having a common denominator, equivalent to $\frac{3}{4}, \frac{5}{9}, \frac{1}{12}$. Since 36 is the l. c. m. of 4, 9, 12, it is called the **least common denominator.**

82. Sometimes, instead of finding the l.c.m., it is more convenient to take as the common denominator the product of all the denominators and multiply each numerator by the product of all the denominators except its own.

Ex. 2. Reduce $\frac{3}{4}$, $\frac{1}{6}$ and $\frac{2}{3}$ to fractions having a common denominator.

$$\frac{3}{4} = \frac{3 \times 6 \times 3}{4 \times 6 \times 3} = \frac{54}{72}.$$

$$\frac{1}{6} = \frac{1 \times 4 \times 3}{4 \times 6 \times 3} = \frac{12}{72}.$$

$$\frac{2}{3} = \frac{2 \times 4 \times 6}{4 \times 6 \times 3} = \frac{48}{72}.$$

Since the common denominator is $4 \times 6 \times 3$, 4 is contained in it 6×3 times and the first numerator will be $3 \times 6 \times 3$, 6 is contained in the common denominator 4×3 times and the second numerator will be $1 \times 4 \times 3$, 3 is contained in the common denominator 4×6 times and the third numerator will be $2 \times 4 \times 6$.

83. Addition and Subtraction of Fractions. Since only the same kinds of units, or the same parts of units, can be added to or subtracted from one another, it is necessary to reduce fractions to a common denominator before performing the operations of addition or subtraction.

Ex. 1. Add $\frac{5}{12}$, $\frac{7}{36}$ and $\frac{1}{84}$.

The l. c. m. of 12, 36 and 84 is 252.

$$\frac{5}{12}=\frac{5\times 21}{12\times 21}=\frac{105}{252},\quad \frac{7}{36}=\frac{7\times 7}{36\times 7}=\frac{49}{252},\quad \frac{1}{84}=\frac{1\times 3}{84\times 3}=\frac{3}{252}.$$

$$\frac{5}{12}+\frac{7}{36}+\frac{1}{84}=\frac{105}{252}+\frac{49}{252}+\frac{3}{252}=\frac{157}{252}.$$

Ex. 2. Add $2\frac{3}{4}$, $1\frac{5}{9}$ and $3\frac{13}{48}$.

$$2\tfrac{3}{4}+1\tfrac{5}{9}+3\tfrac{13}{48}=2+1+3+\tfrac{3}{4}+\tfrac{5}{9}+\tfrac{13}{48}=6+\tfrac{108}{144}+\tfrac{80}{144}+\tfrac{39}{144}=7\tfrac{83}{144}.$$

After a little practice the student should be able to abbreviate the work very much. *Ex.* 1 might be worked briefly, thus:

$$\frac{5}{12}+\frac{7}{36}+\frac{1}{84}=\frac{105+49+3}{252}=\frac{157}{252}.$$

Ex. 2, thus:

$$2\tfrac{3}{4}+1\tfrac{5}{9}+3\tfrac{13}{48}=6+\frac{108+80+39}{144}=7\tfrac{83}{144}.$$

Ex. 3. From $22\frac{5}{42}$ subtract $18\frac{13}{42}$.

$$22\tfrac{5}{42}=21\tfrac{47}{42}.$$

$$21\tfrac{47}{42}-18\tfrac{13}{42}=3\tfrac{34}{42}=3\tfrac{17}{21}.$$

84. Multiplication of Fractions. *The product of two numbers may be found by performing the same operation on one of them as is performed on unity to produce the other.*

Thus, in $3\times 4=12$, unity is taken three times to produce the multiplier 3, hence 4 is taken three times to produce the product 12. Again, in $\frac{2}{3}\times\frac{5}{7}=\frac{10}{21}$, unity is divided into 3 parts and 2 of them are

taken to produce the multiplier $\frac{2}{3}$, hence, $\frac{5}{7}$ is divided into 3 parts, each of which is $\frac{5}{3\times 7}$ (Why?), and 2 of them are taken to produce the product $\frac{2\times 5}{3\times 7}=\frac{10}{21}$.

When the multiplier is a common fraction, the sign ($\times$) should be read "of." Thus, $\frac{2}{3}\times \$5$ means $\frac{2}{3}$ of \$5.

Ex. 1. Multiply $\frac{23}{42}$ by $\frac{21}{69}$.

$$\frac{23}{42}\times\frac{21}{69}=\frac{\overset{1}{\cancel{23}}\times\overset{1}{\cancel{21}}}{\underset{2}{\cancel{42}}\times\underset{3}{\cancel{69}}}=\frac{1}{6}.$$

85. *The student should use cancellation whenever possible. He should never multiply or divide until all possible factors have been removed by cancellation.*

NOTE. Although a knowledge of the principles of multiplication and division of decimals has been assumed in examples given before, it is well to review these principles at this point to make sure that they are thoroughly understood.

Ex. 2. Multiply 0.234 by 0.16.

Solution.
$$0.234\times 0.16=\frac{234}{1000}\times\frac{16}{100}=\frac{234\times 16}{1000\times 100}=\frac{3744}{100000}=0.03744.$$

```
  0.234
  0.16
   234
  1404
0.03744
```

The number of decimal places in the product is the same as the number of zeros in the denominator of the product, that is, it equals the number of decimal places in the multiplicand plus the number in the multiplier. The decimal point in (0.03744) simply provides a convenient way of writing $\frac{3744}{100000}$. It is better, however, to determine the position of the decimal point before beginning the multiplication. This can be done by considering only the last figure at the right of the multiplier and multiplicand. Thus, we see that $0.004\times 0.06=0.00024$. Hence, the order of the product will be hundred thousandths

86. The following simple truths, or **axioms**, are frequently used in arithmetic.

(1) *Numbers that are equal to the same number are equal to each other.*

Thus, if $x = 5$ and $y = 5$, then $x = y$.

(2) *If equals are added to equals, the sums are equal.*

Thus, if $x = 5$, $x + 3 = 5 + 3$.

(3) *If equals are subtracted from equals, the remainders are equal.*

Thus, if $x = 4$, then $x - 2 = 4 - 2$.

(4) *If equals are multiplied by equals, the products are equal.*

Thus, if $\frac{x}{2} = 3$, then $x = 6$.

(5) *If equals are divided by equals, the quotients are equal.*

Thus, if $3x = 6$, then $x = 2$.

87. Division of Fractions. Division may be regarded as the inverse of multiplication. The problem is, therefore, to find one of two factors when the product and the other factor are given.

Thus, $3 \times 4 = 12$, $\therefore 12 \div 3 = 4$, and $12 \div 4 = 3$. Axiom 5.

Again, $\frac{5}{6} \times \frac{7}{8} = \frac{35}{48}$, $\therefore \frac{35}{48} \div \frac{5}{6} = \frac{7}{8}$, and $\frac{35}{48} \div \frac{7}{8} = \frac{5}{6}$.

To divide one fraction by another.

Solution. Let $\frac{35}{48} \div \frac{5}{6} = q$ (a quotient).

Then $\frac{35}{48} = \frac{5}{6} \times q$ (multiplying both members of the equation by $\frac{5}{6}$), and $\frac{35}{48} \times \frac{6}{5} = q$ (multiplying both members of the equation by $\frac{6}{5}$).

$\therefore$ *the quotient is obtained by multiplying the dividend by the reciprocal of the divisor.*

If a number of factors are connected by the signs $\times$ and $\div$, the operations are to be performed from left to right.

Thus, $8 \times 3 \div 4 \times 2 \div 6 = 2$.

The operation indicated by the word "of" following a fraction is to be performed before the operations indicated by $\times$ and $\div$.

Thus, $\frac{1}{2} + 4 \times \frac{1}{3}$ of $6 \div \frac{2}{5} - \frac{4}{9} \div \frac{3}{4}$ of $\frac{2}{3} = 19\frac{11}{18}$.

Observe that the operations $\frac{1}{3}$ of 6 and $\frac{3}{4}$ of $\frac{2}{3}$ are performed first, followed by multiplication and division from left to right, and finally by addition and subtraction.

88. A fraction of the form $\frac{\frac{3}{4}}{\frac{5}{6}}$ is called a **complex fraction** and may be considered as equivalent to $\frac{3}{4} \div \frac{5}{6}$ and treated as a problem in division. In general, however, a complex fraction may be more readily simplified by multiplying both terms by the l. c. m. of the denominators of the two fractions in the numerator and denominator.

Ex. **1.** $\frac{\frac{5}{9}}{\frac{6}{7}} = \frac{63 \times \frac{5}{9}}{63 \times \frac{6}{7}} = \frac{35}{54}$.

Ex. **2.** Divide 38.272 by 7.36.

```
        5.2
7.36)38.272
     36.80
      1.472
      1.472
```

Solution. $38.272 \div 7.36 = \frac{38272}{1000} \div \frac{736}{100}$
$= \frac{38272}{1000} \times \frac{100}{736} = \frac{38272}{736} \times \frac{100}{1000} = \frac{52}{1} \times \frac{1}{10} = 5.2$

The number of decimal places in the quotient will equal the number of zeros in the denominator of the last product. This will be the same as the number of zeros in the denominator of the dividend minus the number of zeros in the denominator of the divisor, or, what is the same thing, the number of decimal places in the dividend minus the number of decimal places in the divisor.

If the number of decimal places in the dividend is less than the number of decimal places in the divisor, we may annex zeros to the dividend till the number of decimal places is the same in both dividend and divisor. The quotient up to this point in the division will be an integer, and, in case it is necessary to carry the division farther, more zeros may be annexed to the dividend. The remaining figures of the quotient will be decimals.

Ex. 3. Divide 52.36 by 3.764.

```
             13.9
3.764)52.360|0
       37.64
       14.720
       11.292
        3.4280
        3.3876
           404
```

EXERCISE 16

1. Change $\frac{2}{3}$ to 9ths; $\frac{8}{21}$ to 168ths.

2. Reduce to lowest terms each of the following fractions: $\frac{9}{27}$, $\frac{111}{370}$, $\frac{72}{999}$, $\frac{1728}{8640}$.

3. Explain the reduction of $7\frac{3}{5}$ to an improper fraction.

4. Explain the reduction of $\frac{1895}{125}$ to a mixed number.

5. Simplify $\dfrac{4\frac{2}{3}}{2\frac{5}{6}}$, $\dfrac{\frac{5}{9}}{\frac{2}{3}}$, $\dfrac{0.5}{\frac{1}{2}}$, $\dfrac{0.75}{\frac{1}{3}}$.

6. Add $\frac{3663}{10989}$, $\frac{1221}{13431}$ and $\frac{5}{1221}$.

7. From $75\frac{5}{42}$ take $12\frac{3}{14}$.

8. Multiply $2\frac{1}{2} \div \frac{3}{4}$ by $\frac{1}{2}$ of $\frac{3}{4} \times \frac{5}{6}$.

9. Find the value of $\frac{5}{9}$ of $\frac{3}{13} \div \frac{2}{3}$ of $\frac{6}{11}$ of $\frac{22}{23}$.

10. Find the value of $\frac{1}{5} \div \frac{3}{4}$ of $\frac{2}{3} + 16 \times \frac{1}{2}$ of $4 \div \frac{2}{3}$ of $\frac{3}{4}$.

11. Find the value of $5\frac{3}{4} - 0.9$ of $2.7 + 25\frac{1}{6} \times 0.02 - \dfrac{2.7}{15}$.

12. Find the value of $\dfrac{3\frac{2}{17} \text{ of } 8\frac{1}{6}}{4\frac{2}{13} \text{ of } 2\frac{1}{16}}$.

13. By what must $\frac{3}{4}$ be multiplied to produce $3\frac{3}{4}$?

14. What number divided by $\frac{18}{5}$ of $\frac{5}{4}$ will give $4\frac{1}{5}$ as a quotient?

15. Simplify: $\dfrac{\frac{1}{2} + \frac{3}{13} \text{ of } \frac{1}{6} - \frac{3}{4} \times \frac{4}{7}}{1 - \frac{1}{5} \text{ of } \frac{10}{13} + \frac{3}{4} \times \frac{4}{7}}$.

16. What fraction added to the sum of $\frac{1}{2}$, $\frac{1}{4}$, and 5.25 will make 6.42?

17. Simplify: $\dfrac{2 - \frac{1}{2} \text{ of } (\frac{3}{4} - \frac{1}{20})}{5 + 0.5 \text{ of } (1 - 0.9)}$

18. How is the value of a proper fraction affected by adding the same number to both numerator and denominator? How is the value of an improper fraction affected?

19. A merchant bought a stock of goods for $2475.50 and sold $\frac{1}{2}$ of it at an advance of $\frac{1}{3}$ of the cost, $\frac{1}{4}$ of it at an advance of $\frac{1}{4}$ of the cost, and the remainder at a loss of $\frac{1}{10}$ of the cost. Did he gain or lose and how much?

20. A ship is worth $90,000 and a person who owns $\frac{5}{12}$ of it sells $\frac{1}{4}$ of his share. What is the value of the part he has left?

21. If 1 is added to both numerator and denominator of $\frac{3}{2}$, by how much is its value diminished?

22. If 1 is added to both numerator and denominator of $\frac{2}{3}$, by how much is its value increased?

89. Cancellation. Much time may be saved in solving problems by writing down a complete statement of the condition given and then canceling common factors if any are present. The student should do this at every stage in the solution of a problem, always factoring and canceling whenever possible, and *never multiplying or dividing till all possible factors have been removed by cancellation.*

Ex. **1.** If $\frac{6}{25}$ of a business block is worth \$6252.66, what is the value of $\frac{11}{15}$ of it?

Solution. $\frac{6}{25}$ is worth \$6252.66.

$\frac{1}{25}$ is worth $\frac{1}{6}$ of \$6252.66.

$\therefore$ the whole is worth $\frac{25}{6}$ of \$6252.66.

$\therefore$ $\frac{11}{15}$ is worth $\frac{11}{15}$ of $\frac{25}{6}$ of \$6252.66.

$$= \frac{11 \times \overset{5}{\cancel{25}} \times \overset{347.37}{\overset{\cancel{1042.11}}{\cancel{\$6252.66}}}}{\underset{3}{\cancel{15}} \times \cancel{6}} = \$19105.35.$$

Ex. **2.** How much must be paid for 59,400 lb. of coal at \$4 per ton of 2000 lb.?

Statement. $\frac{59400 \times \$4}{2000} = \$118.80.$

EXERCISE 17

Find the value of:

1. $\frac{27 \times 72 \times 80}{36 \times 45 \times 30}$.

2. $\frac{144 \times 1728 \times 999}{96 \times 270 \times 33}$.

3. $\frac{1320 \times 432 \times 660}{4400 \times 297 \times 288}$.

4. $\frac{1760 \times 9 \times 125}{55 \times 360}$.

5. How much must be paid for shipping 1200 bbl. of apples at \$35 per hundred barrels?

6. How many bushels of potatoes at 50 ct. a bushel will pay for 500 lb. of sugar at 4 ct. a pound?

7. A merchant bought 12 carloads of apples of 212 bbl. each, 3 bu. in each barrel at 45 ct. per bushel. He paid for them in cloth at 25 ct. per yard. How many bales of 500 yd. did he deliver?

8. How many bushels of potatoes at 55 ct. per bushel must be given in exchange for 22 sacks of corn, each containing 2 bu., at 60 ct. a bushel?

APPROXIMATE RESULTS

90. In scientific investigations exact results are rarely possible, since the numbers used are obtained by observation or by experiments in which, however fine the instrument, the results are only approximate, and there is a degree of accuracy beyond which it is impossible to go.

91. On the other hand, the approximate value of such incommensurable quantities as $\sqrt{2} = 1.414+$, $\pi = 3.14159+$ can be obtained to any required degree of accuracy. The value of π has been computed to 707 decimal places, but no such accuracy is necessary or desirable. The student should always bear in mind that *it is a waste of time to carry out results to a greater degree of accuracy than the data on which they are founded.*

92. It is frequently necessary to determine the value of a decimal fraction correct to a definite number of decimal places. The value of $\frac{23}{24} = 0.95833+$ correct to four decimal places is 0.9583. 0.958 and 0.96 are the values correct to three and two places. The real value of this fraction correct to four places lies between 0.9583 and 0.9584. 0.9583 is 0.00003+ less than the true value, while 0.9584 is 0.00006+ greater. Therefore 0.9583 is nearer the correct value, and is said to be the value correct to four decimal places. Similarly, 0.96 is the value correct to two places.

93. If 5 is the first rejected digit, the result will apparently be equally correct whether the last digit is increased by unity or left unchanged. The value of 0.4235 correct to three places may be either 0.423 or 0.424. However, as the 5 itself is usually an approximation, it can readily be determined which course to pursue by noticing whether the 5 is in excess or defect of the correct value. 0.23649 correct to four places is 0.2365, but correct to three places 0.236 is nearer the true value than 0.237.

94. Addition. *Ex.* Add 0.234673, 0.322135, 0.114342, 0.563217, each fraction being correct to six decimal places.

Solution. It is clear that the last digit in this sum is not correct, since each of the four numbers added may be either greater or less than the correct value by a fraction less than 0.0000005. Hence, the total error in the sum cannot be greater than 0.000002. The required sum must therefore lie between 1.234369 and 1.234365, and in either case the result correct to five places is 1.23437.

0.234673
0.322135
0.114342
0.563217
1.234367

95. The next to the last digit in the sum may be incorrect, as shown in the following example:

Ex. Add 0.131242, 0.276171, 0.113225, 0.342247, each fraction being correct to six decimal places.

Solution. In this case the sum lies between 0.862887 and 0.862883. Hence, it is uncertain whether 0.86289 or 0.86288 is the value correct to five places. 0.8629 is, however, the value correct to four places.

0.131242
0.276171
0.113225
0.342247
0.862885

96. The third digit from the last may be left in doubt, as in the following example:

Ex. Add 5.866314, 3.715918, 0.568286, 4.342233, each fraction being correct to six places.

Solution. Here the true value of the sum lies between 14.492753 and 14.492749. Hence, it is uncertain whether the value correct to four places is 14.4928 or 14.4927. The value correct to three places is 14.493.

5.866314
3.715918
0.568286
4.342233
14.492751

97. Subtraction. *Ex.* Subtract 0.238647 from 0.329528, each fraction being correct to six decimal places.

Solution. Since each fraction cannot differ from the true value by a fraction as large as 0.0000005, the difference cannot be greater or less than the correct value by a fraction as large as 0.000001. Hence, the difference must lie between 0.090882 and 0.090880, and the value correct to five places is 0.09088.

0.329528
0.238647
0.090881

98. Cases will arise where the second and third digits from the last are in doubt, as in addition. The student should determine how far the result may be relied upon in the following examples:

(1) Subtract 0.371492 from 0.764237.
(2) Subtract 0.11132 from 0.23597.
(3) Subtract 15.93133 from 43.71288.

99. Multiplication. From the examples in addition given above the student will notice that it will be sufficient in most cases to carry out the partial products correct to two places more than the required result.

Ex. Find the square of 3.14159 correct to four decimal places.

Solution. The multiplication in full and the contracted form are as follows:

3.14159	3.14159
3.14159	3.14159
9.42477	9.42477
.314159	.314159
.1256636	.125664
.00314159	3142
.001570795	1571
.0002827431	283
9.8695877281	9.8696

After pointing off the first partial product we proceed as indicated in the above contracted form until the multiplication by 3 and 1 are completed. Multiplication by 4 would give a figure in the seventh place. Instead of writing down the figures we add the nearest 10 to the next column. Thus, 4 times 9, 36, add 4 to the next column since 3.6 = 4 approximately. 4 times 5, 20 and 4, 24. 4 times 1, 4 and 2, 6, etc.

In multiplying by the next 1 it is not necessary to take the 9 in the multiplicand into account. So, also, in multiplying by the 5, the 5 and 9 in the multiplicand may both be ignored. And so on until the multiplication is completed.

100. Division. *Ex.* Divide 9376245 by 3724 correct to the units' place.

Solution. The division in full and the contracted form are as follows:

```
        2517                 2517
3724)9376245         3724)9376245
     7448                 7448
     19282                19282
     18620                18620
       6624                 662
       3724                 372
       29005                290
       26068                260
        2937                 30
```

The first two digits in the quotient are 2 and 5 and the second remainder is 662. It is not necessary to bring down any more figures to have a result correct to units since tens divided by thousands will give hundredths. The divisor may also be contracted at this stage of the work. Thus, cutting off the 4, 372 is contained once in the second remainder, 662. Cutting off the 2, 37 is contained 7 times in the next remainder, 290. This gives the units' figure of the quotient.

It will be noticed that the next figure of the quotient is greater than 0.5, therefore the result correct to units is 2518.

The work may be further abridged by omitting the partial products and writing down the remainders only.

```
        2517
3724)9376245
     19282
       662
       290
        30
```


101. *Ex.* Divide 62.473 by 419.6789.*

Solution. First shift the decimal point four places in each so as to have an integral divisor, and then work as follows: The 1 and 4 are obtained without abbreviating and the 8, 8, 5, 9, 0 by cutting off 9, 8, 7, 6, 9 in succession from the divisor.

```
           0.1488590
4196789)624730.0
        20505110
         3717954
          360523
           24780
            3796
              19
```

Ex. Divide 0.0167 by 423.74.*

```
       0.00003941
42374)1.67000
        39878
         1741
           46
            4
```

* From Langley's "Treatise on Computation," p. 68.

EXERCISE 18

1. Divide 100 by 3.14159 correct to 0.01.

2. Find the quotient of 67459633 divided by 4327 correct to five significant figures.

3. Determine without dividing by what number less than 13, 339295680 is exactly divisible.

Determine by casting out the 9's whether the following are correct:

4. $959 \times 959 = 919681$. **5.** $954 \times 954 \times 954 = 868250664$.

6. $33920568 \div 729 = 42829$. **7.** $1019 \times 1019 = 1036324$.

8. $6234751 \div 43265 = 14.41 +$ a remainder 2645.

9. Find the sum of 23.45617, 937.34212, 42.31759, 532.23346, 141.423798 correct to two decimal places.

10. Subtract 987.642 from 993.624 correct to tenths.

11. Find the product of 32.4736×24.7955 correct to five significant figures.

12. Divide 47632 by $3\frac{1}{3}$. **13.** Multiply 23793 by $12\frac{1}{2}$.

MEASURES

102. Measures of Weight. It is curious to note what an important part the grain of wheat or barley has played in the establishment of a unit of weight, both among the ancients and the more modern Europeans. In England, as early as 1266, we find the pennyweight defined as the weight of "32 wheat corns in the midst of the ear"; again about 1600, as "24 barley corns, dry and taken out of the middle of the ear." Still later the artificial grain ($\frac{1}{24}$ pwt. Troy) is defined as "one grain and a half of round dry wheat." The Greeks made four grains of barley equivalent to the keration or carob seed. From this is derived the **carat**, the measure by which diamonds and pearls are weighed. The grain of barley and the carat have been used by all European countries as the basis of existing weights.

103. Great inconvenience was long experienced from this lack of uniformity, so that Parliament in 1824 passed an act adopting the **Imperial Pound Troy** as the **standard of weight**. It was also enacted that of the 5760 grains contained in the pound Troy, the pound avoirdupois should contain 7000. The **international kilogram** is now the fundamental standard of weight in the United States. The *pound* avoirdupois is defined as $\frac{1}{2.2046}$ kilogram (see § 139). The ounce, grain, etc., are subdivisions of the pound.

104. Avoirdupois Weight

16 drams (dr.)	= 1 ounce (oz.)
16 ounces	= 1 pound (lb.)
100 pounds	= 1 hundredweight (cwt.)
2000 pounds	= 1 ton (T.)

112 lb. = 1 long cwt. and 2240 lb. = 1 long ton are used in the customhouse and in weighing coal and iron at the mines.

The *c* in cwt. stands for the Latin word *centum*, a hundred.

Lb. is a contraction of the Latin word *libra*, pound.

Pound is from the Latin word *pondus*, a weight.

Ounce is from the Latin word *uncia*, a twelfth part.

Dram is from the Latin word *drachma*, a handful.

105. Troy Weight

24 grains (gr.)	= 1 pennyweight (pwt.)
20 pennyweights	= 1 ounce Troy
12 ounces Troy	= 1 pound Troy

This weight is used for the precious metals and jewels. The ounce Troy and pound Troy must be carefully distinguished from the ounce and pound avoirdupois. The grain, however, is the same throughout.

437.5 grains = 1 ounce avoirdupois	480 grains = 1 ounce Troy
7000 grains = 1 pound avoirdupois	5760 grains = 1 pound Troy

106. Apothecaries' Weight

20 grains	= 1 scruple (sc. or ℈)
3 scruples	= 1 dram (dr. or ʒ)
8 drams	= 1 ounce (oz. or ℥)
12 ounces	= 1 pound
5760 grains	= 1 pound

This table is used in compounding drugs and medicines. Scruple is from the Latin word *scrupulum*, a small weight.

Of the above measures of weight, avoirdupois is the most generally used.

107. Measures of Length. The ancients usually derived their units of length from some part of the human body. Thus, we find the *fathom* (the distance of the outstretched hands), the *cubit* (the length of the forearm), and later the *ell* (the distance from the elbow to the end of the finger), the *foot* (the length of the human foot), the *span* (the distance between the ends of the thumb and little finger when outstretched), the *palm* (the width of the hand), the *digit* (the breadth of the finger). The Roman foot was subdivided into four palms, and the palm into four digits. The division into inches or *unciæ* (a twelfth part) applied not only to the foot but to anything.

108. For longer measures there was still less uniformity. We find the Hebrew's *half-day's journey*; the Chinese *lih*, the distance a man's voice can be heard upon a clear plain; the Greek *stadium*, probably derived from the length of the race course; the Roman *pace* of five feet; the *furlong*, the length of a furrow. The *mille passus*, a thousand paces, is the origin of the modern *mile*.

109. In 1374 the inch is defined in English law as the length of "three barley corns, round and dry." Later, other arbitrary measures of length were adopted by the government. The **international meter** is now the fundamental standard of length in the United States. The *yard* is defined as $\frac{3600}{3937}$ of a meter (see § 129). The **foot** and **inch** are subdivisions of this standard yard.

110. COMMON MEASURES OF LENGTH

12 inches (in.)	= 1 foot (ft.)
3 feet	= 1 yard (yd.)
5½ yards or 16½ feet	= 1 rod (rd.)
320 rods or 5280 feet	= 1 mile (mi.)

The furlong, equal to 40 rods, is seldom used.

The fathom, equal to 6 feet, and the knot or geographical mile, equal to one minute of the equatorial circumference of the earth (6080 feet), are sometimes used.

111. SURVEYORS' MEASURES OF LENGTH

7.92 inches	= 1 link (li.)
100 links	= 1 chain (ch.) = (4 rd.)
80 chains	= 1 mile

112. MEASURES OF SURFACES

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
30¼ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)
1 square mile	= 1 section.
36 sections	= 1 township (twp.)

113. MEASURES OF SOLIDS

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.)
27 cubic feet	= 1 cubic yard (cu. yd.)

The cord, equal to 128 cubic feet, is a rectangular solid 8 feet long, 4 feet wide, and 4 feet high. The common use of the word is, however, a pile of wood 8 feet long and 4 feet high, the width of the pile varying with the length of the stick.

1 cubic yard	= 1 load
24¾ cubic feet	= 1 perch

114. Measures of Money. Originally, among primitive people, buying and selling was carried on by **barter**, or the actual exchange of commodities. The inconveniences arising from transactions of this kind brought about the adoption of a **medium of exchange**, or **money**. Money, usually consisting of gold and silver, was used at a very early period in the world's history. Gold and silver seem at first to have been exchanged for commodities by weight. Business transactions were then still further simplified by the introduction of coins and paper money. Finally, as in the case of weights and measures, governments adopted definite standards of money value.

115. UNITED STATES MONEY

10 mills	= 1 cent (ct.)
10 cents	= 1 dime (d.)
10 dimes	= 1 dollar ($)
10 dollars	= 1 eagle (E.)

116. ENGLISH MONEY

12 pence (*d.*)	= 1 shilling (*s.*) = $0.2433
20 shillings	= 1 pound (£) = $4.8665

117. FRENCH MONEY

10 centimes	= 1 decime
10 decimes	= 1 franc = $0.193

118. German Money

100 pfennigs = 1 mark (M.) = \$0.238

119. Measures of Number

12 units = 1 dozen (doz.)
12 dozen = 1 gross (gro.)
12 gross = 1 great gross (gt. gro.)
Also 24 sheets of paper = 1 quire
20 quires = 1 ream

120. Liquid Measure

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.) = 231 cu. in.
31½ gallons = 1 barrel (bbl.)

121. Dry Measure

2 pints = 1 quart
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.) = 2150.42 cu. in.

The **Winchester bushel** is the standard measure for dry substances. It is a cylindrical vessel 18½ in. in diameter and 8 in. deep, containing 2150.42 cu. in.

Before the adoption of this and other standards by the English government there was even a greater variety of measures of capacity than of length and weight.

122. Reduction of Compound Numbers. Quantities like 5 mi. 10 rd. 7 yd. 2 ft. and 3 lb. 5 oz. are called *compound numbers*, because they are expressed in several denominations.

Ex. **1.** Reduce 25 yd. 2 ft. 11 in. to inches.

Solution. 1 yd. = 3 ft.
∴ 25 yd. = 25 × 3 ft. = 75 ft.
75 ft. + 2 ft. = 77 ft. 1 ft. = 12 in.
∴ 77 ft. = 77 × 12 in. = 924 in.
924 in. + 11 in. = 935 in.

25
3
75
2
77
12
924
11
935

or

25 yd. = 900 in.
2 ft. = 24 in.
11 in. = 11 in.
∴ 25 yd. 2 ft. 11 in. = 935 in.

Note. The explanation shows that 25 and 77 are the multipliers and 3 ft. and 12 in. the multiplicands; but to shorten the operation, 3 and 12, regarded as abstract numbers, may be used as multipliers, since the product of 25 × 3 = the product of 3 × 25.

Ex. **2.** Reduce 1436 pt. to bushels, pecks, etc.

2	1436	no. of pt.
8	718	no. of qt.
4	89	no. of pk. + 6 qt.
	22	no. of bu. + 1 pk.

Solution. Since there are 2 pt. in 1 qt., in 1436 pt. there are as many quarts as 2 pt. are contained times in 1436 pt., or 718 qt.

Since there are 8 qt. in 1 pk., in 718 qt. there are as many pecks as 8 qt. are contained times in 718 qt., or 89 pk. 6 qt.

Since there are 4 pk. in 1 bu., in 89 pk. there are as many bushels as 4 pk. are contained times in 89 pk., or 22 bu. 1 pk.

∴ 1436 pt. = 22 bu. 1 pk. 6 qt.

EXERCISE 19

1. Reduce 3 A. 5 sq. rd. 12 sq. yd. to square yards.
2. Reduce 11000 sq. rd. to acres.
3. Reduce 2 gt. gro. 5 gro. to dozens.
4. Reduce 972 sheets to reams.
5. Reduce 20 cu. yd. to cubic inches.
6. Reduce 1000 oz. to pounds and ounces (avoirdupois).

7. Reduce 12 lb. 5 oz. 11 pwt. 20 gr. to grains.

8. Reduce 113 T. 7 cwt. 11 lb. to pounds.

9. Reduce 14763051 lb. to tons.

10. Reduce 5 sq. yd. 3 sq. ft. 91 sq. in. to square inches.

11. Reduce 46218385 sq. in. to acres.

12. Reduce 5 bu. 7 pk. 3 qt. to quarts.

13. Reduce 34372 pt. to pecks.

14. Reduce 21 yd. to a decimal of a mile.

15. Reduce 2 pk. 3 qt. 1 pt. to a decimal of a bushel.

16. Reduce 0.0125 A. + 0.25 sq. rd. to square feet.

17. Reduce 0.01 of a cubic yard to cubic inches.

18. Reduce 43629145 in. to miles.

19. Reduce $\frac{2}{3}$ of a peck to pints.

20. Reduce 2 qt. 1 pt. to a fraction of a peck.

21. Reduce 1 mi. 11 ch. to feet.

123. Addition and Subtraction of Compound Numbers. Compound addition and subtraction is the addition and subtraction of compound numbers of the same kind. The processes differ very little from the corresponding processes in the addition and subtraction of abstract numbers.

Ex. 1. Add 4 lb. 7 oz. (Av.), 3 lb. 4 oz., 12 lb. 10 oz., 9 lb. 5 oz.

Solution. The work is as follows:

5, 15, 19, 26 oz. = 1 lb. 10 oz.
1, 10, 22, 25, 29 lb.

4 lb.	7 oz.
3	4
12	10
9	5
29 lb.	10 oz.

Ex. 2. From 41 lb. 4 oz. (Av.) subtract 29 lb. 8 oz.

The work is as follows:

1 lb., or 16 oz. + 4 oz. = 20 oz.
8 and 12 are 20
1 and 29 and 11 are 41.

41 lb.	4 oz.
29	8
11 lb.	12 oz.

124. Multiplication of Compound Numbers.

Ex. Multiply 5 yd. 2 ft. by 7.

7 × 2 ft. = 14 ft. = 4 yd. 2 ft.
7 × 5 yd. = 35 yd.
35 yd. + 4 yd. 2 ft. = 39 yd. 2 ft.

5 yd. 2 ft.
7
39 yd. 2 ft.

125. Division of Compound Numbers. Compound division is of two kinds. The first is the converse of multiplication. In this case the quotient is a compound number of the same kind as the dividend. In the second case the dividend and divisor are both compound numbers of the same kind, and the quotient is an abstract number.

126. The two cases arise from the fact that division may be regarded as the operation of finding one of two factors when the other factor and the product are given.

Thus, 39 yd. 2 ft. is the product of 7 and 5 yd. 2 ft.

$\therefore \frac{39\text{ yd. }2\text{ ft.}}{7} = 5\text{ yd. }2\text{ ft.}$, or $\frac{39\text{ yd. }2\text{ ft.}}{5\text{ yd. }2\text{ ft.}} = \frac{119\text{ ft.}}{17\text{ ft.}} = 7$, the dividend and divisor being reduced to the same denomination before dividing.

Ex. 1. Divide 29 mi. 2 yd. 2 ft. by 8.

Solution. 29 mi. ÷ 8 = 3 mi. + a remainder of 5 mi.
5 mi. = 8800 yd. and 8800 yd. + 2 yd. = 8802 yd.
8802 yd. ÷ 8 = 1100 yd. + a remainder of 2 yd.
2 yd. = 6 ft., and 6 ft. + 2 ft. = 8 ft.
8 ft. ÷ 8 = 1 ft.
∴ 29 mi. 2 yd. 2 ft. ÷ 8 = 3 mi. 1100 yd. 1 ft.

Ex. 2. Divide 139 lb. 8 oz. (Av.) by 4 lb. 8 oz.

Solution. 139 lb. 8 oz. = 2232 oz.
4 lb. 8 oz. = 72 oz.
2232 oz. ÷ 72 oz. = 31.

127. Check. Compound addition and subtraction may be checked in the same way as addition and subtraction of simple numbers. Multiplication may be checked by division, and division by multiplication.

EXERCISE 20

1. How many inches are there in 1 mi. 3 ch.?
2. Add 14 lb. 3 oz., 5 lb. 7 oz., 31 lb. 11 oz.
3. From 17 cu. yd. 11 cu. in. subtract 5 cu. yd. 5 cu. ft.
4. From 11 bu. 1 pk. subtract 4 bu. 5 qt.
5. Multiply 30 A. 11 sq. rd. by 10.
6. Divide 159 A. 29.5 sq. rd. by 2.
7. How many bags containing 2 bu. 1 pk. each can be filled from a bin of wheat containing 256 bu. 2 pk.?
8. How many revolutions will a bicycle wheel 7 ft. 4 in. in circumference make in traveling 25 mi.?
9. How many times can a bushel measure be filled from a bin 8 ft. square and 6 ft. deep? Will there be a remainder?
10. How many gallons of water will a tank 4 ft. 7 in. by 2 ft. 11 in. by 1 ft. 3 in. contain?
11. How many times is 7 ft. 6 in. contained in 195 mi. 280 rd.?

12. How much coal is there in three carloads of 38 T. 3 cwt. 41 lb., 29 T. 7 cwt. 5 lb., 32 T. 17 cwt. 70 lb.?
13. What must be the length of a shed 7 ft. high and 9 ft. wide to contain 50 cd. of 16 in. wood?
14. If a ton of coal occupies 36 cu. ft., what must be the depth of a bin 6 ft. wide by $7\frac{1}{2}$ ft. long in order that it may contain 10 T.?
15. Divide 320 rd. 4 yd. by 10 rd. 2 yd.
16. How many feet are there in $\frac{5}{8}$ of a mile?
17. Reduce 17 pt. to a decimal of a gallon.
18. How many steps does a man take in walking a mile if he advances 2 ft. 10 in. each step?
19. The pound avoirdupois contains 7000 gr. Find the greatest weight that will measure both a pound Troy and a pound avoirdupois. Find the least weight that can be expressed without fractions in both pounds Troy and pounds avoirdupois.
20. A cubic foot of water weighs 1000 oz. avoirdupois. Find the number of grains Troy in a cubic inch of water.

METRIC SYSTEM OF WEIGHTS AND MEASURES

128. Late in the eighteenth century France invented the metric system of weights and measures, but it was not made obligatory until 1837. Previous to this time there existed in France the same lack of uniformity in forming multiples and submultiples of the units of measure as exists in our system at the present time. The metric system is now in use in most civilized countries except the United States and England. It was legalized by Congress in the United States in 1866, but has not been generally adopted. In scientific work the system is quite generally used in all countries.

129. The unit of length is the **meter**. This is the fundamental unit, because from it every other unit of measure or weight is derived; hence the name **metric system**. The meter is theoretically one ten-millionth part of the distance of the pole from the equator. Though an error has since been discovered in the measurement of the distance, the meter has not been changed, and a rod of platinum 39.37 inches in length, deposited in the archives at Paris, is called the standard meter.

1 Decimeter of the Meter Rule
Divided into centimeters, half centimeters and millimeters.

130. The unit of capacity is called the **liter**. It is a cube whose edge is 0.1 of a meter.

131. The unit of weight is the **gram**. The gram is the weight of a cube of distilled water at maximum density, whose edge is 0.01 of a meter.

132. The above units of measure, together with the following prefixes, should be carefully memorized, because from them the whole metric system can be built up.

133. The Latin prefixes, **deci**, **centi**, **milli**, denote respectively 0.1, 0.01, 0.001 of the unit. The Greek prefix **micro** is used to denote 0.000001 of a unit. Thus, decimeter means 0.1 of a meter, and centigram means 0.01 of a gram.

134. The Greek prefixes, **deca**, **hecto**, **kilo**, **myria**, denote respectively 10, 100, 1000, 10000 times the unit. Thus, kilometer means 1000 meters, and hectoliter means 100 liters.

135. In general, nothing beyond practice in arithmetical operations would be gained in reducing from the metric system to our system. Occasionally, however, such reductions are necessary, hence, a few of the common equivalents are given in the tables.

136. Measures of Length

10 millimeters (mm)	= 1 centimeter
10 centimeters (cm)	= 1 decimeter
10 decimeters (dm)	= 1 meter = 39.37 in.
10 meters (m)	= 1 decameter
10 decameters (Dm)	= 1 hectometer
10 hectometers (Hm)	= 1 kilometer
10 kilometers (Km)	= 1 myriameter (Mm)

137. Square Measure

100 square millimeters (mm^2)	= 1 square centimeter
100 square centimeters (cm^2)	= 1 square decimeter
100 square decimeters (dm^2)	= 1 square meter
100 square meters (m^2)	= 1 square decameter
100 square hectometers (Hm^2)	= 1 square kilometer (Km^2)

This table may be extended by squaring each unit of length for the corresponding unit of square measure. The denominations given in the table are the only ones in common use.

In measuring land, the square decameter is called the are, the square hectometer, the **hectare** = 2.47 acres, and the square meter, the **centare**.

138. Cubic Measure

1000 cubic millimeters $^{(mm^3)}$	= 1 cubic centimeter
1000 cubic centimeters $^{(cm^3)}$	= 1 cubic decimeter
1000 cubic decimeters $^{(dm^3)}$	= 1 cubic meter $^{(m^3)}$

This table may be extended by cubing each unit of length for the corresponding unit of cubic measure. The denominations given in the table are the only ones in common use.

The cubic meter is used in measuring wood, and is called the **stere.**

139. Measures of Weight

10 milligrams $^{(mg)}$	= 1 centigram
10 centigrams $^{(cg)}$	= 1 decigram
10 decigrams $^{(dg)}$	= 1 gram
10 grams $^{(g)}$	= 1 decagram
10 decagrams $^{(Dg)}$	= 1 hectogram
10 hectograms $^{(Hg)}$	= 1 kilogram = 2.2 lb.
10 kilograms $^{(Kg)}$	= 1 myriagram
10 myriagrams $^{(Mg)}$	= 1 quintal
10 quintals $^{(Q)}$	= 1 tonneau $^{(T)}$

The **metric ton** or **tonneau** is the weight of one cubic meter of distilled water = 2204.62 pounds.

140. Measures of Capacity

10 milliliters $^{(ml)}$	= 1 centiliter
10 centiliters $^{(cl)}$	= 1 deciliter
10 deciliters $^{(dl)}$	= 1 liter = 1 qt. nearly
10 liters $^{(l)}$	= 1 decaliter
10 decaliters $^{(Dl)}$	= 1 hectoliter = 2.837 bu.
10 hectoliters $^{(Hl)}$	= 1 kiloliter $^{(Kl)}$

EXERCISE 21

1. What is the weight of a liter of water? Give the result in grams.

2. What is the weight of a cubic centimeter of water? of a cubic meter?

3. What is the weight of 15^{Hl} of water?

4. Find the sum of 21.14^{m^3}, 321^{l} and 1.25^{dl}. Give the result in liters.

5. Find in hectares and ares the area of a field 450^{m} long and 200^{m} wide.

6. If gold is 19.36 times as heavy as water, find in kilograms the weight of a bar of gold 10^{cm} long, 30^{mm} wide and 25^{mm} thick.

7. How many square millimeters are there in a square centimeter? in two meters square?

8. Reduce 240064^{mm} to kilometers, etc.

9. Reduce 3463^{ca} to hectares, etc.

10. If 15^{Kg} 7^{g} of beef cost 26 francs $37\frac{1}{4}$ centimes, find the cost per kilogram.

11. How many sacks will be necessary to hold 1245^{Hl} 6^{Dl} of wheat if each sack holds 1^{Hl} 20^{l}?

12. What decimal of a decagram is 6^{g} 4^{cg}?

13. How much wheat is contained in 1396 sacks, each of which contains 1^{Hl} 35^{l}?

14. If the distance from the equator to the pole is 1000 myriameters, how many meters are there in a degree?

15. What will be the price of 47^{Ha} 5^{a} 65^{ca} of land at 89.76 francs per are?

16. Mercury is 13.598 times as heavy as water. Find the weight of 567.859^{cm^3} of mercury.

17. If a man steps 80^{cm} at each step, how many steps will he take in walking 10^{Km}?

18. Olive oil is 0.914 as heavy as water. Find the cost of a hectoliter at 3 francs a kilogram.

19. A piece of land 1236 meters square sold for 240 francs per hectare. How much did the land bring?

20. A person bought $\frac{3}{4}$ of a piece of land containing 2^{Ha} 15^{a} at 45 francs an are; he sold $\frac{3}{5}$ of what he bought for 5000 francs. How much did he gain?

21. A spring furnishes 5^{l} of water in 2 min. How long will it take the spring to fill a vessel holding $32\frac{3}{4}$ liters?

22. Three fountains furnish $3\frac{1}{2}^{l}$, $2\frac{2}{3}^{l}$ and $7\frac{3}{4}^{l}$ of water each minute respectively. The three together fill a tank in 2 hr. and 43 min. How many hectoliters of water does the tank contain?

23. If 8^{a} of land are bought for 19200 francs and sold for 25.20 francs per square meter, how much is gained by the transaction?

24. If sea water is 1.026 times as heavy as distilled water and olive oil is 0.914 as heavy, how much more than an equal volume of olive oil will a hectoliter of sea water weigh?

141. Measures of Angles and Time. The sexagesimal division of numbers is undoubtedly of Babylonian origin. The Babylonian priests in their astronomical work reckoned the year as 360 days. They supposed the sun to revolve around the earth once each year and hence divided the circumference of the circle into 360 parts, each of which represented the apparent daily path of the sun. They

probably knew the construction of the regular hexagon by applying the radius to the circumference six times. It was then natural to take one of the 60 parts thus cut off as a unit and to further subdivide this unit into 60 equal parts, and so on, according to their method of sexagesimal fractions. This is the origin of our degree, minute and second. The names *minutes* and *seconds* are taken from the Latin *partes minutæ primæ* and *partes minutæ secundæ*.

142. The principal measures of time are the **day** and the **year**. The day is the average time in which the earth revolves on its axis. The division of the day into 24 hours, of the hour into 60 minutes, and of the minute into 60 seconds is probably due to the Babylonians. The solar year is the time in which the earth travels once around the sun. It contains 365.2426 days.

143. In B.C. 46 Julius Cæsar reformed the calendar and decreed that there should be three successive years of 365 days followed by a year of 366 days to account for the difference of 0.2426 of a day between the year of 365 days and the solar year. The difference between four years of 365 days and four years of 365.2426 days is only 0.9704 of a day, so that if a whole day is added every fourth year there is added 0.0296 of a day too much. In 1582 Pope Gregory XIII corrected this by striking 10 days from the year, — calling Oct. 5th Oct. 15th, — and he decreed that three leap years were to be omitted in every four hundred years. Every year whose number is divisible by four is a leap year unless it is a year ending a century, as 1900, when it is a leap year only if divisible by 400. 1800, 1900, 2100, are not leap years, but 1600, 2000, 2400, are. The Gregorian calendar was adopted at once in Roman Catholic countries and in England in 1752. At the same time in England the beginning of the year was changed from March 25 to Jan. 1. Russia and some other countries still use the Julian calendar. Since 1582 they have had three more leap years (1700, 1800 and 1900) than countries using the Gregorian calendar, and hence are now 13 days behind other countries. What we call Jan. 23 is Jan. 10 with them.

144. Dates given according to the Julian calendar are called Old Style (O. S.) and dates according to the Gregorian calendar are called New Style (N. S.)

LONGITUDE AND TIME

145. Longitude is distance east or west of the **prime meridian.** The meridian of the Royal Observatory at Greenwich, England, is the prime meridian generally adopted, and longitude is reckoned east or west 180° from that meridian.

Since the earth makes one complete revolution on its axis in 24 hours, each place in its surface passes through 360° in that time. Hence:

360° of longitude correspond to 24 hr. of time.
1° of longitude corresponds to $\frac{1}{360}$ of 24 hr., or $\frac{1}{15}$ hr., or 4 min.
1′ of longitude corresponds to $\frac{1}{60}$ of 4 min., or 4 sec.
1″ of longitude corresponds to $\frac{1}{60}$ of 4 sec., or $\frac{1}{15}$ sec.

And

24 hr. of time correspond to 360° of longitude.
1 hr. of time corresponds to $\frac{1}{24}$ of 360°, or 15°.
1 min. of time corresponds to $\frac{1}{60}$ of 15°, or 15′.
1 sec. of time corresponds to $\frac{1}{60}$ of 15′, or 15″.

Ex. 1. The difference in time between two places is 2 hr. 25 min. 13 sec. What is the difference in longitude?

Solution. $2 \times 15° = 30°$,

$25 \times 15' = 375' = 6° 15'$,

$13 \times 15'' = 195'' = 3' 15''$,

$30° + 6° 15' + 3' 15'' = 36° 18' 15''$.

Check by reducing 36° 18′ 15″ to hours, minutes, seconds, as in the following example.

Ex. 2. The difference in longitude of two places is 46° 32′ 45″. What is the difference in time?

Solution. $46 \times \frac{1}{15}$ hr. = 3 hr. 4 min.

32×4 sec. = 128 sec. = 2 min. 8 sec.

$45 \times \frac{1}{15}$ sec. = 3 sec.

3 hr. 4 min. + 2 min. 8 sec. + 3 sec. = 3 hr. 6 min. 11 sec.

Check by reducing 3 hr. 6 min. 11 sec. to degrees, minutes and seconds as in *Ex.* 1.

EXERCISE 22

1. In what direction does the sun appear to move as the earth revolves on its axis?

2. How many degrees pass under the sun's rays in 5 hr.?

3. When it is noon at Chicago, what time is it at a place 15° 15′ east of Chicago? 45° 30′ 45″ west?

4. What is the difference in longitude between two places, the difference in time being 1 hr. 4 min.?

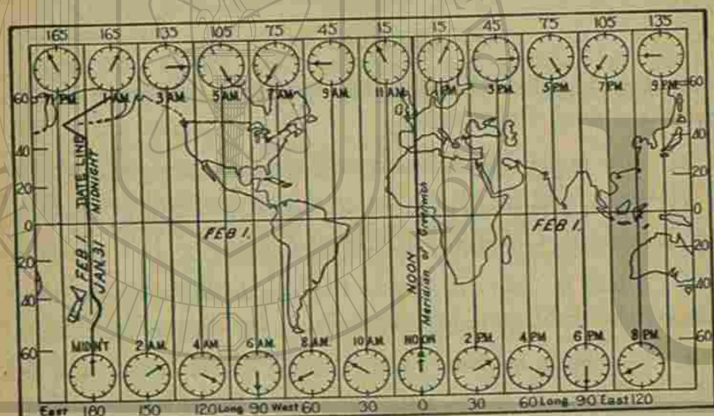
5. A person travels from Detroit until his watch is 45 min. fast. In what direction and through how many degrees has he traveled?

6. What is the difference in time between two places whose longitudes are 75° and 60°?

7. When it is 9 A.M. local time at Washington, it is 8 hr. 7 min. 4 sec. at St. Louis; the longitude of Washington being 77° 1′ W., what is the longitude of St. Louis?

146. International Date Line. Suppose that two men, starting from the prime meridian on Monday noon, travel the one eastward and the other westward, each traveling just as fas' as the earth rotates. The man who goes west as fast as the earth turns east keeps exactly beneath the sun all the time; and it seems to him to be still Monday noon when he reaches his starting point again twenty-four hours later. He has *lost* a day in his reckoning by traveling westward around the earth.

The other man travels eastward over the earth as fast as the earth itself turns eastward, and therefore he moves away from the sun twice as fast as the prime meridian does. After twelve hours' travel he reaches the meridian of 180°, but twelve hours rotation has carried this meridian beneath the sun, and so the traveler reaches it at noon. In twenty-four hours the man reaches his starting point on the prime



meridian, but twenty-four hours' rotation has brought this meridian beneath the sun again, so the traveler reaches it on the second noon after his start; he therefore supposes it to be Wednesday noon, though really it is but twenty-four hours after Monday noon. He has *gained* a day in his reckoning by traveling eastward around the earth. To correct such errors in their dates, navigators usually *add* a day to their reckoning when they sail westward across the meridian of 180°, and *subtract* a day when they cross it to the eastward. The line where the adjustment is made, corresponding in general with the meridian of 180°, is called the international date line.

The map represents the earth when it is noon Feb. 1 at Greenwich.

It is, therefore, one hour earlier in the day for each 15° west of Greenwich and one hour later in the day for each 15° east of Greenwich. Hence 180° west of Greenwich it is midnight of Jan. 31, and 180° east of Greenwich it is midnight of Feb. 1.

When it is 6 A.M. Feb. 1 at Greenwich, at 90° W. it is midnight of Jan. 31, and at 90° E. it is noon of Feb. 1; at 180° W. it is 6 P.M. Jan. 31 and at 180° E. it is 6 P.M. Feb. 1. In this case it is Jan. 31 in all longitudes from 90° W. westward to the date line, and Feb. 1 in all longitudes from 90° W. eastward to the date line.

When it is midnight of Jan. 31 at Greenwich, at 90° W. it is 6 P.M. Jan. 31 and at 90 E. it is 6 A.M. Feb. 1; at 180° W. it is noon Jan. 31 and at 180° E. it is noon Feb. 1. In this case it is Jan. 31 in all longitudes from Greenwich westward to the date line, and Feb. 1 in all longitudes from Greenwich eastward to the date line.

When it is 6 P.M. Jan 31 at Greenwich, at 90° W. it is noon Jan. 31, and at 90° E. it is midnight Jan. 31; at 180° W. it is 6 A.M. Jan. 31, and at 180° E. it is 6 A.M. Feb. 1. In this case it is Jan. 31 in all longitudes from 90° E. westward to the date line, and Feb. 1 in all longitudes from 90° E. eastward to the date line.

1. When it is noon February 1 at Greenwich, what date is it at Paris? at New York City? at San Francisco?

2. Imagine the midnight line of Jan. 31 as a dark line moving westward parallel to the meridian. Everywhere on this line it is midnight. Behind this line it is Feb. 1; in front, Jan. 31. On what part of the earth's surface is it Feb. 1, and on what part Jan. 31 when this imaginary midnight line has reached the prime meridian? 90° E.? 140° E.?

3. What date will be in front of this line when it reaches 180°? What date will be behind it?

4. After crossing the 180th meridian and passing on to 175° E., what date is before the line and what date behind it?

5. What change must be made in the calendar of a ship crossing this line going westward? Going eastward?

147. Table of longitudes for use in solving problems:

Ann Arbor, Mich.	83° 43′ 48″ W.	London	0° 5′ 38″ E.
Albany, N.Y. . .	73° 44′ 48″ W.	Lisbon	9° 11′ 10″ W.
Boston	71° 3′ 30″ W.	Melbourne . .	144° 58′ 42″ E.
Berlin	13° 23′ 43″ E.	New Orleans . .	90° 3′ 28″ W.
Brussels	4° 22′ 9″ E.	New York . . .	74° 0′ 3″ W.
Chicago	87° 36′ 42″ W.	Paris	2° 20′ 15″ E.
Cincinnati . . .	84° 26′ 0″ W.	Peking	116° 26′ 0″ E.
Cambridge, Eng.	0° 5′ 41″ E.	Rome	12° 27′ 14″ E.
Cape Town . . .	18° 28′ 45″ E.	San Francisco .	122° 26′ 15″ W.
Calcutta	88° 19′ 2″ E.	St. Louis . . .	90° 12′ 11″ W.
Detroit	83° 5′ 7″ W.	Sydney	151° 11′ 0″ E.
Dublin	6° 2′ 30″ W.	Tokyo	139° 42′ 30″ E.
Honolulu . . .	157° 52′ 0″ W.	Washington . .	77° 1 W.

EXERCISE 23

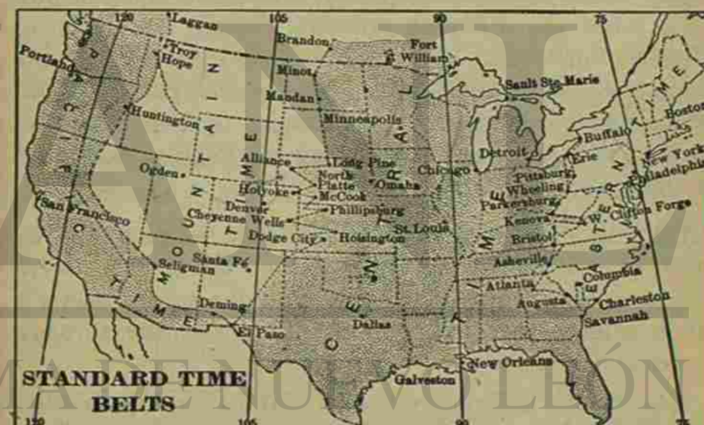
1. Determine the time and date at Ann Arbor, Berlin, Cape Town and Peking when it is midnight May 15 at Greenwich.

2. When it is noon March 1 at Rome, what time and date is it at San Francisco? at Sydney? at Detroit?

3. If a man were to travel westward around the earth in 121 da., in how many days would he actually make the trip by the local time of the places he passes through? In how many days would he make the trip traveling eastward?

4. When it is noon Sunday, Jan. 31, on the 90th meridian west, what part of the world has Sunday? What is the day and date on the other part?

5. When it is 3 P.M. Feb. 5 on the 45th meridian east, what part of the world has Feb. 5, and what is the date on the other part?



STANDARD TIME BELTS

148. Standard Time. In order to secure uniform time over considerable territory, in 1883 the railroad companies of the United States decided to adopt standard time. They divided the country into four time belts, each of approximately 15° of longitude in width. The time in the various belts will therefore differ by hours, while the minute and second hand of all timepieces will remain the same. The correct time is distributed by telegraph throughout the United States from the Naval Observatory at Washington each day.

149. The **Eastern** time belt lies approximately $7\frac{1}{2}°$ each side of the 75th meridian, and has throughout the local time of the 75th meridian. Similarly, the **Central, Mountain** and **Pacific** time belts lie approximately $7\frac{1}{2}°$ each side of the 90th, 105th and 120th meridians, and the time throughout in each belt is determined by the local time of the 90th, 105th and 120th meridians.

150. These divisions are not by any means equal or uniform, since the different railroads change their time at the most convenient places. Consequently, variations are made from the straight line to include such places. Thus, while most roads change from Eastern to Central time at Buffalo, the Canadian roads extend the Eastern belt much farther west.

151. Standard time has more recently been adopted by most of the leading governments of the world. With few exceptions the standard meridian chosen represents a whole number of hours from the prime meridian through Greenwich.

Great Britain, Belgium and Holland use the time of the meridian through Greenwich. France uses the time of the meridian (2° 20′ E.) through Paris. Cape Colony uses the time of the meridian 22° 30′ E. Germany, Italy, Austria, Denmark, Norway and Sweden use the time of the 15th meridian east. Roumania, Bulgaria and Natal use the time of the 30th meridian east. Western Australia uses the time of the 120th meridian east. Southern Australia and Japan use the time of the 135th meridian east. New South Wales, Victoria, Queensland, and Tasmania use the time of the 150th meridian east. New Zealand uses the time of 170° 30′ E.

EXERCISES 24

1. When it is 7 P.M. at Philadelphia, what time is it at London? at Paris? at Berlin?

2. What is the difference in time between Boston and Rome? Melbourne and Tokyo?

3. It is 7 A.M. March 1 at St. Louis. What is the time and date at Tokyo?

4. When it is 1.30 P.M. at Buffalo, what time is it at Cleveland?

5. What is the difference between the local and standard time of Boston? of Chicago? of St. Louis? of New York?

6. The local time of Detroit is 27.658 min. faster than the standard time. Find the longitude of Detroit.

7. When it is noon local time at Boston, what is the standard time at Ogden, Utah?

THE EQUATION

152. An **equation** is a statement of equality of two numbers or expressions. Thus, $5 = 3 + 2$ and $3 \times 5 = 15$ are equations.

153. The equation is one of the most powerful instruments used in mathematics and can be used to decided advantage in the solution of many arithmetical problems.

154. The book written by the Egyptian priest Ahmes, and referred to elsewhere, is one of the very oldest records of the extent of mathematical knowledge among the ancients. In this book we find a very close relation between arithmetic and algebra. A number of problems are given leading to the simple equation. Here the unknown quantity is called *hau* or "heap," and the equation is given in the following form: *heap* its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, gives 33, or $\frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x + x = 33$. (Cajori, "History of Elementary Mathematics," p. 23.)

Ex. 1. Find a number such that, if 30 is subtracted, $\frac{2}{3}$ of the original number will remain.

The given relation may be expressed as follows:

The number diminished by 30 equals $\frac{2}{3}$ of the number.
From this relation we are to find the number.
Let x = the number.
Then $x - 30$ = the number diminished by 30, and $\frac{2}{3}x = \frac{2}{3}$ of the number.

$\therefore x - 30 = \frac{2}{3}x$ is the equation expressing the relation between the number given in the conditions of the problem and the letter representing the number to be found.

Solution. Since we wish to obtain an equation with x alone on the left side and only numerical quantities on the right side, we proceed as follows:

Subtracting $\frac{2}{3}x$ from both sides of the equation, we have

$$x - 30 - \frac{2}{3}x = \frac{2}{3}x - \frac{2}{3}x, \qquad \text{Axiom 3.}$$

or

$$\frac{1}{3}x - 30 = 0.$$

Adding 30 to both sides of the equation, we have

$$\frac{1}{3}x - 30 + 30 = 30, \qquad \text{Axiom 2.}$$

or

$$\frac{1}{3}x = 30.$$

Multiplying both terms by 3, we have

$$x = 90.$$

$\therefore$ 90 is the required number.

To *check* the result, substitute 90 for the unknown number in the problem.

This gives us

$$90 - 30 = \frac{2}{3} \text{ of } 90,$$

or

$$60 = 60.$$

$\therefore x = 90$ is the correct result.

Ex. 2. The sum of two numbers is 20 and their difference is 4. What are the numbers?

Solution. Let x = the larger number.

Then $x - 4$ = the smaller number.

$$\therefore x + x - 4 = 20.$$

$$2x = 24, \qquad \text{Axiom 2.}$$

or $x = 12$ = the larger number. Axiom 5.

Check. $12 - 4 = 8$ = the smaller number.

$12 + 8 = 20. \quad 12 - 8 = 4.$

EXERCISE 25

1. The sum of two numbers is 31 and their difference is 11. What are the numbers?

2. The difference of two numbers is 60, and if both numbers are increased by 5 the greater becomes four times as large as the smaller. What are the numbers?

3. Find two numbers such that their difference is 95 and the smaller divided by the greater is $\frac{1}{6}$.

4. After spending $\frac{2}{3}$ of his money a man pays bills of \$25, \$40 and \$14, and finds that he has \$132 left. How much money had he at first?

5. Find a number such that its half, third and fourth parts shall exceed its fifth part by 106.

6. A father wishes to divide \$28,000 among his two sons and a daughter so that the elder son shall receive twice as much as the younger, and the younger son twice as much as the daughter. Find the share of each.

POWERS AND ROOTS

155. Archimedes (287–212 B.C.) in his measurements of the circle computed the approximate value of a number of square roots, but nothing is known of his method. A little later, Heron of Alexandria also used the approximation $\sqrt{a^2 + b} = a + \frac{b}{2a}$, *e.g.* $\sqrt{85} = \sqrt{9^2 + 4} = 9 + \frac{4}{18}$.

156. The Hindus included powers and roots among the fundamental processes of arithmetic. As early as 476 A.D. they used the formulæ $(a + b)^2 = a^2 + 2ab + b^2$ and $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ in the extraction of square and cube root, and they separated numbers into periods of two and three figures each.

157. The Arabs also extracted roots by the formula for $(a + b)^n$. They introduced a radical sign by placing the initial letter of the word *jird* (root) over the number.

158. The **power** of a number is the product that arises by multiplying the number by itself any number of times. The second power is called the **square** of the number and the number itself is called the **square root** of its second power. The third power is called the **cube** of the number, and the number itself is called the **cube root** of its third power. Thus, 4 is the square of 2, and 2 is the square root of 4. 125 is the cube of 5, and 5 is the cube root of 125.

159. Since the square root of a number is one of the two equal factors of a perfect second power, numbers that are not exact squares have no square roots. However, they are treated as having approximate square roots. These approximate square roots can be found to any required

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160. According to the definition, only abstract numbers have square roots.

161. The square, the cube, fourth power, etc., of 2 are expressed by 2^2, 2^3, 2^4, etc. The small figure which denotes the power is called the **exponent**.

162. The square root is denoted by the symbol $\sqrt{\ }$, the cube root by $\sqrt[3]{\ }$, the fourth root by $\sqrt[4]{\ }$, etc.

163. The following results are important:

$$1^2 = 1,$$
$$10^2 = 100,$$
$$100^2 = 10000,$$
$$1000^2 = 1000000.$$

The squares of all numbers between 1 and 10 lie between 1 and 100, the squares of all numbers between 10 and 100 lie between 100 and 10000, etc. Hence, the square of a number of one digit is a number of one or two digits, the square of a number of two digits is a number of three or four digits, etc. It will be noticed that the addition of a digit to a number adds two digits to the square.

164.
$$1^3 = 1,$$
$$10^3 = 1000,$$
$$100^3 = 1000000, \text{ etc.}$$

It will be noticed in this case that the addition of a digit to a number adds three digits to its cube.

165.
$$0.1^2 = 0.01,$$
$$0.01^2 = 0.0001,$$
$$0.001^2 = 0.000001, \text{ etc.}$$

Hence, the square of a decimal number contains twice as many digits as the number itself.

166.
$$0.1^3 = 0.001,$$
$$0.01^3 = 0.000001,$$
$$0.001^3 = 0.000000001, \text{ etc.}$$

Hence, the cube of a decimal number contains three times as many digits as the number itself.

167. Laws of exponents:

Since $2^2 = 2 \times 2$ and $2^3 = 2 \times 2 \times 2$,

then $2^2 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$.

This may be written in the form

$$2^2 \times 2^3 = 2^{2+3} = 2^5.$$

Or, in general, $a^m \times a^n = a^{m+n}$. **I.**

Also,

$$(2^3)^4 = (2 \times 2 \times 2)(2 \times 2 \times 2)(2 \times 2 \times 2)(2 \times 2 \times 2) = 2^{3+3+3+3} = 2^{12}.$$

This may be written in the form

$$(2^3)^4 = 2^{3 \times 4} = 2^{12}.$$

Or, in general, $(a^m)^n = a^{m \times n}$. **II.**

Also, $2^5 \div 2^3 = \dfrac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 \times 2 = 2^2.$

This may be written in the form

$$2^5 \div 2^3 = 2^{5-3} = 2^2.$$

Or, in general, $a^m \div a^n = a^{m-n}$. **III.**

Also, since $\frac{2^2}{2^2} = 2^{2-2} = 2^0$ (Principle III), and $\frac{2^2}{2^2} = 1$, $\therefore 2^0 = 1$.

Or, since $2^2 \times 2^0 = 2^{2+0} = 2^2$ (Principle I), and $2^2 \times 1 = 2^2$, $\therefore 2^0 = 1$.

Or, in general, $a^0 = 1$. **IV.**

Also, since $\frac{2}{2^2} = 2^{1-2} = 2^{-1}$ (Principle III), and $\frac{2}{2^2} = \frac{2}{4} = \frac{1}{2}$, $\therefore 2^{-1} = \frac{1}{2}$.

Or, since $2^2 \times 2^{-1} = 2$ (Principle I), and $2^2 \times \frac{1}{2} = 2$, $\therefore 2^{-1} = \frac{1}{2}$.

Or, in general, $a^{-n} = \frac{1}{a^n}$. **V.**

That is, any number with a negative exponent is equal to the reciprocal of the same number with a numerically equal positive exponent.

Also, $(2^{\frac{1}{2}})^2 = 2$, since $(2^{\frac{1}{2}})^2 = 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$.

$\therefore 2^{\frac{1}{2}} = \sqrt{2}$ (extracting the square root of both members of the equation $(2^{\frac{1}{2}})^2 = 2$).

And $(2^{\frac{2}{3}})^3 = 2^2$, or $2^{\frac{2}{3}} = \sqrt[3]{2^2}$.

Or, in general,

$$(a^{\frac{m}{n}})^n = a^m, \text{ or } a^{\frac{m}{n}} = \sqrt[n]{a^m}. \qquad \textbf{VI.}$$

Ex. Show that $\frac{3^3}{3^2} = 3^1 = 3$; $\frac{3^2}{3^2} = 3^0 = 1$; $\frac{3}{3^2} = 3^{-1} = \frac{1}{3}$; $10^{-1} = 0.1$; $10^{-2} = 0.01$; $10^{-5} = 0.00001$.

Show that $8^{\frac{2}{3}} = 4$; $27^{\frac{2}{3}} = 9$; $2^{\frac{12}{4}} = 8$; $81^{\frac{3}{4}} = 27$; $32^{\frac{6}{5}} = 64$.

EXERCISE 26

1. How many figures are there in the square of a number of 3 figures? of 4 figures?

2. How many figures are there in 31^2? in 32^2?

3. How many figures are there in the cube of a number of 2 figures? of 3 figures?

4. How many figures are there in 30^3? in 32^3?

5. How many figures are there in the fourth power of a number of 3 figures? in the fifth power of a number of 2 figures?

6. How many figures are there in the cube of a number of 5 figures?

7. How many figures are there in the cube of a number of 4 figures?

8. How many figures are there in $\sqrt{5929}$? in $\sqrt{1038361}$?

9. How many figures are there in $\sqrt{0.04}$? in $\sqrt{0.36}$?

10. How many figures are there in $\sqrt{37.21}$? in $\sqrt{4.8841}$?

11. How many figures are there in $\sqrt[3]{1030301}$?

12. How many figures are there in $\sqrt[3]{10793861}$?

13. Show by multiplication that $(a+b)^2 = a^2 + 2ab + b^2$, and by use of this formula square 32 and 65.

14. Show by multiplication that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, and by use of the formula cube 41 and 98.

15. Square 234, first considering it as $230 + 4$ and then as $200 + 34$.

16. Show that no number ending in 2, 3, 7 or 8 can be a perfect square.

17. Prove that the cube of a number may end in any digits.

168. Square Root. The square root can often be determined by inspection if the number can readily be separated into prime factors. Thus:

$$32400 = 2^4 \times 3^4 \times 5^2.$$
$$\therefore \sqrt{32400} = 2^2 \times 3^2 \times 5 = 180.$$

Ex. By separating into prime factors find the square root of (*a*) 64, (*b*) 17424, (*c*) 7056, (*d*) 99225, (*e*) 680625, (*f*) 2800625, (*g*) 11025, (*h*) 81, (*i*) 1764, (*j*) 9801.

169. Since $43^2 = (40+3)^2 = 40^2 + 2 \times 40 \times 3 + 3^2 = 1849$, by reversing the process we can find $\sqrt{1849}$.

	$40^2 + 2 \times 40 \times 3 + 3^2$	$40 + 3$
	40^2	
$2 \times 40 + 3$	$2 \times 40 \times 3 + 3^2$	
	$2 \times 40 \times 3 + 3^2$	

170. In the above we notice that 40 is the square root of the first part. After subtracting 40^2 the remainder is $2 \times 40 \times 3 + 3^2$. The **trial divisor**, 2×40, is contained in the remainder 3 times. By adding 3 to 2×40 the **complete divisor**, $2 \times 40 + 3$, is formed. The complete divisor is contained exactly 3 times in the remainder. By this division the 3, the second figure of the root, is found.

171. The above is equivalent to the following:

	4 3
	1849
$40^2 =$	1600
$2 \times 40 + 3$	249
$= 83$	249

Separating 1849 into periods of two figures each (why?) we find that 1600 is the greatest square in 1800. 4 is therefore the first figure in the root. Subtracting 1600 and using 2×40 as the trial divisor, the next figure in the root is found to be 3. Completing the divisor by adding the 3, it is found to be exactly contained in the remainder. If the number contains more than two periods, the process is repeated.

172. The whole process of extracting the square root of a number is contained in the formula $(a+b)^2 = a^2 + 2ab + b^2$. The process of extracting the cube root is contained in $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. In general, the process of extracting any root can be derived from the corresponding power of $(a+b)$.

Ex. Extract the square root of 4529.29.

The a of the formula will always represent the part of the root already found and the b the next figure.

	6 7 3
$(a+b)^2 = a^2 + 2ab + b^2$	4529.29
a^2	$= 3600$
$2a = 120$	929.29
$b = 7$	
$(2a+b)b = 127 \times 7 =$	889.
$2a = 134$	40.29
$b = 0.3$	
$(2a+b)b = 134.3 \times 0.3 =$	40.29

In practice the work may be arranged as follows:

	6 7 3
	4529.29
	36
120	929.
127	889.
134	40.29
134.3	40.29

EXERCISE 27

1. In extracting the square root, why should the number be separated into periods of two figures each?

2. Where do you begin to separate into periods?

3. Separate each of the following into periods: 312, 4.162, 0.0125, 30000.4.

4. Will the division by $2a$ always give the next figure of the root?

5. Why is $2a$ called the trial divisor?

6. Why is $2a+b$ called the complete divisor?

7. In the above example explain how $2a$ can equal both 120 and 134.

8. Explain how $2ab+b^2$ can equal both 889 and 40.29.

9. Show how square root may be checked by casting out the 9's.

10. Extract the square root of each of the following, using the formula: (*a*) 2916, (*b*) 5.3361, (*c*) 65.61, (*d*) 0.003721, (*e*) 1632.16, (*f*) 289444, (*g*) 0.597529, (*h*) 103.4289, (*i*) 978121.

11. Extract the square root of 14400.

12. By first reducing to a decimal, extract the square root of $\frac{2}{3}$, $1\frac{1}{4}$, $\frac{3}{8}$, $1\frac{1}{7}$, $6\frac{4}{5}$.

13. By first making the denominator a perfect square, extract the square root of $\frac{2}{5}$, $\frac{3}{7}$, $\frac{7}{13}$, $\frac{5}{11}$.

$$\frac{2}{5}=\frac{10}{25};\text{ hence }\sqrt{\frac{2}{5}}=\sqrt{\frac{10}{25}}=\frac{3.162+}{5}=0.632+.$$

14. Which of the two methods given in Ex. 12 and 13 is to be preferred?

15. Extract to 0.01 the square root of 16, 1.6, 0.016.

173. Cube Root.

Since $37^3=(30+7)^3=30^3+3\times 30^2\times 7+3\times 30\times 7^2+7^3=50653$, by reversing the process we can find $\sqrt[3]{50653}$.

$$\begin{array}{r|l} & 30^3+3\times 30^2\times 7+3\times 30\times 7^2+7^3(30+7 \\ & 30^3 \\ \hline 3\times 30^2+3\times 30\times 7+7^2 & 3\times 30^2\times 7+3\times 30\times 7^2+7^3 \\ & 3\times 30^2\times 7+3\times 30\times 7^2+7^3 \\ \hline \end{array}$$

174. In the preceding we notice that 30 is the cube root of the first part.

After subtracting 30^3 the remainder is $3\times 30^2\times 7+3\times 30\times 7^2+7^3$. The **trial divisor**, 3×30^2, is contained in the remainder 7 times.

By adding $3\times 30\times 7+7^2$ the **complete divisor** is formed. This complete divisor is contained 7 times in the remainder. By this division the 7, the second figure in the root, is found.

175. The above is equivalent to the following:

$$\begin{array}{rr} & 3\ \ 7 \\ & 50653 \\ 30^3= & 27000 \\ \hline & 23653 \\ 3\times 30^2+3\times 30\times 7+7^2=3379 & \\ 3379\times 7= & 23653 \end{array}$$

Separating 50653 into periods of three figures each (Why?) we find that 27000 is the greatest cube in 50000. The first figure of the root is therefore 3. Subtracting 27000 and using 3×30^2 as the trial divisor, the next figure in the root (since 8, when the trial divisor is completed, proves to be too large) is found to be 7. Completing the divisor by adding $3\times 30\times 7+7^2$, it is found to be contained exactly 7 times in the remainder. If the number contains more than two periods, the process is to be repeated.

Ex. Extract the cube root of 362467.097.

As in extracting the square root of a number, the a of the formula will always represent the part of the root already found and the b the next figure.

$$\begin{array}{lr} & 7\ \ 1.\ \ 3 \\ (a+b)^3=a^3+3a^2b+3ab^2+b^3 & 362467.097 \\ a^3= & 343000 \\ \hline 3a^2=14700 & 19467.097 \\ b=\quad 1 & \\ (3a^2+3ab+b^2)=14911 & 14911 \\ \hline 3a^2=15123 & 4556.097 \\ b=\quad 0.3 & \\ (3a^2+3ab+b^2)=15186.99 & 4556.097 \\ \hline \end{array}$$

In practice the work may be arranged as follows:

	7 1. 3
	362467.097
	343
14700	19467.
14911	14911.
15123	4556.097
15186.99	4556.097

EXERCISE 28

1. Separate each of the following numbers into periods: 2500, 2.5, 3046.2971, 0.0125, 486521.3.

2. In extracting the cube root of 208527857, does the division by $3a^2$ give the second figure of the root correctly? Why is $3a^2$ called the trial divisor? Why is $3a^2+3ab+b^2$ called the complete divisor?

3. In the above example explain how $3a^2$ can equal both 14700 and 15123. Explain how $3a^2+3ab+b^2$ can equal both 14911 and 15186.99.

4. Show how cube root may be checked by casting out the 9's.

5. Extract the cube root of each of the following, using the formula: (*a*) 472729139, (*b*) 278.445077, (*c*) 1054.977832, (*d*) 19683, (*e*) 205379, (*f*) 25153.757.

6. Extract the cube root of $\frac{1331}{3375}$, $\frac{216}{343}$, $\frac{10648}{13824}$.

7. By reducing to a decimal, extract the cube root of $\frac{2}{3}$, $\frac{4}{9}$, $\frac{51}{56}$, $16\frac{2}{3}$, $33\frac{1}{3}$.

8. By first making the denominator a perfect cube, extract the cube root of $\frac{7}{8}$, $\frac{11}{14}$, $\frac{3}{7}$, $\frac{5}{9}$.

9. Find correct to 0.01 the cube root of 12.5, 125, 1.25.

MENSURATION

176. Certain measurements have been in very common use in the development of arithmetical knowledge from the earliest times.

177. The Babylonians and Egyptians used a great variety of geometrical figures in decorating their walls and in tile floors. The sense perception of these geometrical figures led to their actual measurement and finally to abstract geometrical reasoning.

178. The Greeks credited the Egyptians with the invention of geometry and gave as its origin the measurement of plots of land. Herodotus says that the Egyptian king, Sesostris (about 1400 B.C.), divided Egypt into equal rectangular plots of ground, and that the annual overflow of the Nile either washed away portions of the plot or obliterated the boundaries, making new measurements necessary. These measurements gave rise to the study of geometry (from *ge*, earth, and *metron*, to measure).

179. Ahmes, in his arithmetical work, calculates the contents of barns and the area of squares, rectangles, isosceles triangles, isosceles trapezoids and circles. There is no clew to his method of calculating volumes. In finding the area of the isosceles triangle he multiplies a side by half of the base, giving the area of a triangle whose sides are 10 and base 4 as 20 instead of 19.6, the result obtained by multiplying the altitude by half of the base. For the area of the isosceles trapezoid he multiplies a side by half the sum of the parallel bases, instead of finding the altitude and multiplying that by half the sum of the two parallel sides. He finds the area of the circle by subtracting from the diameter $\frac{1}{9}$ of its length and squaring the remainder. This leads to the fairly correct value of 3.1604 for π.

180. The Egyptians are also credited with knowing that in special cases the square on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. They were careful to locate their temples and other public buildings on north and south, In practice the work may be arranged as follows:

		7 1. 3
		362467.097
		343
14700		19467.
14911		14911.
15123		4556.097
15186.99		4556.097

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180. The Egyptians are also credited with knowing that in special cases the square on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. They were careful to locate their temples and other public buildings on north and south, and east and west lines. The north and south line they determined by means of the stars. The east and west line was determined at right angles to the other, probably by stretching around three pegs, driven into the ground, a rope measured into three parts which bore the same relation to each other as the numbers 3, 4 and 5. Since $3^2 + 4^2 = 5^2$, this gave the three sides of a right triangle.

181. The ancient Babylonians knew something of rudimentary geometrical measurements, especially of the circle. They also obtained a fairly correct value of π.

182. It was the Greeks who made geometry a science and gave rigid demonstrations of geometrical theorems.

183. The importance of certain measurements gives mensuration a prominent place in arithmetic to-day. The rules and formulæ of the present chapter will be developed without the aid of formal demonstration.

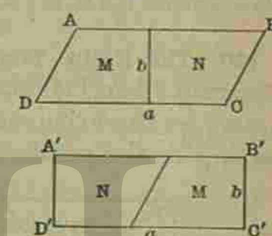
184. The Rectangle. If the unit of measure *CYXN* is 1 sq. in., then the strip *CYMD* contains 5×1 sq. in. $= 5$ sq. in., and the whole area contained in the three strips will be 3×5 sq. in., or 15 sq. in.

185. The dimensions of a rectangle are its **base** and **altitude**, and the area *is equal to the product of the base and altitude. That is, the number of square units in the area is equal to the product of the numbers that represent the base and altitude.* If we denote the area by A, the number of units in the base by b, and the numbers of units in the altitude by a, then $A = ab$ and any one of the three quantities, A, b, a, can be determined when the other two are given.

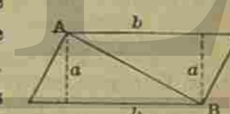
Thus, if the area of a rectangle is 54 sq. in. and the altitude is 6 in., the base can be determined from $6b = 54$, or $b = 9$ in.

186. If the dimensions of a rectangle are equal, the figure is a square and the area is equal to the second power of a number denoting the length of its side, or $A = a^2$. For this reason the **second power** of a number is called its **square**.

187. The Parallelogram. Any parallelogram has the same area as a rectangle with the same base and altitude, as can be shown by dividing the parallelogram *ABCD* into two parts, *M* and *N*, and placing them as in *A′B′C′D′*, thus forming a rectangle with the same base and altitude as the given parallelogram. Therefore, *the area of a parallelogram is equal to the product of its base and altitude*, or $A = ab$.

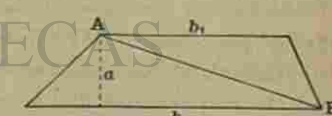


188. The Triangle. Since the line *AB* divides the parallelogram into two equal triangles with the same base and altitude as the parallelogram, the area of the triangle is equal to one half of the area of the parallelogram. But the area of the parallelogram is equal to the product of its base and altitude. Therefore, *the area of the triangle is one half the product of its base and altitude*, or $A = \frac{1}{2}ab$.



Thus, the area of a triangle with base 6 in. and altitude 5 in. is $\frac{1}{2}$ of 6×5 sq. in. $= 15$ sq. in.

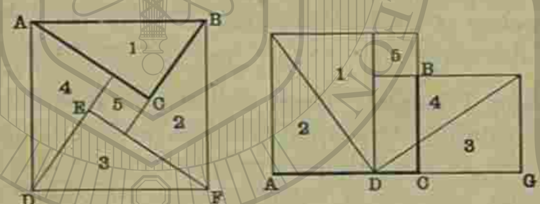
189. The Trapezoid. The line *AB* divides the trapezoid into two triangles whose bases are the upper and lower bases of the trapezoid and whose common altitude is the



altitude of the trapezoid. The areas of the triangles are respectively $\frac{1}{2}ab_1$ and $\frac{1}{2}ab_2$, and since the area of the trapezoid equals the sum of the areas of the triangles, therefore the area of the trapezoid is $\frac{1}{2}ab_1 + \frac{1}{2}ab_2 = \frac{1}{2}a(b_1 + b_2)$, or *the area of a trapezoid is equal to one half of the product of its altitude and the sum of the upper and lower bases*, or $A = \frac{1}{2}a(b_1 + b_2)$.

Thus, the area of a trapezoid whose bases are 20 ft. and 17 ft. and whose altitude is 6 ft. is $\frac{1}{2}$ of $6 \times (20 + 17)$ sq. ft. = 111 sq. ft.

190. The Right Triangle. The Hindu mathematician Bhaskara (born 1114 A.D.) arranged the figure so that the square on the hypotenuse contained four right triangles, leaving in the middle a small square whose side equals



the difference between the sides of the right triangle. In a second figure the small square and the right triangles were arranged in a different way so as to make up the squares on the two sides. Bhaskara's proof consisted simply in drawing the figure and writing the one word "Behold." From these figures it is evident that *the area of the square constructed on the hypotenuse will equal the sum of the areas of the squares constructed on the two sides.* In general, if the sides of the right triangle are a and b and the hypotenuse is c, $a^2 + b^2 = c^2$.

Thus, the hypotenuse of the right triangle whose sides are 5 and 12 is $\sqrt{25 + 144} = 13$.

This theorem is known by the name of the Pythagorean theorem, because it is supposed to have been first proved by the Greek mathematician Pythagoras, about 500 B.C.

191. If either side and the hypotenuse of a right triangle are known, the other side can be found from the equation $a^2 + b^2 = c^2$.

Thus, if one side is 3 and the hypotenuse is 5, the other side is $\sqrt{25 - 9} = 4$.

EXERCISE 29

1. The two sides of a right triangle are 6 in. and 8 in. Find the length of the hypotenuse.

2. Find the area of an isosceles triangle, if the equal sides are each 10 ft. and the base is 4 ft.

3. Find the area of an isosceles trapezoid, if the bases are 10 ft. and 18 ft. and the equal sides are 8 ft.

4. What is the area in hectares, etc., of a field in the form of a trapezoid of which the bases are 475^m and 580^m and the altitude is 1270^m?

5. Show that the altitude of an equilateral triangle, each of whose sides is a, is $\frac{a}{2}\sqrt{3}$.

6. The hypotenuse of a right triangle with equal sides is 10 ft. Find the length of the two equal sides.

7. The diagonal of a square field is 80 rd. How many acres does the field contain?

8. Find correct to square centimeters the area of an equilateral triangle each side of which is 1^m in length.

192. The Circle. If the circumference (c) and the diameter (d) of a number of circles are carefully measured, and if the quotient $\frac{c}{d}$ is taken in each case, the quotients will be found to have nearly the same value. If absolutely correct measurements could be made, the quotient in each case would be the same and equal to 3.14159+, *i.e.* the ratio of the circumference of a circle to its diameter is the same for all circles. The ratio is denoted by the Greek letter π (pī). The value of π found in geometry is 3.14159+. In common practice π is taken as 3.1416. The value of π cannot be exactly expressed by any number, but can be found correct to any desired number of decimal places.

193. Since $\frac{c}{d}=\pi$ and $d=2r$, where r stands for the radius of the circle, then $c=\pi d=2\pi r$, and $d=\frac{c}{\pi}$ and $r=\frac{c}{2\pi}$. Hence, if the radius, diameter or circumference of a circle is known, the other parts can be found.

Thus, the circumference of a circle whose radius is 10 in. is $2 \times 3.1416 \times 10$ in. = 62.832 in.

EXERCISE 30

1. Find the circumference of a circle whose diameter is 20 in.

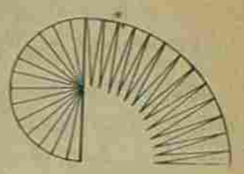
2. Find the radius of a circle whose circumference is 250 ft.

3. If the length of a degree of the earth's meridian is 69.1 mi., what is the diameter of the earth?

4. If the radius of a circle is 8 in., what is the length of an arc of 15° 20′?

5. The diameter of a circle is 10 ft. How many degrees are there in an arc 16 ft. long?

194. The Area of a Circle. The circle may be divided into a number of equal figures that are essentially triangles. The sum of the bases of these triangles is the circumference of the circle, and the altitudes are radii of the circle. Treating these figures as triangles, their areas will be $\frac{1}{2}c \times r$. Therefore, since $c=2\pi r$, $A=\frac{1}{2}$ of $2\pi r \times r=\pi r^2$. It is proved in geometry that this result is exactly correct.



The area of a circle whose radius is 5 ft. is $3.1416 \times 5 \times 5$ sq. ft. = 78.54 sq. ft.

EXERCISE 31

1. Find the area of a circle whose radius is 10 in.

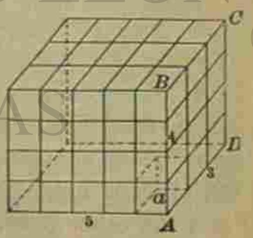
2. Find the area of a circle whose circumference is 25 ft.

3. Find the radius of a circle whose area is 100 sq. ft.

4. The areas of two circles are 60 sq. ft. and 100 sq. ft. Find the number of degrees in an arc of the first that is equal in length to an arc of 45° in the second.

5. Find the side of a square that is equal to a circle whose circumference is 50 in. longer than its diameter.

195. The Volume of a Rectangular Parallelopiped. If the unit of measure a is 1 cu. in., then the column AB is 4 cu. in., and the whole section $ABCD$ will contain 3 of these columns, or 3×4 cu. in. Since there are five of these sections in the parallelopiped, the entire volume (V) is $5 \times 3 \times 4$ cu. in., or 60 cu. in. Therefore, *the vol-*



ume of a rectangular parallelopiped is equal to the products of its three dimensions. That is, the number of cubic units in the volume is equal to the product of the three numbers that represent its dimensions.

196. If the dimensions of the rectangular parallelopiped are a, b and c, it can be shown in the same way that $V = abc$. Any of these four quantities, V, a, b, c, can be determined when the other three are known.

Ex. If the volume of a rectangular parallelopiped is 36 cu. in. and two of the dimensions are 6 in. and 2 in., the third dimension is $\frac{36}{6 \times 2} = 3$. $\therefore$ 3 in. is the other dimension.

197. If the dimensions of a rectangular parallelopiped are equal, the figure is a cube and the volume is equal to the third power of the number denoting the length of its edge (a), or $V = a^3$. For this reason the **third power** of a number is called its **cube**.

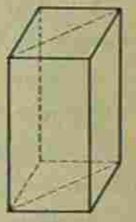
198. It is proved in geometry that any parallelopiped has the same volume as a rectangular parallelopiped with the same base and altitude.

EXERCISE 32

1. Find the volume of a cube 3 in. on an edge.
2. Find the volume of a rectangular parallelopiped whose edges are 3^{cm}, 5^{cm} and 11^{cm}.
3. The volume of a rectangular parallelopiped is 100 cu. in. The area of one end is 20 sq. in. Find the length.
4. How many cubic feet of air are there in a room 12 ft. 6 in. long, 10 ft. 8 in. wide and 9 ft. high?
5. Find the weight of a rectangular block of stone at 135 lb. per cubic foot, if the length of the block is $9\frac{1}{2}$ ft. and the other dimensions are 2 ft. and 5 ft.

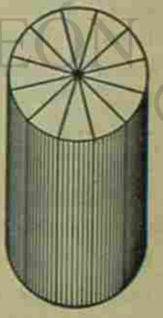
6. If a cubic foot of water weighs 1000 oz., find the edge of a cubical tank that will hold 2 T.
7. Show why the statement that the volume of a rectangular parallelopiped is equal to the product of its three dimensions is the same as the statement that its volume is equal to the product of its altitude and the area of its base.

199. The Volume of a Prism. A rectangular parallelopiped can be divided into two equal triangular prisms with the same altitude and half the base. Hence, the volume of the prism is half the volume of the parallelopiped. But the base of the parallelopiped is twice the base of the prism, therefore, *the volume of a triangular prism is equal to the product of its altitude and the area of its base.*



200. Since any prism can be divided into triangular prisms, as in the figure, it follows that *the volume of any prism is equal to the product of its altitude and the area of its base.*

201. The Volume of a Cylinder. The cylinder may be divided into a number of solids that are essentially prisms, as indicated in the figure. The sum of the bases of these prisms is the base of the cylinder and the altitude of the prisms is the same as the altitude of the cylinder. Therefore, *the volume of a cylinder is the product of its altitude and the area of its base.* $V = a \times \pi r^2$.



EXERCISE 33

1. Find the volume of a prism with square ends, each side measuring 1 ft. 8 in., and the height being 12 ft.

2. Find the volume of a prism whose ends are equilateral triangles, each side measuring 11 in. and the height being 20 in.

3. Find the volume of a cylinder if the diameter of its base is 20 in. and the altitude is 30 in.

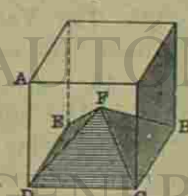
4. How many cubic yards of earth must be removed in digging a well 45 ft. deep and 3 ft. in diameter?

5. A cubic foot of copper is to be drawn into a wire $\frac{1}{10}$ of an inch in diameter. Find the length of the wire.

6. How many revolutions of a roller $3\frac{1}{2}$ ft. in length and 2 ft. in diameter will be required in rolling a lawn $\frac{3}{4}$ of an acre in extent.

7. Show how to find the surface of a cylinder by dividing it into figures that are essentially parallelograms. Show how to find the surface of a prism.

202. The Volume of a Pyramid. Let AB be a cube and F the middle point of the cube, then by connecting F with B, C, D and E a pyramid with a square base is formed. It is evident that by drawing lines from F to each of the vertices, the cube will consist of six such pyramids. Hence, the volume of the pyramid is $\frac{1}{6}$ of the volume of the cube. The volume of the cube is the product of its altitude and the area of its base $BCDE$. Therefore, the volume of the pyramid is $\frac{1}{6}$ of the product of the altitude of the cube and the area of its base. But the base



of the pyramid is the base of the cube and its altitude is $\frac{1}{2}$ of the altitude of the cube, *hence, the volume of the pyramid is one third of the product of its altitude and the area of its base.* In geometry this is proved true of *any* pyramid.

Ex. If the altitude of a pyramid is 45^m, and a side of its square base is 60^m, its volume is $\frac{1}{3}$ of $45 \times (60^2)^{m^2} = 54000^{m^3}$.

203. The Volume of a Cone. The cone may be divided into a number of equal figures that are essentially pyramids as indicated in the figure. The sum of the bases of these pyramids is the base of the cone, and their altitudes are the same as the altitude of the cone. Therefore, *the volume of a cone is equal to one third of the product of its altitude and the area of its base.*

Ex. If the altitude of a cone is 10 ft. and the radius of its base 4 ft., its volume is $\frac{1}{3}$ of $10 \times 3.1416 \times 4^2$ cu. ft. = 167.55 cu. ft.

EXERCISE 34

1. Show that the pyramid with a square base can be divided into two equal pyramids with triangular bases and the same altitude as the original pyramid, and hence show how any pyramid may be similarly divided.

2. Find the volume of a cone if the diameter of the base is 16 in. and the altitude is 12 in. Find the volume if the diameter of the base is 16 in. and the slant height is 12 in.

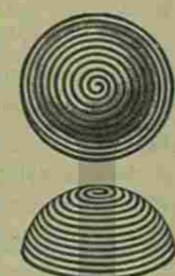
3. Show how to find the surface of a cone by dividing it into figures that are essentially triangles. Show how to find the surface of a pyramid.

4. Find the volume of a pyramid if the area of its base is 4 sq. ft. and its altitude is 2 ft. Find the volume if the base is 2 feet square and the slant height is 2 ft.

5. How much canvas is necessary for a conical tent 8 ft. high, if the diameter of the base is 8 ft.?

6. The radius of a cylinder is 8 ft. and its altitude is 10 ft. Find the altitude of a cone with the same base and volume.

204. The Surface of a Sphere. The surface of a sphere is proved in geometry to be equal to the area of 4 great circles or $4\pi r^2$, r being the radius of the sphere. This can be shown by winding a firm cord to cover a hemisphere and a great circle as indicated in the figure. It will be found that twice as much cord is used to cover the hemisphere as the great circle, therefore, to cover the whole sphere 4 times as much would be required.



Ex. A sphere with a radius of 6 in. has a surface of $4 \times 3.1416 \times 6^2$ sq. in. $= 452.39+$ sq. in.

205. The Volume of a Sphere. The sphere may be divided into a number of figures that are essentially pyramids, as indicated in the figure. The sum of the bases of these pyramids is the surface of the sphere and the altitude of each pyramid is its radius. Therefore, the volume of all these pyramids is equal to $\frac{1}{3}r \times 4\pi r^2 = \frac{4}{3}\pi r^3$.

Ex. The volume of a sphere whose radius is 3 in. is $\frac{4}{3} \times 3.1416 \times 3^3$ cu. in. $= 113.1$ cu. in.

206. Board Measure. In measuring lumber the **board foot** is used. It is a board 1 ft. long, 1 ft. wide and 1 in. or less thick. Lumber more than 1 in. thick is measured by the number of square feet of boards 1 in. thick to which it is equal.

Thus, a board 10 ft. long, 1 ft. wide and $1\frac{1}{2}$ in. thick, contains 15 board feet.

Lumber is usually sold by the 1000 board feet. A quotation of $17 per M, means $17 per 1000 board feet.

EXERCISE 35

1. Find the cost of 12 boards 16 ft. long, 6 in. wide, and 1 inch thick at $18 per M.

2. How many board feet are there in a stick of timber 16 ft. by 16 in. by 10 in.?

3. How much is a stick of timber 15 ft. by 2 ft. by 1 ft. 4 in. worth at $22 per M?

4. How many board feet are used in laying the flooring of two rooms, each 32 ft. by 20 ft., allowing $\frac{1}{3}$ for waste in sawing and in tongue and groove.

5. What is the cost of 25 $2\frac{1}{2}$-in. planks 16 ft. long by 1 ft. wide at $22.50 per M?

6. What is the cost of 15 joists 12 ft. by 10 in. by 4 in. at $23 per M?

207. Wood Measure. The unit of wood measure is the **cord**. The cord is a pile of wood 8 ft. by 4 ft. by 4 ft.

A pile of wood 1 ft. by 4 ft. by 4 ft. is called a **cord foot.**

A cord of stove wood is 8 ft. long by 4 ft. high. The length of stove wood is usually 16 in.

EXERCISE 36

1. Find the number of cords of wood in a pile 32 ft. by 4 ft. by 4 ft.

2. At $5.75 per cord, how much will a pile of wood 52 ft. by 4 ft. by 4 ft. cost?

3. How much will a pile of stove wood 94 ft. long 4 ft. high be worth at $2.75 per cord?

208. Carpeting. A yard of carpet refers to the running measurement, regardless of the width. The cheaper grades of carpet are usually 1 yd. wide, and the more expensive, such as Brussels, Wilton, etc., are $\frac{3}{4}$ of a yard wide.

In carpeting, it is usually necessary to allow for some waste in matching the figures in patterns. Dealers count this waste in their charges. In computing the cost of carpets, dealers charge the same for a fractional width as for a whole one.

Carpets may often be laid with less waste one way of the room than the other; hence, it is sometimes best to compute the cost with the strips running both ways, and by comparison determine which involves the smaller waste.

EXERCISE 37

1. How many yards of Brussels carpet $\frac{3}{4}$ of a yard wide will be required to cover the floor of a room 15 ft. by 13 ft. 6 in., the waste in matching being 4 in. to each strip except the first? Which will be the more economical way to lay the carpet?

2. How much will it cost to cover the same room with Brussels carpet if a border $\frac{5}{8}$ of a yard wide is used, the carpet and border being $1.25 per yard, and the waste being 4 in. to each strip of carpet except the first, and $\frac{5}{8}$ of a yard of border at each corner?

3. How much will it cost to cover the same room with ingrain carpet 1 yd. wide, at 67$\frac{1}{2}$ ct. per yard, the waste being 6 in. to each strip except the first?

4. At 1.12\frac{1}{2}$ per yard, how much will it cost to carpet a flight of stairs of 14 steps, each step being 8 in. high and 11 in. wide?

5. A room is 17 ft. by 14 ft. 9 in. Will it be cheaper to run the strips lengthwise or across the room? If the room is covered with carpet $\frac{3}{4}$ of a yard wide at $1.35 per yard, how much will it cost? Allow 1 yd. for waste in matching.

209. Papering. Wall paper is sold in single rolls 8 yd. long, or in double rolls 16 yd. long. It is usually 18 in. wide.

There is considerable waste in cutting and matching paper. Whole rolls may be returned to the dealer, but part of a roll will not usually be taken back. Paper for border is usually sold by the yard.

EXERCISE 38

1. How many rolls of paper and how many yd. of border are used in papering the walls and ceiling of a room 14 ft. by 13 ft. and 8 ft. high above the baseboard, deducting $\frac{1}{2}$ of a roll for each of 2 windows and 2 doors, the width of the border being 18 in., and 1 roll being allowed for waste in matching?

2. How much will it cost to paper the room mentioned in Ex. 1 if the paper is 12 ct. a roll and the border is 5 ct. a yd.? The paper hanger works 8 hr. at 30 ct. an hr.

3. At 25 ct. per roll, how much will it cost to paper the walls and ceiling of a room 18 ft. square and 9 ft.

high above the baseboard, allowing ½ of a roll for each of 2 doors and 3 windows, the border being 18 in. wide and costing 12 ct. a yd.? The paper hanger works 11 hr. at 30 ct. an hr., and 1½ rolls are allowed for waste in matching.

210. Painting and Plastering. The **square yard** is the unit of painting and plastering.

There is no uniform practice as to allowances to be made for openings made by windows, doors, etc., and the baseboard. To avoid complications, a definite written contract should always be drawn up.

EXERCISE 39

1. How much will it cost to plaster the walls and ceiling of a room 15 ft. by 13 ft. 6 in., and 9 ft. high, at 27½ ct. per square yard, deducting half of the area of 2 doors, each 7 ft. by 3½ ft., and 2 windows, each 6 ft. by 3¼ ft.?

2. How much will it cost to paint the walls and ceiling of the same room at 12½ ct. per square yard, the same allowance being made for openings?

3. At 20 ct. per square yard, how much will it cost to paint a floor 18 ft. by 16 ft. 6 in.?

4. Allowing ⅕ of the surface of the sides for doors, windows and baseboard, how much will it cost to plaster the sides and ceiling of a room 22 ft. by 18 ft. and 9½ ft. high, at 22½ ct. per square yard?

211. Roofing and Flooring. A **square** 10 ft. on a side, or 100 sq. ft., is the unit of roofing and flooring.

The average shingle is taken to be 16 in. long and 4 in. wide. Shingles are usually laid about 4 in. to the weather.

Allowing for waste, about 1000 shingles are estimated as needed for each square, but if the shingles are good, 850 to 900 are sufficient. There are 250 shingles in a **bundle**.

EXERCISE 40

1. At $8.60 per square, how much will it cost to shingle a roof 50 ft. by 22½ ft. on each side?

2. How much will it cost to lay a hard-wood floor in a room 30 ft. by 28 ft., if the labor, nails, etc. cost $22.50, lumber being $28 per M, and allowing 57 sq. ft. for waste?

3. Allowing 900 shingles to the square, how many bundles will be required to shingle a roof 70 ft. by 28 ft. on each side? How much will the shingles cost at $3.75 per M?

4. At $12.50 per square, how much will the slate for a roof 40 ft. by 24 ft. on each side cost?

212. Stonework and Masonry. The **cubic yard** or the **perch** is the unit of stonework.

A **perch of stone** is a rectangular solid 16½ ft. by 1½ ft. by 1 ft., and therefore contains 24¾ cu. ft.

A common brick is 8 in. by 4 in. by 2 in. Bricks are usually estimated by the thousand, sometimes by the cubic foot, 22 bricks laid in mortar being taken as a cubic foot.

There is no uniformity of practice in making allowances for windows and other openings. There should be a definite written contract with the builder covering this point. The corners, however, are counted twice on account of the extra work involved in building them. It is also generally considered that the work around openings is more difficult, so that allowance is frequently made here.

EXERCISE 41

1. If 60 ct. per cubic yard was paid for excavating a cellar 30 ft. by 20 ft. by 7 ft., and $4.75 a perch was paid for building the four stone walls, 18 in. thick and extending 2 ft. above the level of the ground, what was the total cost?

2. How many bricks will be used in building the walls of a flat-roofed building 90 ft. by 60 ft. and 20 ft. high, if the walls are 18 in. thick and 500 cu. ft. are allowed for openings?

3. How much will it cost to build the walls described in Ex. 2, if the bricks are $8.50 per M, and the mortar and brick-laying cost $3.50 per M?

4. How many perch of stone will be needed for the walls of a cellar 30 ft. by $22\frac{1}{2}$ ft. and 9 ft. deep from the top of the wall, the wall being 18 in. thick? How many perch will be needed for a cross wall of the same thickness, allowing for half of a door 7 ft. by 4 ft.? How much will the stone cost at $4.50 a perch?

213. Contents of Cisterns, Tanks, etc. The **gallon** or the **barrel** is the unit of measure for cisterns, tanks, etc.

The liquid gallon contains 231 cu. in. and the barrel $31\frac{1}{2}$ gal.

EXERCISE 42

1. How many gallons of water will a tank 10 ft. long, 3 ft. wide and 3 ft. deep contain? How many barrels?

2. How many gallons of water will a cistern 10 ft. deep and 10 ft. in diameter contain? How many barrels?

3. How many barrels will a cylindrical tank 5 ft. high and 3 ft. in diameter contain?

4. How many barrels of oil will a tank 40 ft. long and 6 ft. in diameter contain?

5. Show that to find the approximate number of gallons in a cistern it is necessary only to multiply the number of cubic feet by $7\frac{1}{2}$ and subtract from the product $\frac{1}{400}$ of the product. Apply this method to each of the above exercises.

6. How many gallons will a cask contain, the bung diameter being 24 in., the head diameter 20 in. and the length 34 in.?

Suggestion. The average or mean diameter is $\frac{24\text{ in.} + 20\text{ in.}}{2} = 22$ in.

214. Measuring Grain in the Bin, Corn in the Crib, etc. There are 2150.42 cu. in. in every bushel, stricken measure, and 2747.71 cu. in. in every bushel, heaped measure.

EXERCISE 43

1. How many bushels of wheat does a bin 8 ft. by 7 ft. by 6 ft. contain?

2. Show that multiplying by 0.8 will give the approximate number of stricken bushels in any number of cubic feet, and dividing by 0.8 will give the approximate number of cubic feet in any number of stricken bushels.

3. Show that multiplying by 0.63 will give the approximate number of heaped bushels in any number of cubic feet, and dividing by 0.63 will give the approximate number of cubic feet in any number of heaped bushels.

4. How deep must a bin 10 ft. by 8 ft. be to hold 500 bushels of wheat?

5. A farmer builds a corncrib 20 ft. long, 10 ft. high, 8 ft. wide at the bottom and 12 ft. wide at the top. How many heaped bushels of corn in the ear will the crib hold when level full? If the ridge of the roof is 3 ft. above the top level, how many bushels will the crib hold when filled to the ridge?

Suggestion. The average width of the crib is $\frac{12 \text{ ft.} + 8 \text{ ft.}}{2} = 10$ ft.

6. How many stricken bushels of shelled corn are there in the above crib if 3 half bushels of ears make one bushel of shelled corn?

215. Measuring Hay in the Mow or Stack. The only correct way to measure hay is to weigh it. However, it is sometimes convenient to be able to estimate the number of tons in a mow or stack. The results of such estimations can be only approximately correct, as different kinds of hay vary in weight. In well-settled mows or stacks, as nearly as can be estimated, 15 cu. yd. make one ton. When hay is baled, 10 cu. yd. make a ton.

EXERCISE 44

1. Approximately how many tons of hay are there in a mow 40 ft. by 22 ft. and 15 ft. deep?

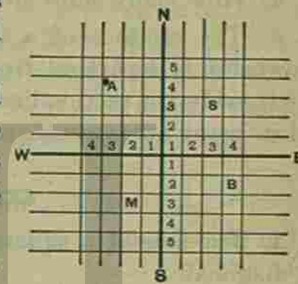
2. Approximately how many tons of hay are there in a circular stack 21 ft. high and averaging 80 ft. in circumference?

3. Approximately how many tons of hay are there in a rick averaging 35 ft. long, 15 ft. wide and 20 ft. high?

216. Land Measure. The unit of land measure is the acre. In the Eastern states, the land was divided, as convenient, when settled, and the description of tracts of land refer to

such natural objects as near-by bowlders, ponds, established roads, etc. But all states whose lands have been surveyed since 1802 are divided by a system of meridians and parallels into **townships** 6 miles square. Each township contains 36 square miles or **sections**. Each section contains 2 **half sections** or 4 **quarter sections**.

Public lands are located with reference to a north and south line called the **principal meridian** and an east and west line called the **base line**. The north and south rows of townships are called **ranges** and these rows are numbered from the principal meridian. The townships are numbered from the base line. A township is therefore designated by its number and the number of its range.



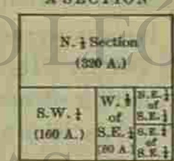
Thus, *A* is township 4 N., Range 3, W. What is *B*? *M*? *S*?

The 36 sections of a township are numbered as in the following diagram. The corners of all sections are permanently marked by stones, or otherwise.

A TOWNSHIP

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

A SECTION



The divisions of sections into half sections, quarter sections, etc., are shown in the diagram.

Thus, the N. E. ¼ of N. E. ¼, section 6, means the northeast quarter of the northeast quarter of section 6.

EXERCISE 45

1. How many acres are there in a section? In the S. W. $\frac{1}{4}$ of S. W. $\frac{1}{4}$, section 16? In S. $\frac{1}{2}$ of N. E. $\frac{1}{4}$, section 36? Locate these sections.

2. What will be the cost of a quarter section of land at $55 an acre?

3. How many rods of fence are necessary to inclose a quarter section?

4. How many acres are there in a township?

5. The sections of a township are separated and the township is separated from adjacent townships by a road 45 ft. wide, the section lines being in the middle of the road. How many acres are there in the roads of the township?

EXERCISE 46

1. The side of a square is 100 ft. Find the length of a diagonal.

2. One side of a right-angled triangle is 16 yd. and the other side is $\frac{3}{5}$ of the hypotenuse; what is the length of the hypotenuse?

3. Find the volume of a pyramid whose base and faces are all equilateral triangles with sides 10 in. long.

4. The largest pyramid in the world has a square base with sides 764 ft. Its four faces are equilateral triangles. Find the number of acres covered by its base, the number of square yards in its four faces, and the height of the pyramid.

5. A cistern 22 ft. long, 10 ft. wide and 8 ft. deep is to be filled with water from a well 8 ft. in diameter and 40 ft. deep. If no water flows into the well while filling the cistern, find how far the water in the well is lowered.

6. Two persons start from the same place at the same time. One walks due east at the rate of 3 mi. an hour, and the other due south at the rate of $3\frac{1}{2}$ mi. an hour. In how many hours will they be 30 mi. apart?

7. What is the circumference of the earth if its diameter is 7916 mi.?

8. Air being 0.00129206 as heavy as water, find in kilograms the weight of the air in a room 23^m long, 16^m wide and 10^m high.

9. A rectangular sheet of tin of uniform thickness is 85^{cm} wide and 2.7^m long, and weighs 356^g. Find its thickness if tin is 7.3 times as heavy as water.

10. A plate of iron weighs 277.54^{Kg}, and is 137^{cm} long, 643^{mm} wide, 43.1^{mm} thick. How much heavier than water is iron?

11. A tank is 2^m long, 5^{dm} wide and 8^{cm} deep. How many liters of water will it contain, and how much will the water weigh?

12. Sulphuric acid is 1.84 times as heavy as water. How many kilograms will a tank hold that is 2^m long, 75^{cm} wide and 50^{cm} deep?

13. A block of marble is 2 ft. long, 10 in. wide and 8 in. thick. What is the edge of a cubical block of equal volume?

14. If 1 T. of hard coal occupies a space of 36 cu. ft., how many tons will a bin 10 ft. long, $7\frac{1}{2}$ ft. wide and 9 ft. deep hold?

15. How much space will a car load of hard coal consisting of 38 T. 14 cwt. 75 lb. occupy, if one ton occupies 36 cu. ft.?

16. How long must a bin 20 ft. wide and 20 ft. deep be to hold the above car load of coal?

17. Find correct to 0.001 the diagonal of a square whose side is 10 in., and the diagonal of a cube whose edge is 10 in.

18. What will be the expense of painting the walls and ceiling of a room whose height is 10 ft. 4 in., length 16 ft. 6 in. and width 12 ft. 3 in., at 15 ct. per square yard?

19. At 11 ct. per square foot, how much will it cost to make a cement walk 5 ft. wide around a school yard in the shape of a rectangle, 18 rd. by 26 rd.?

20. Two corridors of a public building intersect at right angles near the center of the building. If the corridors are 160 ft. and 140 ft. long respectively, and 20 ft. wide, how much will it cost to cover them with a hard-wood floor at $24 per thousand feet?

21. At $18 per M, how much will it cost to cover the floor of a barn 30 ft. long and 20 ft. wide with 2-inch planks?

22. How much will it cost to fence the school yard mentioned in Ex. 19, with 1-inch boards, 6 in. wide, at $17.50 per M; the fence to be 4 boards high and built 2 ft. inside the walk?

23. How many board feet are there in 150 rafters, 14 ft long, 4 in. wide and 2 in. thick?

24. How many bunches of shingles will be required to shingle a barn with a roof 60 ft. long and rafters 18 ft. long, the shingles being laid 4 in. to the weather with a double row at the bottom?

25. What is the value of a log that will cut 36 1-inch boards, each 16 ft. long and 12 in. wide at $1\frac{3}{4}$ ct. per square foot?

26. How many board feet are there in a stick of timber $18\frac{1}{2}$ ft. long, 16 in. wide and 12 in. thick?

27. How many bricks will be used in building the walls of a building 120 ft. long, 60 ft. wide and 45 ft. high, outside measurement, if the walls are 18 in. thick and no allowance is made for doors and windows?

28. How many cubic centimeters of lead are there in a piece of lead pipe 1^m long, the outer diameter being 5^{cm}, and the thickness of the lead being 10^{mm}?

29. A race track 30 ft. wide with semicircular ends is constructed in a field 1050 ft. by 400 ft. Find the inside and outside lengths of the track. Also find the area of the track and the area of the field inside the track.

30. Find the volume and convex surface of a right cone, the diameter of the base being 16 in. and the altitude 18 in.

31. Find the volume and surface of a sphere whose diameter is 6 in.

32. Find the least possible loss of material in cutting a cube out of a sphere of wood 9 in. in diameter.

33. Find the least possible loss of material in cutting a sphere out of a cubical block of wood with edges 9 in. long.

34. Find the cost of making a road 200 yd. in length and 24 ft. wide; the soil being first excavated to the depth of 14 in., at a cost of 20 ct. per cubic yard; crushed stone being then put in 8 in. deep at a cost of 40 ct. per cubic yard, and gravel placed on top 6 in. thick at a cost of 45 ct. per cubic yard.

35. A map of Kansas is made on a scale of 1 in. to 100 mi. The map measures 4 in. by 2 in. Find the area of the state.

GRAPHICAL REPRESENTATIONS

217. Graphical methods of representing relations between different measurements are so extensively used in many lines of work that it seems best to give a brief treatment of the subject here. Such graphical representations as are given in the following exercises show relations pictorially in a much clearer manner than can be shown by a mere statement of figures.

Ex. **1.** Explain graphically the relation between an inch and a centimeter. The two lines drawn accurately to scale represent graphically the relation between the inch and the centimeter.

1 in.

1 cm.

Ex. **2.** Draw a line 1.5 in. long and find the number of centimeters in it.

Ex. **3.** Explain graphically the relation between the pound and the kilogram, given $1^{Kg} = 2.2$ lb.

1 lb.

$1^{Kg} = 2.2$ lb.

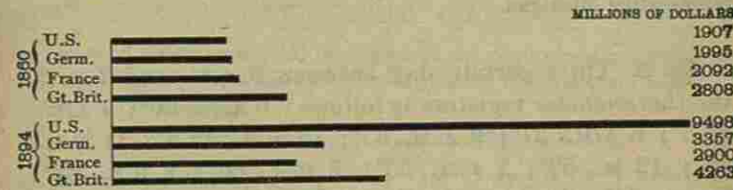
Ex. **4.** Explain graphically the relation between a pint and a liter, given $1^{l} = 1.76$ pt.

Ex. **5.** From a diagram find (*a*) the number of centimeters in 4 in., (*b*) the number of liters in a gallon, (*c*) the number of pounds in 5^{Kg}.

Ex. **6.** The values of manufactures produced in the United States, Germany, France and Great Britain in 1860 were \$1907000000, \$1995000000, \$2092000000,

\$2808000000 respectively, and in 1894 they were \$9498000000, \$3357000000, \$2900000000, \$4263000000 respectively.

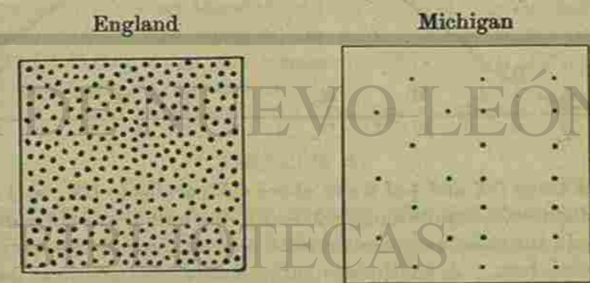
These facts may be represented graphically as follows:



These measurements, drawn accurately to a scale, show at a glance the comparative growth in manufactures produced in the different countries mentioned from 1860 to 1894.

Ex. **7.** The areas of England and Michigan are 50839 and 58915 square miles respectively. The populations are approximately 31000000 and 2421000. Represent graphically the comparative sizes and the comparative density in population of the two.

The square roots of the numbers representing the areas correct to units' place are 225 and 243 respectively. The ratio between these



two numbers reduces to 5 to 5.4. If some convenient unit of measure be taken, and squares be constructed with sides equal to 5 and 5.4 of these units, these squares will represent graphically the comparative areas. The comparative density in population will be represented by the number of dots that appear in each square, it being assumed that a dot represents 100000 in population. There will then be 310 dots in the square representing England and 24 in the square representing Michigan.

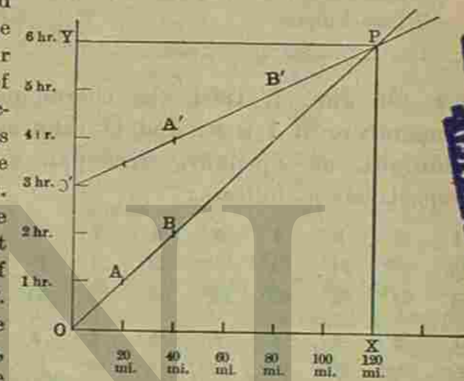
Ex. 8. On a certain day between 6 A.M. and 7 P.M. the thermometer registers as follows: 6 A.M., 20°; 7 A.M., 22.5°; 8 A.M., 27°; 9 A.M., 35°; 10 A.M., 42.5°; 11 A.M., 48°; 12 M., 52°; 1 P.M., 55°; 2 P.M., 60°; 3 P.M., 62°; 4 P.M., 60°; 5 P.M., 50°; 6 P.M., 42°; 7 P.M., 35°. Illustrate graphically this variation in temperature.

Draw two straight lines perpendicular to each other. Measure off on the horizontal line OX equal spaces, each representing 1 hr., and on the perpendicular line OY equal spaces, each one representing 10°. The temperature at 6 A.M. is shown at O; at 7 A.M. at A, a distance

of 1 unit along OX and $\frac{1}{4}$ of a unit above OX parallel to OY; at 8 A.M. at B, a distance of 2 units along OX and .7 of a unit above OX parallel to OY. In the same way points may be located showing the temperature at each hour. A continuous curve drawn through these points is the **temperature curve** for the day from 6 A.M. to 7 P.M. This curve shows at a glance the variation in temperature between the hours given.

Ex. 9. Two trains leave a certain place traveling in the same direction, one at the rate of 20 mi. an hour, and the other at the rate of 40 mi. an hour. If the second train leaves 3 hr. after the first, when and where will it pass the first?

Let each space along OX represent 20 mi., and each space along OY represent 1 hr. At the end of the first hour the first train is at A; at the end of the second hour at B; and at the end of the sixth hour at P. At the end of the fourth hour the second train, which starts from O', 3 spaces above O, since it starts 3 hr. later, is at A'; at the end of the fifth hour at B'; and at the end of the sixth hour at P. The point P, where the line OP and $O'P$ cross, is the place where the second train overtakes the first. If from P perpendiculars PX and PY are dropped upon OX and OY, then the distances OX and OY will represent the space traveled and the time that has elapsed since the starting of the first train till the second one overtakes it. OX contains 6 distance spaces, and represents 120 mi., while OY contains 6 time spaces, and represents 6 hr.



EXERCISE 47

For convenience in constructing the graphical representations required in the following exercises, the student should provide himself with paper ruled in small squares.

1. Illustrate graphically the comparative areas and the comparative density in population in the following cases:

	Area	Population
(a) Alaska	590884	63592
Greenland	837837	12000
(b) Mexico	767258	13606000
Texas	265780	3048710
(c) United States (including foreign possessions)	3806279	84907156
British Empire	11391036	383165494

2. On Jan. 1, 1904, the thermometer registered the temperature at 1 A.M., and at each succeeding hour till midnight, at Ypsilanti, Michigan and Havana, Cuba, respectively as follows:

1	2	3	4	5	6	7	8	9	10	11	N.
23°	24°	24°	24°	22°	22°	21°	19°	18°	20°	21°	22°
64°	64°	63°	63°	63°	63°	62°	64°	67°	68°	70°	72°

1	2	3	4	5	6	7	8	9	10	11	Mt.
22°	22°	22°	21°	17°	16°	16°	15°	13°	12°	12°	12°
73°	73°	73°	73°	72°	70°	69°	68°	67°	66°	65°	65°

Illustrate each graphically.

3. The mean temperature for January (average for the 31 da. of the month) for the same hours and places as in Ex. 2 was as follows:

1	2	3	4	5	6	7	8	9	10	11	N.
14.2°	14.3°	14.2°	14.2°	14.1°	14.3°	14.2°	14.5°	15.3°	17.1°	18.1°	19.7°
67.2°	67.0°	66.7°	66.4°	66.0°	65.6°	65.4°	66.6°	68.9°	71.0°	73.1°	74.0°

1	2	3	4	5	6	7	8	9	10	11	Mt.
20.4°	20.7°	20.6°	19.9°	18.8°	18.1°	17.5°	16.5°	16.0°	15.4°	14.5°	16.6°
74.7°	74.9°	75.0°	74.7°	74.2°	72.9°	71.6°	70.6°	69.8°	69.0°	68.4°	67.9°

Illustrate graphically.

4. Illustrate graphically, as in Ex. 9, the point where and time at which the two trains given in the annexed time-table pass each other.

Going East A.M.	Miles	Station	Miles	Going West A.M.
10.00	284	Detroit	0	12.35
8.54	247	Ann Arbor	37	1.25
8.00 Lv. / 7.50 Ar.	214	Jackson (Ar. / Lv.)	76	2.20 / 2.25
6.10	164	Battle Creek	121	3.30
4.55	141	Kalamazoo	144	4.10
3.25 Lv. / 3.15 Ar.	93	Niles (Ar. / Lv.)	192	5.28 / 5.33
1.55	56	Michigan City	228	6.32
12.40	13	Kensington	271	7.30
12.00 night	0	Chicago	284	8.00 A.M.

5. A cyclist starts at 7 A.M. from a town and rides 2 hr. at the rate of 10 mi. an hour. He rests 1 hr. and then returns at the rate of 9 mi. an hour. A second cyclist leaves the same place at 8 A.M. and rides at the rate of 6 mi. an hour. When and where will they meet?

6. Two cyclists start from the same place at the same time. The first rides for 2 hr. at the rate of 9 mi. an hour, rests 15 min., and then continues at 6 mi. an hour. The second one rides without stopping at the rate of 7 mi. an hour. Where will the second cyclist overtake the first?

7. The average yield of wheat per acre in the United States for the years from 1893 to 1903 in bushels was as follows: 11.4, 13.2, 13.7, 12.4, 13.4, 15.3, 12.3, 12.3, 15.0, 14.5, 12.9. The highest Chicago cash price per bushel for the same years given in cents was: 64.5, $63\frac{5}{8}$, $64\frac{3}{4}$, $93\frac{1}{8}$, 109, 70, $69\frac{1}{2}$, $75\frac{5}{8}$, $79\frac{1}{2}$, $77\frac{3}{4}$, 87. Illustrate graphically, putting the two curves in one figure.

8. The average yield of corn per acre in the United States for the years from 1893 to 1903 in bushels was as follows: 22.5, 19.4, 26.2, 28.2, 23.8, 24.8, 25.3, 25.3, 16.7, 26.8, 26.5. The highest Chicago cash price per bushel for the same years given in cents was: $36\frac{1}{2}$, $47\frac{1}{4}$, $26\frac{3}{4}$, $23\frac{3}{4}$, $27\frac{1}{2}$, 38, $31\frac{1}{4}$, $40\frac{1}{4}$, $67\frac{1}{4}$, $57\frac{1}{4}$, $43\frac{3}{4}$. Illustrate graphically, putting the two curves in one figure.

9. The average summer daily temperature in Paris at the foot and top of the Eiffel tower in 1900 was as follows:

2	4	6	8	10	N.	2	4	6	8	10	Mt.
57.2°	55.4°	58.1°	63.5°	67.8°	69.8°	70.1°	69.8°	68°	62.1°	60.7°	58.9°
57.4°	55.7°	57.2°	58.1°	60.1°	63.5°	63.9°	64°	64.4°	61.2°	60.7°	59.1°

Illustrate graphically, putting the two curves in one figure.

RATIO AND PROPORTION

218. The **ratio** of one number to another of the same kind is their quotient. The former number is called the **antecedent**, and the latter the **consequent**. The terms of the ratio therefore bear the same relation to each other as the terms of a fraction. Thus, the ratio of a to b may be written $a : b$ (read the ratio of a to b), $\frac{a}{b}$ or $a \div b$. The forms $a : b$, and $\frac{a}{b}$, are generally used. The ratio of 3 ft. to 5 ft. is 3 : 5. This may also be expressed by $\frac{3}{5}$ or 0.6.

219. The ratio is always an abstract number, since it is the relation of one number to another of the same kind. There can be no ratio between 5 hr. and $10, nor between 7 lb. and 6 ft. But there can be a ratio between 3 ft. and 6 in., since the quantities are of the same kind. Both terms must, however, be reduced to the same unit. Thus, 3 ft. = 36 in., and 36 in. : 6 in. = $\frac{36}{6} = 6$.

The ratio $\frac{b}{a}$ is called the **inverse** or **reciprocal** of the ratio $\frac{a}{b}$.

EXERCISE 48

1. How is the value of a ratio affected by multiplying or dividing both terms by the same number?

2. How is the value affected by multiplying or dividing the antecedent? by multiplying or dividing the consequent?

Express the ratio of:

3. 100 to 25.
4. $16\frac{2}{3}$ to 100.
5. $33\frac{1}{3}$ to 100.
6. $2^{m}\,4^{cm}$ to 50^{cm}.
7. \$15 to 50 cents.
8. $7\frac{1}{2}$ to $37\frac{1}{2}$.
9. $\frac{2}{3}$ to $16\frac{2}{3}$.
10. $12\frac{1}{2}$ to 100.

11. 14 hr. 30 min. 3 sec. to a day.

12. 2 mo. 10 da. to a year.

13. What number has to 10 the ratio 2? to 5 the ratio 0.3?

14. If $x : 3 = 5$, find x.

15. If $x : \frac{1}{2} = 2$, find x.

16. Which ratio is the greater, $\frac{5}{13}$ or $\frac{6}{17}$? $\frac{12}{13}$ or $\frac{16}{17}$?

17. The ratio of the circumference of a circle to its diameter being 3.1416, find the diameter of a circle whose circumference is 125 ft. correct to inches.

18. A map is drawn on the scale of 1 in. to 75 mi. In what ratio are the lengths diminished? In what ratio is the area diminished?

19. Two rooms are 14 ft. long, 12 ft. wide, and 12 ft. long, 10 ft. wide respectively. What is the ratio of the cost of carpeting them?

20. What is the ratio of a square field 20 rd. on a side to one 25 rd. on a side?

21. What is the ratio of the circumferences of two circles whose diameters are 2 in. and 4 in.? of two circumferences whose diameters are 5 in. and 7 in.? of two circumferences whose diameters are d and d'? Hence in general the ratio of two circumferences is equal to what?

22. What is the ratio of the areas of two circles whose radii are 3 in. and 5 in.? of the areas of two circles whose radii are 4 in. and 6 in.? of the areas of two circles whose radii are r and r'? Hence in general the ratio of the areas of two circles is equal to what?

23. What is the ratio of the volumes of two spheres whose radii are 2 in. and 3 in.? of the volumes of two spheres whose radii are 5 in. and 6 in.? of the volumes of two spheres whose radii are r and r'? Hence in general the ratio of the volumes of two spheres is equal to what?

220. Specific Gravity. The specific gravity of a substance is the ratio of its weight to the weight of an equal volume of some other substance taken as a standard.

221. Distilled water at its maximum density, 4° C., is the standard of specific gravity for solids and liquids.

222. Since 1^{cm^3} of water weighs 1 gram, the same number that expresses the weight of any substance in grams will also express its specific gravity. Thus, 1^{cm^3} of water weighs 1^{g}; hence, 1 is the specific gravity of water. 1^{cm^3} of lead weighs 11.35^{g}; hence, this being 11.35 times as heavy as an equal volume of water, the specific gravity of lead is 11.35.

SPECIFIC GRAVITIES OF SUBSTANCES

Copper	8.92	Tin	7.29	Sea Water	1.026
Iron (cast)	7.21	Anthracite Coal	1.30	Sulphuric Acid	1.841
Gold	19.26	Cork	0.24	Milk	1.032
Lead	11.35	Pine	0.65	Alcohol	0.84
Platinum	21.50	Oak	0.845	Ice	0.92
Mercury	13.598	Beech	0.852	Rock Salt	2.257

1 cu. ft. of water weighs about 1000 oz., or 62.5 lb.

Ex. 1. A mass of cast iron weighs 3500 lb. How many cubic feet does it contain?

Since 1 cu. ft. of water weighs 62.5 lb., 1 cu. ft. of iron weighs 7.21×62.5 lb.

$\therefore \frac{3500}{7.21 \times 62.5} = 7.77$, the number of cubic feet.

Ex. 2. In France wood is sold by weight. How much does 1 stere of beech wood weigh, allowing $\frac{1}{3}$ for space not filled?

Since 1^{m^3} of water weighs 1000^{Kg}, 1 stere of beech wood weighs $0.852 \times 1000^{Kg} - \frac{1}{3}$ of $0.852 \times 1000^{Kg} = 568^{Kg}$.

EXERCISE 49

1. What is the ratio of the weight of 1 stere to 1 cord of oak wood, allowing $\frac{1}{3}$ for waste space?

2. Allowing $\frac{1}{3}$ for waste space, how many tons of coal will a bin 9 ft. long, 8 ft. wide and 8 ft. deep hold?

3. What is the weight of a cubic decimeter of each of the substances in the above table? of a cubic foot?

4. A flask will hold 6 oz. of water. How much alcohol will it hold? how much mercury?

5. To what depth will a cubic foot of cork sink in sea water? in alcohol?

6. How much does a piece of copper 20^{cm} long, 15^{cm} wide and 5^{mm} thick weigh?

7. If 1 lb. of rock salt is dissolved in 1 cu. ft. of water without increasing its volume, what will be the specific gravity of the solution?

8. How much does a boat weigh that displaces 7000 cu. ft. of water?

9. If a boat is capable of displacing 3000 cu. ft., what weight will be required to sink it?

223. Proportion. A **proportion** is an equality of ratios and is expressed in the following way:

$$\frac{a}{b} = \frac{c}{d},$$

$$a : b = c : d,$$

$$a : b :: c : d.$$

224. The method of solving problems by proportion is often called the **Rule of Three**, since problems which give three quantities so related that two of them sustain the same ratio to each other as the third to the quantity required, can readily be solved by proportion.

225. Thus, if any three of the four terms of a proportion are known, the other one can be found.

If $\frac{x}{3} = \frac{5}{7}$, then, $x = 3 \times \frac{5}{7} = 2\frac{1}{7}$.

Check by putting $2\frac{1}{7}$ for x, then $\frac{2\frac{1}{7}}{3} = \frac{5}{7}$, or $\frac{15}{21} = \frac{5}{7}$.

226. The first and last terms of a proportion are called the **extremes**, and the second and third terms the **means**.

227. *In any proportion the product of the means is equal to the product of the extremes.*

If $\frac{a}{b} = \frac{c}{d}$, then by clearing of fractions $ad = bc$. This proves the proposition, since a and d are the extremes, and b and c the means.

228. If 1 lb. of sugar costs 4 ct., 2 lb. will cost 8 ct. and 4 ct. : 8 ct. = 1 lb : 2 lb. At the same rate 3 lb. would cost 12 ct., etc. The ratio of costs in each case is equal to the ratio of the weights. The cost of sugar is said to be **directly proportional** to its weight.

Ex. The Washington monument is 555 ft. high. What is the height of a post that casts a shadow 1 ft. 9 in. when the monument casts a shadow 192 ft. 6 in.?

Solution by proportion.

Let $x =$ the height of the post.

Then $\frac{x}{555} = \frac{1.75}{192.5}$.

$$\therefore x = \frac{\overset{111}{\cancel{555}} \times \overset{0.05}{\cancel{1.75}}}{\underset{\underset{1.1}{\cancel{38.5}}}{\cancel{192.5}}} = 5.04 \text{ ft.}$$

Solution by unitary analysis.

A shadow 192 ft. 6 in. long is cast by a monument 555 ft. high.

$\therefore$ a shadow 1 ft. long will be cast by a post $\frac{555}{192.5}$ ft. high.

$\therefore$ a shadow 1 ft. 9 in. long will be cast by a post $\frac{1.75 \times 555}{192.5}$ ft. high, or 5.04 ft. high.

229. If 1 man can do a certain piece of work in 6 days, 2 men working at the same rate will do the work in 3 days, 3 men will do it in 2 days, etc. 2 men do the work in $\frac{1}{2}$ the time that 1 man will do it; 3 men in $\frac{1}{3}$ the time, etc. Hence, as the number of men increases, the time diminishes in the same ratio. If 2 men do the work in 3 days, 3 men will do it in $\frac{2}{3}$ of 3 days, or 2 days. Therefore the ratio of the number of men, $\frac{3}{2}$, is equal to the corresponding ratio of time inverted. Hence, the number of men is said to be **inversely proportional** to the time.

Ex. The crew and passengers of a steamship consisted of 1500 persons. The ship had sufficient provisions to last 12 weeks when the survivors of a wreck were taken on board. The provisions were then consumed in 10 weeks; how many were taken on board?

Solution by proportion.

Let x equal the total number on board.

Then $\frac{x}{1500} = \frac{12}{10}$,

or $x = \frac{150\cancel{0} \times 12}{1\cancel{0}} = 1800$,

and $1800 - 1500 = 300$, the number taken on board.

Solution by unitary analysis.

There are provisions for 12 weeks for 1500 persons.

$\therefore$ there are provisions for 1 week for 12×1500 persons.

$\therefore$ there are provisions for 10 weeks for $\frac{150\cancel{0} \times 12}{1\cancel{0}}$ persons or 1800 persons.

$\therefore 1800 - 1500 = 300$, the number taken on board.

EXERCISE 50

State which of the following are directly proportional and which are inversely proportional:

1. The price of bread, the price of flour.
2. The number of workmen, the amount of work done in a given time.
3. The number of workmen, the time required to do a given amount of work.
4. The height of the thermometer, the temperature.
5. The velocity of a train, the time required to go a given distance.
6. The number of horses bought for a given sum, the price per horse.
7. The price of freight, the distance carried.
8. The area of a circle, the length of its diameter.

9. In how many ways can the terms of the proportion $2:3=8:12$ be arranged without destroying the proportion?

10. The assessed value of a certain town is $7500000, and bonds for $6000 are issued. What part of this does a person worth $10000 pay?

11. A shadow cast by a post 6 ft. high is 9 ft. 3 in. How long is the shadow cast by a church steeple 150 ft. high?

12. A merchant fails for $12,300 and his property is worth $5720. How much will he pay a creditor whom he owes $2500?

13. A clock is set at noon on Monday; at 6 P.M. on Wednesday it is 2 minutes and 20 seconds too slow. Supposing the loss of time to be constant, what is the correct time when the clock strikes 12 on Sunday noon?

14. There are two kinds of thermometers used in this country, Fahrenheit, used to register temperature, and Centigrade, used largely in scientific work. The freezing point of water is 32° and 0° respectively, while the boiling point is 212° and 100° respectively. 68° Fahrenheit corresponds to what temperature Centigrade and 54° Centigrade to what temperature Fahrenheit?

15. There is another kind of thermometer known as Réaumur, the freezing and boiling points being 0° and 80° respectively. Express in Réaumur scale 70° on each of the other two.

16. The boiling point of alcohol is 78° Centigrade; what is the boiling point of alcohol on each of the other two?

17. A grain of gold can be beaten into a leaf of 56 sq. in. How many of these leaves will make an inch in height if 1 cu. ft. of gold weighs 1215 lb.?

18. Divide 60 into two parts proportional to 2 and 3.

19. Divide 90 into parts proportional to 2, 3 and 4.

20. Two men start in business with a capital of $7500. One of them furnishes $4000 and the other $3500. At the end of a year the profits are $3250. How much is each man's share?

21. A man starts in business with a capital of $5000 and in 3 months admits a partner with a capital of $4500. At the end of the year the profits amount to $3750. How much is each man's share?

22. A piece of work was to have been done by 10 men in 20 days, but at the end of two days 3 men left. How long did it take the remaining 7 men to complete the work?

23. If the interest on $325 is $72.50 in a given time, how much is the interest on $850 for the same time?

24. Two cog wheels work together; one has 36 cogs and the other 14. How many revolutions does the smaller one make while the larger one makes 28 revolutions?

METHOD OF ATTACK

230. In solving any arithmetical problem the student will find the following suggestions useful:

(1) The first essential is a thorough understanding of the proper relations between the conditions given. This requires some form of **analysis** leading to a complete **statement** of the conditions.

(2) The solution should involve no **unnecessary work.** Cancellation and other convenient short methods should be used if possible.

(3) All arithmetical work should be carefully **checked.** The student must realize that **accuracy** is of the highest importance and that to secure accuracy his work must always be checked. Any arithmetical work that has an error in it is valueless. The check also gives the student a means of knowing for himself whether he has a correct result or not. He has no need of answers to his problems.

Ex. 1. If the time of the beat of a pendulum varies as the square root of its length, and the length of a pendulum that beats seconds is 39.2 in., find the length of a pendulum that beats 50 times a minute.

Solution. The given pendulum beats 60 times per minute, the required pendulum beats 50 times per minute.

Since the longer the pendulum the more slowly it beats, the required pendulum is longer than the given one.

Therefore, the square root of the lengths of the pendulums are in the ratio $\frac{60}{50}$, or $\frac{6}{5}$.

Let l = the length of the required pendulum.

Then, $$\frac{\sqrt{l}}{\sqrt{39.2}} = \frac{6}{5},$$

or $$\frac{l}{39.2} = \frac{6^2}{5^2},$$

or $$l = \frac{6 \times 6 \times 39.2}{5 \times 5}\text{ in.} = \frac{6 \times 6 \times 39.2 \times 4}{100}\text{ in.} = 56.448\text{ in.}$$

Check either by changing the order of the factors and performing the multiplication again, or by casting out the nines.

Ex. 2. The greatest possible sphere is cut from a cube, one of whose edges is 3 ft. Find the portion of the cube cut away.

Solution. The volume of the cube is 3^3 cu. ft.

The volume of the sphere is $\frac{4}{3}\pi \times (\frac{3}{2})^3$ cu. ft.

Therefore the portion cut away is 3^3 cu. ft. $- \frac{4}{3}\pi \times (\frac{3}{2})^3$ cu. ft.

Without performing the operations indicated the student can by cancellation and combination of terms write the result thus,

$$3^2\left(3 - \frac{\pi}{2}\right)\text{cu. ft.} = 3^2\left(\frac{6 - 3.1416}{2}\right)\text{cu. ft.} = 9 \times 1.4292\text{ cu. ft.} = 12.8628\text{ cu. ft.}$$

Check as before.

Ex. 3. Find the area of a square field whose diagonal is 50 rods.

Solution. Let x = one side of the square field.

Then $$x^2 + x^2 = 50^2,$$

or $$2x^2 = 50^2.$$

$\therefore x^2$, or the area of the field in square rods, $= \frac{50^2}{2}$ sq. rd. $= 7\frac{13}{16}$ acres.

Check each step in the work.

Ex. 4. Find the area of the circle which is equal in area to two circles whose radii are 5 in. and 7 in.

Solution. Let r = the radius of the required circle.

Then its area in square inches $= \pi r^2 = \pi \times 5^2 + \pi \times 7^2 = \pi(5^2 + 7^2)$ $= \pi \times 74$, or 232.48 sq. in.

Check each step in the work.

Here, instead of multiplying π by 25 and then by 49 and adding the results, time is saved by adding 25 and 49 and multiplying π by the sum, 74.

231. The **foot pound** is used as a unit of work. This unit is defined as the amount of work required to overcome the resistance of one pound through a space of one foot. The **rate** of work is generally defined by using the term *horse power.* An engine of one horse power can do 33000 foot pounds of work in one minute, *i.e.* can overcome a resistance of 33000 pounds through a space of one foot in one minute.

Ex. 5. What horse power is an engine exerting that draws a train with a uniform speed of 40 miles an hour against a resistance of 1000 pounds?

Solution. The amount of work done in one hour is $1000 \times 40 \times 5280$ foot pounds.

The amount of work done in one minute is $\frac{1000 \times 40 \times 5280}{60}$ foot pounds.

Therefore, the rate of doing work is $\frac{\overset{2}{\cancel{1000}} \times \cancel{40} \times \overset{16}{\cancel{5280}}}{\cancel{60} \times \underset{3}{\cancel{33000}}}$ horse power

$= \frac{10 \times 2 \times 16}{3}$ horse power $= 106\frac{2}{3}$ horse power.

Check each step.

232. The student will notice that in each of the above exercises, first, *the relations between the given conditions are carefully established;* and second, *a complete statement of these conditions is written out and the work shortened as much as possible by cancellation or otherwise, before the processes of multiplication and division are used.* Frequently students in solving such problems will perform the operations indicated at each step, thus doing a large amount of unnecessary work. By carefully studying these model solutions the student will see where the unnecessary work can be avoided.

As indicated in Art. 41, it is a good plan, whenever possible, to estimate the result mentally and to compare this rough estimate with the result found by solving the problem. This will prevent large errors and such errors as arise from misplacing the decimal point.

EXERCISE 52

1. Find the area bounded by 6 equal coins whose centers are at the vertices of a regular hexagon, the diameter of each coin being 2.38^{cm}.

2. A crescent is bounded by a semi-circumference of a circle whose radius is 15 inches, and by the arc of another circumference whose center is on the first arc produced. Find the area and perimeter of the crescent.

3. A horse is tied with a 50 ft. rope to one corner of a barn 30 ft. by 40 ft. Find the area he can graze over.

4. A well 30 ft. deep and 4 ft. in diameter is to be dug. If a cubic foot of earth weighs 72 lb., how much work is to be done?

5. A horse drawing a wagon along a level road at the rate of 2 mi. an hour does 29216 foot pounds of work in 3 min. What pull in pounds does he exert in drawing the wagon?

6. A uniform heavy bar, 12 ft. long and weighing 80 lb., rests on 2 props in the same horizontal plane, so that 2 ft. project over one of the props; find the distance between the props so that the pressure on one may be double that on the other; also find the pressures.

7. It is proved in geometry that similar volumes are to each other as the cubes of their like dimensions. If a cubical bin whose edge is 4 ft. holds 52 bu. of wheat, how many bushels will a bin 6 ft. on an edge hold?

8. The temperature remaining the same, the space occupied by a gas varies inversely as the pressure. At a constant temperature a mass of air occupies 25 cu. ft. under a pressure of 10 lb. to the square inch; what space will it occupy under a pressure of 26 lb. to the square inch?

9. A cubic foot of water weighs 1000 oz., and the pressure of the air is 336 oz. per square inch; find the pressure on a square foot at a depth of 10 ft. below the surface of a pond.

10. If the specific gravity of mercury is 13.598 and the weight of a cubic inch of water is 252.6 grains, find the pressure of air per square inch in pounds when the mercury in the barometer stands at 30.5 in.

11. An iceberg (specific gravity 0.925) floats in sea water (specific gravity 1.025). Find the ratio of the part out of water to the part immersed.

12. A piece of lead placed in a cylindrical vessel, the radius of whose base is 1.2^{dm}, causes the liquid in the vessel to rise 3^{cm}. What is the volume of the piece of lead, and how much does it weigh if lead is 11.35 times as heavy as water?

MISCELLANEOUS EXERCISE 52

Express the ratio of:

1. A cubic decimeter to a liter.
2. A cubic centimeter to a cubic millimeter.
3. A cubic decimeter to a cubic meter.
4. A kilogram to a centigram.
5. A meter to a yard.
6. A quart to a liter.
7. A kilogram to a pound.
8. A milligram to a kilogram.
9. A kilogram to 40 grams.
10. A kilometer to 200 centimeters.

Find the value of:

11. $(60 - \frac{12}{6}) \times 3.$

12. $\frac{120}{12 \times 50} + 1.$

13. $(\frac{143}{11} - 5) \times 6.$

14. $(\frac{369}{9} + 2) \times 4.$

15. $\left(\frac{522}{6} - \frac{2727}{22} + \frac{144}{130} + \frac{8 \times 9}{4 \times 5}\right) \times 12.$

16. $\frac{(26 \div 13) \times 7}{2 + 15 \div 3}.$

17. $\frac{5 \times 8 - 17 \times 2}{17 - 14}.$

18. $\frac{\frac{2}{3} \times 4\frac{1}{2}}{\frac{1}{3} + 1\frac{5}{6}}.$

19. $\frac{7\frac{1}{2} + 3\frac{1}{3}}{1\frac{5}{6}}$ of $\frac{3\frac{1}{3}}{1\frac{1}{2} \times \frac{3}{4}}.$

20. What is a decimal fraction?

21. How is the units' place distinguished?

22. What is the place value of a digit one place to the right of units? three places to the right?

23. What is the importance of the symbol 0 in the decimal scale of notation?

24. If a decimal fraction is multiplied by a digit in units' place, do the place values of the digits in the product differ from the place value of the digits in the multiplicand? If the decimal fraction is multiplied by the same digit two orders lower, is there a difference in the place value of the digits in the product?

25. If a decimal fraction is divided by a digit in units' place, do the place values of the digits in the quotient differ from the place values of the digits in the dividend? If the decimal fraction is divided by the same digit three orders higher, what is the difference in the place values of the digits in the quotient?

26. What is a divisor of a number? a common divisor of two or more numbers? the greatest common divisor of two or more numbers?

27. What is a multiple of a number? a common multiple of two or more numbers? the least common multiple of two or more numbers?

28. What is a prime number? What is a prime factor of a number? When are two numbers prime to each other?

29. What is the shortest piece of rope that can be cut exactly into pieces 12, 15 or 20 ft. long?

30. Find the l. c. m. of the first five odd numbers, also of the first six even numbers.

31. Find the g. c. d. of 125, 340 and 735.

32. Evaluate $3\frac{1}{2} + 5\frac{1}{3} + 7\frac{1}{4} + 9\frac{1}{5}$.

33. Evaluate $\frac{7}{8} + \frac{7}{12} + \frac{7}{16} + \frac{7}{20} - \frac{5}{6} - \frac{5}{10} - \frac{5}{14}$.

34. Evaluate $4\frac{5}{18} + 2 \times 5\frac{5}{9} - 3 \times \frac{3}{4} \div \frac{1}{2}$.

35. A cubic foot of water weighs 1000 oz. How many tons, etc., of water are there in a canal 30 ft. wide, 8 ft. deep and 10 mi. long?

36. How many feet per second are equal to 40 mi. an hour?

37. Find the square root of 0.4; the cube root of 0.27.

38. If I walk 7.2^{Km} in 1 hr., how far shall I go in 6 hr. and 20 min. at the same rate?

39. How many cubic centimeters of air are there in a room $9\frac{1}{4}^{m}$ long, $6\frac{1}{2}^{m}$ wide and 3.15^{m} high?

40. What is the area of a cube that has the same volume as a box 2 ft. 6 in. by 2 ft. 3 in. by 2 ft.?

41. How many cubic meters of water pass under a bridge in one minute when the river is 20^{m} wide, 4^{m} deep and is running 3^{Km} per hour?

42. Write three numbers of four figures each that are divisible by both 8 and 3.

43. Write three numbers of six figures each that are divisible by both 9 and 11.

44. Replace the zeros in 205006 so that the number may be divisible by both 9 and 11.

45. What is the cost per hour of lighting a room with 40 burners, each consuming $2\frac{1}{2}$ cu. in. of gas per second, the price of gas being $1.25 per thousand cubic feet?

46. A roller used in rolling a lawn is $6\frac{1}{2}$ ft. in circumference and $2\frac{3}{4}$ ft. wide. If the roller makes 10 revolutions in crossing the lawn once and must pass back and forth 12 times to cover the whole lawn, find the area of the lawn.

47. Find the sum of $\frac{1}{3} + \frac{1}{6} + \frac{3}{20} + \frac{4}{30}$ correct to four decimal places.

48. Find each of the following products correct to five significant figures:

(a) 20.361×40.482. (b) 1.5674×75.429.
(c) 824.763×45. (d) 103.64×0.033.
(e) 0.423×0.00765.

49. Find each of the following quotients correct to 0.01:
(a) $22 \div 3.1416$; (b) $42.567 \div 21.268$; (c) $0.4 \div 0.75$;
(d) $237.64 \div 2.1473$; (e) $2 \div 9.97$.

50. Find the cost of carpeting a room 12 ft. 3 in. long and 10 ft. 9 in. wide with carpet 27 in. wide at $1.12 a yard.

51. Find the cost of 8 T. 1450 lb. of coal at $7.25 a ton.

52. Multiply 7644 by $33\frac{1}{3}$ and divide the result by $16\frac{2}{3}$.

53. Divide 8350 by 25 and multiply the result by $12\frac{1}{2}$.

Find the value of:

54. 0.0001×0.0001; 6.74×21.023.

55. 1.1×0.011; 7.6×0.76.

56. $2.5 \times 25 \times 250$, $2.5 \times 0.25 \times 0.025$.

57. 0.002×3.01; $0.0005 \times 0.01 \times 5000000$.

58. $15.625 \div 25$; $0.15625 \div 2.5$.

59. $8 \div 0.002$; $50 \div 0.25$.

60. $9.065 \div 0.049$; $0.005 \div 0.01$.

61. $0.00128 \div 8.192$; $1708.4592 \div 0.00024$.

Find correct to 4 decimal places:

62. $0.138138 + 0.1425876 + 2.060606 + 0.008964$.

63. $7.427525 - 2.347596$.

64. $0.33\frac{1}{3} \times 0.37\frac{1}{2}$.

65. $0.33\frac{1}{3} \div 0.37\frac{1}{2}$.

66. $0.0404 \div 7692$.

67. If the length of Jupiter's day is 9 hr. 56 min., how many more days has Jupiter than the earth in one year?

68. If $500 can be counted in one minute, how long will it take to count $1000000?

69. What is the difference between the daily income of a man whose salary is $1200 a year and of one whose salary is $1600?

70. Counting 12 hr. a day, how long would it take to count a billion at the rate of 750 a minute?

71. How many days old was a person Oct. 5, 1904, who was born July 27, 1861?

72. The ancient Roman mile is 0.917 of the English mile. Express the diameter of the earth (7926 English miles) in Roman miles.

73. The diameter of a fly wheel is found by measurement to be 20.12 in. Find its circumference.

74. The specific gravity of copper is 8.92; of gold, 19.26; of lead, 11.35. Find the weight of a lump of each equal in bulk to a liter of water.

75. The diameter of the earth is 7926 mi. The sun's diameter is 111.454 times the earth's diameter. Find the sun's diameter correct to miles.

76. A lump of iron containing 12 cu. ft. is drawn out into a rod 50 ft. long. What is the diameter of the rod?

77. The true length of the year is 365.2426 da. What error is made by calculating the year as 365 da., and adding a day every leap year, omitting three leap years in four centuries?

78. The edge of a cube is 12 in. What is the edge of a cube three times its volume?

79. How many miles an hour does a person walk who takes two steps a second and 1900 steps to the mile?

80. Express in words 0.12071 and 12000.00071.

81. How many steps 0.8 of a meter long will a person take in walking 10^{Km}?

82. A clock which gains one minute in 10 hr. is correct on Monday noon. What is the correct time when it indicates Monday noon of the next week?

In scientific work, when numbers depend upon measurements and therefore cannot be expressed with absolute accuracy the index notation is frequently used. Thus, the wave length of blue light, determined by the physicist to be 0.000431^{mm} would usually be written 4.31×10^{-4mm}. The distance from the sun to the earth is determined by the astronomer to be approximately 93000000 mi. In index notation it would be written 9.3×10^7 mi.

83. Express the following in the index notation:
0.0000025; 36500000000; $\frac{1}{200000000}$; 41100000.

84. Express in the common notation 1.1×10^{-6}; 3.6×10^6; 4.321×10^{-8}; 5×10^{-4}; 5×10^6.

85. From 3542_6 subtract 2131_6.

86. Find the sum of 34.6_{12}, 121.51_{12}, and 25.11_{12} and express it in the decimal notation.

87. If brass weighs 525 lb. per cubic foot, find the weight of a circular brass plate 21 in. in diameter and $\frac{1}{2}$ in. thick.

88. If a cubic foot of gold may be made to cover uniformly 432000000 sq. in., find the thickness of the gold.

89. If a gallon of water contains 231 cu. in., and a cubic foot of water weighs 1000 oz., how much does a pint of water weigh? How many gallons will weigh a ton?

90. Four circles each 1 ft. in diameter are so placed that two of them touch two of the others, and the remaining two both touch three of the others; find the area of the figure whose angles are at the four centers.

91. What (standard) time is it in Boston when it is 4.30 P.M. in San Francisco?

92. A ship's clock is corrected at 1 o'clock each day. If the ship passes over 10° 30′ each day, what change must be made in the clock (*a*) if the ship is sailing from W. to E.; (*b*) from E. to W.?

93. Find the remainders (without dividing) after 471321 has been divided by all of the numbers (except 7) from 2 to 12 inclusive.

94. Show without dividing that 133056 is divisible by 792.

95. A ship's clock is corrected every day at 1 P.M.; how much must it be put back or forward at 1, if the ship has passed over 11° of longitude from east to west?

96. When it is noon (standard time) Wednesday, Dec. 7, at Chicago, what time and date is it at Rome? at Tokyo?

97. A meter is defined as 1×10^{-7} of the distance from the pole to the equator. Find the circumference of the earth in kilometers.

98. Find the circumference of the earth in miles if the meter is equal to 39.37 in.

99. If 1 cu. ft. of water weighs 1000 oz., and platinum is 20.337 times as heavy as water, how many feet of platinum wire $\frac{1}{30000}$ of an inch in diameter would weigh a grain?

PERCENTAGE

233. $\frac{1}{2} = \frac{50}{100} = 0.50 = 50\% = 50$ per cent,

and $\frac{1}{3} = \frac{33\frac{1}{3}}{100} = 0.33\frac{1}{3} = 33\frac{1}{3}\% = 33\frac{1}{3}$ per cent.

These are different ways of denoting the same **fractional part**. In business operations it is customary to express fractions in hundredths, but in stating problems the denominator 100 is omitted and the per cent symbol, %, or the expression **per cent** is used. Percentage is therefore only an application of the decimal fraction and not a separate department of arithmetic.

234. The word **percentage** is derived from the Latin *per centum*, meaning *by the hundredths*.

235. The number denoting how many hundredths are taken is called the **rate per cent**. Thus, if 5% of a number is to be taken, 5 is called the **rate per cent**, and 5% the rate.

236. The following examples illustrate several closely related operations frequently used in business transactions.

Ex. 1. What is 8% of $750?

Solution. 8% of $750 = 0.08 of $750 = $60.

Ex. 2. 12 is what per cent of 240?

Solution. Let $x\%$ = the rate.

Then $x\%$ of $240 = 12$,

$$x\% = \tfrac{12}{240} = \tfrac{1}{20} = 0.05 = 5\%.$$

Ex. 3. 20 is 6% of what number?

Solution. Let x = the number.

Then 6% of $x = 20$,

$$x = \frac{20}{0.06} = 333\tfrac{1}{3}.$$

EXERCISE 53

1. Express the following fractions in per cent, also as decimals: $\frac{1}{2}$, $\frac{3}{4}$, $\frac{9}{10}$, $\frac{5}{6}$, $\frac{1}{8}$, $\frac{2}{5}$, $\frac{1}{7}$, $\frac{13}{10}$, $\frac{12}{5}$.

2. 3 is what per cent of 4? 8 is what per cent of 4? 18 is what per cent of 27? 25 is what per cent of 200? 7 is what per cent of 2?

3. The population of a town is 7200. What is the population five years later if it has increased 7% in that time?

4. A town of 11750 inhabitants decreases 12% in ten years. What is its population after this loss?

5. Express the following as decimals: $\frac{1}{8}\%$, $33\frac{1}{3}\%$, 0.5%, 125%.

6. What is $\frac{1}{2}\%$ of 75? $\frac{1}{3}\%$ of 100? 0.4% of $\frac{5}{8}$? $\frac{2}{3}\%$ of $\frac{45}{8}$?

7. Write as per cent $1\frac{1}{2}$, $2\frac{3}{4}$, $\frac{1}{3}$, $0.00\frac{1}{3}$, 10, 2, 0.25, 2.5, $0.16\frac{2}{3}$.

8. The attendance in a certain school increased in one year from 318 to 425; find the rate per cent of increase.

9. In a certain school there are 291 boys and 315 girls. What percentage of the attendance is boys and what percentage is girls?

10. In a certain town the total school enrollment is 962; of this 156 are in the high school. What percentage of the whole enrollment is in the high school?

11. If 0.8% of those living at the age of 24 die within a year, how many out of 6625 persons of this age die during that period?

12. At the age of 15, 735 out of 96285 die within a year. What is the rate per cent of deaths?

13. At the age of 25, 718 out of 89032 die within one year. Is the death rate higher or lower than at the age of 15?

14. A man owns a farm worth $7500. His annual taxes are $68.50. How much must he make in order to clear 6% from his farm each year?

15. A house depreciates in value each year at the rate of 5% of its value at the beginning of the year, and its value at the end of three years is $4225; find the original value.

16. A man sold two horses for $200 each; on the purchase price of one he made 20%, and on the other he lost 25%. Did he gain or lose and how much?

17. The wholesale grocer buys coffee at 25 ct. per pound and sells it at 30 ct. The retail grocer buys it at 30 ct. and sells it at $37\frac{1}{2}$ ct. What per cent does each make?

18. If a person spends 60% of his income and saves $1000, what is his income?

19. Which investment returns the larger per cent, flour costing $1.98 per hundred pounds and sold for $2.10, or sugar costing $3\frac{1}{2}$ ct. a pound and sold for $4\frac{1}{2}$ ct.?

20. A man owning a $\frac{2}{3}$ interest in a store sold $\frac{1}{2}$ of his interest. What per cent of his share did he sell, and what per cent of the store did he still own?

21. A merchant sold out his stock of goods at a discount of 10% of the cost and realized $14756.34. How much did his goods cost him?

22. A house rents for $300 a year, which represents 6% of its value. How much is it worth?

23. In 1880 the population of the United States was 50152866, in 1890 it was 63069756, and in 1900 it was 75994575. During which decade was the per cent of increase greater and how much?

24. What is the difference, in square yards, between $\frac{3}{4}$ of an acre and $\frac{3}{4}$% of an acre?

25. The population of a city is 14553, and is 35% more than it was 10 yr. ago. What was the population then?

26. On Nov. 1, 1897, the amount of money in circulation in the United States was: gold (including gold certificates), $576000000; silver (including silver certificates), $496000000; paper, $634000000. Nov. 1, 1902, the corresponding amounts were $967000000, $623000000 and $736000000. What was the per cent of increase in each case during the 5 yr., and what was the total per cent of increase?

27. The following tables show the total receipts and disbursements of three of the largest life insurance companies in the United States for the year 1902:

Total Income	Expenses and Taxes	Death Claims	Other Disbursements
1073636984	183485217	252617938	316541543
782424835	156329328	163663466	185702274
330651136	54403289	60459793	69056722

Find the per cent of the total income remaining in the hands of each company at the end of the year. Find the per cent of expense to income and of death claims to income in each case.

28. In 1890 the total foreign population in the United States was 9249547, of whom 2784894 were born in Germany and 1871509 in Ireland. The population of the United States in 1890 being 63069756, what per cent of the population was born in Germany, and what per cent in Ireland?

29. In 1890 the total number of negroes in the United States was 7470000, which was 11.8% of the total population at that time. Determine the population correct to thousands.

30. In 1898 the total value of the exports from the United States was $1231482330, the total value of imports was $616049654. By what per cent did the value of the exports exceed the value of the imports?

COMMERCIAL DISCOUNTS

237. Manufacturers, publishers and wholesale dealers have a fixed price list for their products. Their customers are allowed certain discounts from their list price, determined by the current market value. Thus, a book may be published at $1.50 with a discount of 20% to dealers. The $1.50 is the **list price** and 20% is the **discount**. The list price less the discount ($1.50 − 20% of $1.50 = $1.20) is the **net price,** or **cost.**

238. To avoid the inconvenience and expense of issuing a new catalogue whenever the market values change, business houses generally print a new trade price list giving new discounts, without issuing a new catalogue. The discount is changed either by increasing or diminishing the single rate of discount already allowed, according as the cost of production is diminished or increased. If the discount is to be increased, the change is generally made by quoting a further discount. Thus, in a catalogue of electrical goods a 32 candle power lamp is quoted at $1.20. In trade price list A, accompanying the catalogue, a discount of 50% is allowed on small orders. In trade price list B, issued later on account of a change in the cost of production, a discount of 50% and 15% is allowed. A dealer buying 10 lamps according to trade price list A would pay 10 × $1.20 − 50% of 10 × $1.20 = $6, while according to trade price list B he would pay $6 − 15% of $6 = $5.10.

The discount is frequently increased in case of large orders. Thus, in the above trade price list, a discount of 50% is allowed on all orders for less than 25 lamps, a discount of 50% and 20% is allowed on all orders for 25 to 100 lamps, and a discount of 50%, 20% and 10% on orders for 100 or over.

239. Bills are generally made out payable in 30, 60 or 90 days, subject to a certain discount for cash, or if paid before due. Business houses usually print on their bill heads their terms of discount for cash, *e.g.* "Terms: 60 days, or 2% discount for cash." "Terms: net 90 days, or 3% in 10 days."

Ex. **1.** On March 12, 1903, E. C. Horner & Co. bought of James Bros., Chicago, 50 plows, listed at $6.50, less 25% and 10%. Terms: 90 days, 3% in 10 days.

Bill Rendered

CHICAGO, ILL., *March* 12, 1903.

E. C. HORNER & CO.

Bought of JAMES BROS.

Terms: 90 days; 3% 10 days.

50 Plows	@ $6.50	$325	00		
	Discount, 25%	81	25		
		243	75		
	Discount, 10%	24	38	$219	37

If Horner & Co. avail themselves of cash payment, they will deduct 3% of $219.37 = $6.58, and send the remainder, $212.79, to James Bros. If the bill is not paid till the 90 days expire, they will send $219.37.

Ex. **2.** Find the cost of a bill of goods amounting to $75 less 20%, 5% and 2% for cash.

Solution. Let x = the cost.

Then $x = 0.98 \times 0.95 \times 0.80$ of $75 = $55.86.

Analysis. $75 is the list price.

Then $75 − 20% of $75 = 0.80 of $75 is the amount left after the first discount. And 0.80 of $75 − 5% of 0.80 of $75 = 0.95 × 0.80 of $75 is the amount left after the second discount. And 0.95 × 0.80 of $75 − 2% of 0.95 × 0.80 of $75 = 0.98 × 0.95 × 0.80 of $75 is the amount left after the third discount.

∴ 0.98 × 0.95 × 0.80 of $75 = $55.86 is the net price or cost.

Second Solution.

5)$75 = list price.
$15 = 20% discount.
20)$60
$3 = 5% discount
50)$57
$ 1.14 = 2% discount for cash.
$55.86 = cost of the goods.

Ex. **3.** What must be the list price of goods in order to realize $243 after deducting discounts of 25%, 10% and 10%?

Solution. Let x = the list price.

Then 0.90 × 0.90 × 0.75 of x = $243.

$$\therefore x = \frac{\$243}{0.90 \times 0.90 \times 0.75} = \frac{\$\overset{3}{\cancel{243}} \times \overset{400}{\cancel{10000}}}{\cancel{9} \times \cancel{9} \times \underset{25}{\cancel{75}}} = \$400.$$

EXERCISE 54

1. Find the net amount of the bill to render in each of the following cases:

(*a*) $750 less $33\frac{1}{3}$%.
(*b*) $1250 less 25% and 15%.
(*c*) $525 less 20%, 10% and 5%.
(*d*) $525 less 5%, 10% and 20%.
(*e*) $5050.75 less 50% and 10%.

2. March 1, 1903, the Manhattan Electrical Supply Co. sold George J. Fiske & Co. the following bill of goods, 60 da., 2% 10 da.: 2 electrical gongs at $17.22 each, less 40% and 10%; 2 hotel annunciators at $15 each, less 60%; 2 spools of wire at 75 ct. each, less 50% and 10%. Find the amount to be remitted if paid March 11, and write the bill rendered.

3. A piano listed at $750 was sold at a discount of 40% and 10%. If the freight was $4.87 and drayage $3, what was the net cost of the piano?

4. Find the net cost of a piece of Rogers's statuary listed at $65 and discounted at 35%, 20%, 10% and 5%.

5. A merchant buys $1750 worth of goods at a discount of $33\frac{1}{3}$% and 10%. If he sells the goods at the list prices, what is the rate of gain on the cost?

6. A car load of flour weighing 195 hundredweight cost a grocer $1.85 a hundredweight. If he is allowed a discount of 1% for cash and sells the flour for $2.10 a hundredweight, how much does he make?

7. Which is the greater, a discount of 10%, 10% and 10%, or a discount of 20%, 5% and 5%?

8. A merchant buys goods at a discount of 40% and 10% and sells at a discount of 30% and 5%. What is his gain per cent?

9. A certain publishing house allows a discount of $16\frac{2}{3}$% on all orders under $100, $16\frac{2}{3}$% and 10% on all orders between $100 and $500, and $16\frac{2}{3}$%, 10% and 5% on all orders above $500. If three dealers wish to send in orders amounting to $60, $175 and $350 respectively, how much will each one gain if they combine their orders?

10. Which is the better discount for a buyer to take:

(*a*) $33\frac{1}{3}$%, 10% and 5%, or 40%?
(*b*) 10%, 10% and 5%, or 25%?
(*c*) 40% and 15%, or 40%, 10% and 5%?
(*d*) 50% and 15%, or 60%?

11. How much above the cost must a book marked $2 be sold, if 10% is taken from the marked price and a profit of 10% on the cost is still made?

12. One firm offers to sell $500 worth of galvanized pipe at a discount of 40%, 10% and 5%, and another firm offers a discount of $33\frac{1}{3}$%, 20% and 10%. Which is the better rate of discount and what is the difference in dollars?

13. Office furniture amounting to $750 was inventoried at the end of the first year at 25% below cost and at the end of the second year at 15% below inventory. What was the loss in value?

14. If a grocer buys sugar at 3.42 ct. per pound and sells it at 4 ct., what is his gain per cent?

15. A dealer marked his goods at $33\frac{1}{3}$% above cost, but sold at a certain per cent discount and still made 15% on the cost. What was the rate per cent of discount?

16. What three equal rates of discount are equivalent to a single rate of 27.1%?

MARKING GOODS

240. Most merchants use a private mark to indicate the cost and selling price of goods. They usually select some word or phrase containing 10 different letters and use it as a **key**. These letters are used to represent the 9 digits and 0. In this way the cost and selling price will be understood only by those who know the key.

Two different keys are generally selected, one to mark the cost and the other to mark the selling price. One or more extra letters, called repeaters, are used to avoid the repetition of a figure and to prevent giving any clew to the private mark used. The cost is usually written above and the selling price below a line.

241. The words *equinoctial* (omitting the last *i*) and *importance* are adapted for use as keys, since they both contain 10 different letters. These words give the following keys:

1	2	3	4	5	6	7	8	9	0	
e	*q*	*u*	*i*	*n*	*o*	*c*	*t*	*a*	*l*	
i	*m*	*p*	*o*	*r*	*t*	*a*	*n*	*c*	*e*	Repeaters *x* and *y*.

Thus, if a merchant pays $29.98 per dozen for hats, and sells them for $3.50 each, he would mark them $\frac{\$\,qa.xt}{\$\,p.re}$.

EXERCISE 55

1. Explain why, if the cost of a dozen articles is divided by 10, the result will give the retail price of one article with a profit of 20% added.

2. Explain why, to make a profit of $33\frac{1}{3}\%$, the cost of a dozen articles may be divided by 10 and $\frac{1}{9}$ of the result added.

3. Determine short methods of finding the retail price of one article when the cost per dozen is given and the dealer wishes to make a profit of 35%; $37\frac{1}{2}\%$; 40%; 50%; 60%.

4. A merchant buys shirts for $12.50 per dozen. For what price must he sell them to make 50%? 40%?

5. A merchant retails neckties at 50 ct. and makes 50%. How much did they cost him per dozen?

Using *equinoctal* and *importance* as keys, mark the cost and selling price of the following articles:

6. Gloves costing $5 per dozen and selling for $6.50.

7. Hats costing $22.50 per dozen and selling at 20% gain.

8. Caps costing $7.50 per dozen and selling at $33\frac{1}{3}\%$ gain.

9. Shoes costing $1.98 and selling at 25% gain.

10. Rubber boots costing $2.68 and selling at $3.75.

11. Make a key of the letters contained in the words *Cumberland* and *Charleston* spelled backward, and mark the articles given in Ex. 6 to 10.

12. A merchant sold a bill of goods that cost $125; the asking price was 30% in advance of the cost, from which a wholesale discount of 15% was allowed. What was the per cent gain?

13. An invoice of hats costing $112 is marked so as to sell at 40% profit. Does the merchant gain or lose if the hats are sold at 30% discount from the marked price?

COMMISSION AND BROKERAGE

242. Farmers, produce dealers, manufacturers and others frequently find it more convenient to employ a third person to dispose of their goods, instead of selling direct to consumers. The person who sells the goods is called a **commission merchant**, an **agent** or a **broker**. The pay received for such services is called **commission** or **brokerage**.

243. Produce is usually shipped to a commission merchant, and sold by him in his own name. The proceeds less the commission, or the **net proceeds**, are sent to the **shipper** or **consignor**. If a commission merchant is buying goods for a customer, he charges the cost plus the commission. The amount of commission varies in different lines of business.

244. A broker buys and sells without having possession of the goods, and generally does not make contracts in his own name.

245. Commission, or brokerage, is usually computed at a certain per cent of the amount realized on sales, or invested for the customer. In buying and selling certain kinds of merchandise, it is customary to pay a certain price per unit of measurement or weight; as grain per bushel, hay per ton, etc.

EXERCISE 56

What is the commission on:

1. \$750.50 at 2%?
2. \$12368 at $\frac{1}{2}$%?
3. \$75429.75 at $\frac{1}{8}$%?
4. \$350.45 at 10%?
5. \$3764 at $\frac{1}{4}$%?
6. \$5250 at $7\frac{1}{2}$%?
7. The sale of 1000 bu. of grain at $\frac{1}{2}$ ct. a bushel?
8. The sale of 25 T. of hay at 50 ct. a ton?
9. The sale of 40 head of cattle at 50 ct. a head?
10. The sale of 1500 bales of cotton at 25 ct. a bale?
11. The sale of 22 horses @ \$125 a head at 2%?

Find the amount to invest and the commission when the following remittances and rates of commission are given:

12. \$1030 at 3%.
13. \$5025 at $\frac{1}{2}$%.
14. \$8020 at $\frac{1}{4}$%.
15. \$6300 at 5%.
16. \$1100 at 10%.
17. \$2562 at $2\frac{1}{2}$%.

Find the net proceeds and commission on each of the following sales:

18. 200 bbl. of apples @ \$3, less freight \$62.50, commission 5%.

19. 5000 bu. of wheat @ 72 ct., less \$102.50 freight, \$25 storage, $\frac{1}{4}$% insurance and 2% commission.

20. 500 bbl. of beef @ \$19.50, less 48 ct. a barrel freight, \$7.50 storage and $2\frac{1}{2}$% commission.

21. 1500 doz. eggs @ 22 ct., less \$9.50 express and 10% commission.

22. 12 bales of cotton averaging 475 lb. @ $9\frac{1}{2}$ ct. a pound, less \$42.50 freight, \$1.25 a bale storage and $2\frac{1}{2}$% commission.

23. An agent charges \$20 for advertising the sale of a farm, and 3% commission. He sells the farm for \$7500. What are the net proceeds and the agent's commission?

24. A collector is given a bill of \$1350 to collect at 5% commission. He succeeds in collecting 85 ct. on the dollar. How much is due his employer, and what is his commission?

25. A miller orders his agent to buy him 2500 bu. of wheat @ 80 ct. If the agent charges 3% commission, and freight and drayage charges are \$95.75, what is the total cost of the wheat?

26. A merchant sends his agent \$1836 to buy an equal number of yards of each of three grades of muslin at 3, 4 and 5 ct. a yard respectively, after deducting 2% commission. How many yards of each kind does he get, and what is the agent's commission?

27. A manufacturer sold \$20000 worth of goods through his agent at 2% commission, and instructed him to purchase raw material with the proceeds at 1% commission. Find the net proceeds of the sale, the amount invested in raw material, and the agent's entire commission.

28. A dealer sent two car loads of hay weighing 27 T. to his broker in New York, who sold it for \$16 a ton, and remitted \$418.50. If the dealer paid \$8.50 a ton, the freight cost $16\frac{2}{3}$ ct. a hundredweight, and storage was \$12.50, how much did he make, and what was the broker's commission per ton?

29. A book agent sells, during July and August, 77 copies of a certain book at 40% commission. If he sells 20 copies in full leather binding @ \$6.50, 25 copies in half leather @ \$5.25, and 32 copies in cloth @ \$4, how much does he make if his expenses average \$1.25 a day?

INTEREST

246 **Interest** is money paid for the use of money.

247. The sum loaned is called the **principal**.

248. The **rate of interest** is the rate per cent per annum of the principal paid for the use of money.

In the absence of a specific contract the rate of interest is fixed by law in most states. The rate thus determined is called the **legal rate**. By special contract interest may be received at a higher rate than the legal rate. The maximum contract rate is fixed by law in most states. Interest in excess of the maximum contract rate is called **usury**. The penalty for usury is fixed by law in states where it is forbidden.

249. The principal plus the interest is called the **amount**.

250. The practical business problem of most frequent occurrence in interest is *to find the interest when the principal, rate, and time are given.*

251. In computing interest without tables it is usually the custom to reckon the year as 360 da., the month as $\frac{1}{12}$ of a year and the day as $\frac{1}{30}$ of a month or $\frac{1}{360}$ of a year. (See, however, § 254.)

Ex. Find the interest and amount of \$720 for 2 yr. 6 mo. 15 da. at 6%.

Solution. 2 yr. 6 mo. 15 da. = $\frac{915}{360}$ yr.
The interest for one year = 0.06 of \$720.
∴ the interest for $\frac{915}{360}$ yr. = $\frac{915}{360} \times \frac{6}{100} \times \$720 = \$109.80$.
The amount is \$720 + \$109.80 = \$829.80.

EXERCISE 57

Find the interest and amount of:

1. \$100 for 1 yr. 4 mo. at 6%.
2. \$125 for 6 yr. 1 mo. 20 da. at 7%.
3. \$150 for 5 yr. 9 mo. 11 da. at 7%.
4. \$50 for 4 yr. 11 mo. 10 da. at 6%.
5. \$1000 for 5 mo. 10 da. at 7%.
6. \$350 for 3 yr. 9 mo. at 7%.
7. \$1500 for 2 mo. at 6%.
8. \$25 for 1 yr. at 6%.
9. \$1200 for 2 yr. 6 mo. at 5%.
10. \$500 for 2 yr. 3 mo. 15 da. at 4%.

252. A *short method* of computing interest at 6% is based on the year of 12 mo. of 30 da. each. This method is sometimes called the 6% method.

The interest on \$1 for 1 yr. at 6% $= \$0.06$.
The interest on \$1 for 1 mo. at 6% $= \frac{1}{12}$ of \$0.06 $= \$0.005$.
The interest on \$1 for 1 da. at 6% $= \frac{1}{30}$ of \$0.005 $= \$0.000\frac{1}{6}$.

Ex. Find the interest on \$250 for 2 yr. 4 mo. 12 da. at 6%.

The interest on \$1 for 2 yr. at 6% $= 2 \times \$0.06 = \$0.12$.
The interest on \$1 for 4 mo. at 6% $= 4 \times \$0.005 = \$0.02$.
The interest on \$1 for 12 da. at 6% $= 12 \times \$0.000\frac{1}{6} = \$0.002$.
∴ the interest on \$1 for 2 yr. 4 mo. 12 da. at 6% $= \$0.142$.
∴ the interest on \$250 for 2 yr. 4 mo. 12 da. $= 250 \times \$0.142 = \$35.50$.

253. To find the interest at 5%, subtract $\frac{1}{6}$ of the interest at 6%; at 7%, add $\frac{1}{6}$ of the interest at 6%, etc.

EXERCISE 58

1. By the 6% method find the interest on \$100 for 2 yr. 3 mo. 10 da. at 4%; at $4\frac{1}{2}$%; at $6\frac{1}{2}$%; at $7\frac{1}{2}$%; at 3%; at $3\frac{1}{2}$%.

Find the interest on:

2. \$325 for 1 yr. 2 mo. at 6%.
3. \$450 for 2 yr. 3 mo. 14 da. at $5\frac{1}{2}$%.
4. \$315.75 for 2 mo. 15 da. at 8%.
5. \$2000 for 30 da. at 6%.
6. \$115.50 for 3 mo. 10 da. at $7\frac{1}{2}$%.
7. \$387.50 for 6 mo. at 5%.
8. \$524.70 for 60 da. at 6%.
9. \$97.30 for 3 mo. 10 da. at 7%.
10. \$80.60 for 1 yr. 6 mo. 15 da. at 3%.

254. Exact Interest. To find the exact interest we must take the exact number of days between dates and reckon 365 da. to a year. Exact interest is used by the United States government, by some banks, and to some extent in other business transactions.

Ex. Find the exact interest on \$2500 from April 10 to Sept. 5 at 5%.

148 = the number of days from April 10 to Sept. 5.

∴ the interest on \$2500 for 148 da. at 5% $= \frac{5 \times 148 \times \$2500}{100 \times 365} = \$50.68$.

EXERCISE 59

Find the exact interest on:

1. \$575 from July 5 to Sept. 5 at 7%.
2. \$125 from Jan. 1 till Nov. 1 at 6%.
3. \$10000 from March 10 till June 1 at 5%.
4. \$375.30 from April 25 till Aug. 1 at 6%.
5. Find the amount of \$375 at 6% exact interest from Nov. 11, 1903, till July 27, 1905.
6. May 10, 1903, \$500 is loaned at 6%. Find the amount due Sept. 1, 1905, exact interest.
7. If \$500 is loaned at 6% on July 28, 1905, when will it amount to \$720?
8. What is the difference between the exact interest and the common interest on \$1000 from July 1 till Nov. 1 at 6%? If the exact number of days between dates and 360 days to the year are taken, how much does the common interest differ from the exact interest?
9. Show that the difference between the common interest and the exact interest for a fractional part of a year is $\frac{1}{73}$ of the former and $\frac{1}{72}$ of the latter.
10. Hence, show that exact interest for a fractional part of a year may be obtained by subtracting $\frac{1}{73}$ part from the common interest, and the common interest may be obtained from the exact interest by adding $\frac{1}{72}$ part of itself.

Exact interest is the fairest, but on account of its inconvenience without tables is not generally used.

255. The following is a section of an interest table for the year of 365 da. at 6%:

DAYS	1000	2000	3000	4000	5000	6000	7000	8000	9000
60	9.863	19.726	29.589	39.452	49.315	59.178	69.041	78.904	88.767
61	10.027	20.055	30.082	40.110	50.137	60.164	70.192	80.219	90.247
62	10.192	20.384	30.575	40.767	50.959	61.151	71.342	81.534	91.726
63	10.356	20.712	31.068	41.425	51.781	62.137	72.493	82.849	93.205
64	10.521	21.041	31.562	42.082	52.603	63.123	73.644	84.164	94.685
65	10.685	21.370	32.055	42.740	53.415	64.110	74.795	85.479	96.164
66	10.849	21.699	32.548	43.397	54.247	65.096	75.945	86.795	97.644
67	11.014	22.027	33.041	44.055	55.068	66.082	77.096	88.110	99.123
68	11.178	22.356	33.534	44.712	55.890	67.068	78.947	89.425	100.603
69	11.342	22.685	34.027	45.370	56.712	68.055	79.397	90.740	103.562
70	11.507	23.014	34.521	46.027	57.534	69.041	80.548	92.055	103.562
71	11.671	23.342	35.014	46.685	58.356	70.027	81.699	93.370	105.041
72	11.836	23.671	35.507	47.342	59.178	71.014	82.849	94.685	106.521
73	12.000	24.000	36.000	48.000	60.000	72.000	84.000	96.000	108.000
74	12.164	24.329	36.493	48.658	60.822	72.986	85.151	97.315	109.479
75	12.329	24.658	36.986	49.315	61.644	73.973	86.301	98.630	110.959
76	12.493	24.986	37.479	49.973	62.466	74.959	87.452	99.945	112.438
77	12.658	25.315	37.973	50.630	63.288	75.945	88.603	101.260	113.918
78	12.822	25.644	38.466	51.288	64.110	76.932	89.753	102.575	115.397
79	12.986	25.973	39.959	51.945	64.932	77.918	90.904	103.890	116.877
80	13.151	26.301	39.452	52.603	65.753	78.904	92.055	105.205	118.356

YEARS	1000	2000	3000	4000	5000	6000	7000	8000	9000
1	60	120	180	240	300	360	420	480	540
2	120	240	360	480	600	720	840	960	1080
3	180	360	540	720	900	1080	1260	1440	1620
4	240	480	720	960	1200	1440	1680	1920	2160
5	300	600	900	1200	1500	1800	2100	2400	2700
6	360	720	1080	1440	1800	2160	2520	2880	3240

Ex. By the use of the table find the interest on \$4650 for 2 yr. 67 da. at 6%.

	for 2 yr.		for 67 da.
The interest on \$4000 =	\$480	+	\$44.06
The interest on 600 =	72	+	6.61
The interest on 50 =	6	+	0.55

The interest on \$4650 = \$558 + \$51.22 = \$609.22.

EXERCISE 60

By the use of the table find the interest **on**:

1. \$500 for 65 da.
2. \$1000 for 60 da.
3. \$5225 for 73 da.
4. \$10575 for 1 yr. 60 da.
5. \$1846 for 2 yr. 80 da.
6. \$1710 for 75 da.
7. \$1250 for 63 da.
8. \$2120 from July 2 till Sept. 5.
9. \$648.60 from Jan. 10 till March 15.
10. \$1410 from May 1 till July 10.

256. In any problem in interest there are four elements involved, the **principal**, the **rate**, the **time** and the **interest**. When any three of these are given, the other can be found. As indicated above, the practical business problem is to find the interest when the principal, rate and the time are given. However, the principles involved in the following illustrative examples are frequently met with in business:

Ex. 1. What principal will produce \$72 interest in 1 yr. 6 mo. at 6%?

Solution. Let x = the principal.

Then $\frac{3}{2} \times 6\%$ of $x = \$72$.

$$\therefore x = \frac{2 \times \$72}{3 \times 0.06} = \$800.$$

Ex. 2. At what rate will \$800 produce \$72 interest in 2 yrs.?

Solution. Let $x\%$ = the rate.

Then $x\% \times 2$ of $\$800 = \$72$.

$$\therefore x\% = \frac{\$72}{2 \times \$800} = \tfrac{9}{200} = 4\tfrac{1}{2}\%.$$

Ex. 3. In what time will \$1000 produce \$70 interest at 4%?

Solution. Let x = the time.

Then $x \times 4\%$ of $\$1000 = \$70$.

$$x = \frac{\$70}{0.04 \times \$1000} = \tfrac{7}{4}.$$

$$\therefore x = \tfrac{7}{4} \text{ yr., or 1 yr. 9 mo.}$$

Ex. 4. What principal will amount to \$1238 in 6 mo. 10 da. at 6%?

Solution. Let x = the principal.

Then $x + 6\% \times \frac{190}{360}$ of $x = \$1238$ = the amount.

$$\therefore x = \frac{\$1238}{1 + 0.06 \times \frac{190}{360}} = \$1200.$$

EXERCISE 61

Find the rate at which:

1. \$750 will produce \$67.50 interest in 1 yr. 6 mo.
2. \$2000 will produce \$105 interest in 9 mo.

Find the time in which:

3. \$250 will produce \$25 interest at 5%.
4. \$1200 will produce \$90 interest at 6%.

5. \$850 will produce \$106.25 at 5%.

6. \$2000 will produce \$105 at 7%.

What principal will produce:

7. \$108 interest in 1 yr. 6 mo. at 6%?

8. \$61.25 interest in 2 yr. 6 mo. at 7%?

9. \$262.50 interest in 1 yr. 6 mo. at 5%?

What principal will amount to:

10. \$575 in 2 yr. 6 mo. at 6%?

11. \$1050 in 1 yr. at 5%?

12. \$1570 in 1 yr. 2 mo. at 4%?

13. A man with \$25000 invested in his business makes $12\frac{1}{2}$% profit annually. He sells out and invests the \$25000 at 5% and works on a salary of \$2000 per annum. Does he make or lose by the change and how much?

14. A man invests \$20000 in business and makes \$6000 in one year on his sales. If the total expenses of running the business are \$3500, what rate does he make on his money?

15. A house and lot costs \$1800 and rents for \$16 a month. If taxes, insurance and repairs cost \$72 a year, what rate is earned on the investment?

16. Jan. 1, 1900, \$450 are deposited in a savings bank at 3%. Find the amount due July 3, 1900.

257. Compound Interest. In **compound interest** the interest is added to the principal at the end of each interest period. Then the amount becomes the new principal for the next interest period.

Unless otherwise stated, interest is compounded annually, though it may be compounded semiannually, quarterly, etc., by agreement. In most states compound interest cannot be collected by law, but payment of it does not constitute usury.

Ex. Find the compound interest on \$500 for 3 yr. 4 mo. 15 da. at 4%.

Solution. \$500 = principal first year.
0.04
20.00 = interest first year.
500
\$520.00 = amount first year = principal second year.
0.04
20.80 = interest second year.
520
\$540.80 = amount second year = principal third year.
0.04
21.63 = interest third year.
540.80
\$562.43 = amount third year = principal fourth year

Interest on \$562.43 for 4 mo. 15 da. at 4% = \$8.44.

\$562.43 + \$8.44 = \$570.87 = amount for 3 yr. 4 mo. 15 da.
500
\$70.87 = compound interest for 3 yr. 4 mo. 15 da

258. The chief use of compound interest is among large investors, such as life insurance companies, building and loan associations, private banking establishments, etc., who wish to compute the income from reinvestment of interest when due. For such work compound interest tables are used.

The following is a section of such a table:

Periods	1 Per Cent	1½ Per Cent	2 Per Cent	3 Per Cent	4 Per Cent
1	1.0100000	1.015000	1.020000	1.030000	1.040000
2	1.0201000	1.030225	1.040400	1.060900	1.081600
3	1.0303010	1.045678	1.061208	1.092727	1.124864
4	1.0406040	1.061364	1.082432	1.125509	1.169859
5	1.0510100	1.077284	1.104081	1.159274	1.216653
6	1.0615201	1.093443	1.126162	1.194052	1.265319
7	1.0721353	1.109845	1.148686	1.229874	1.315932
8	1.0828567	1.126493	1.171660	1.266770	1.368569
9	1.0936852	1.143390	1.195093	1.304773	1.423312
10	1.1046221	1.160541	1.218994	1.343916	1.480244
11	1.1156683	1.177949	1.243374	1.384234	1.539454
12	1.1268250	1.195618	1.268242	1.425761	1.601032
13	1.1380932	1.213552	1.293607	1.468534	1.665074
14	1.1494742	1.231756	1.319479	1.512590	1.731676
15	1.1609689	1.250232	1.345868	1.557967	1.800944
16	1.1725786	1.268985	1.372786	1.604706	1.872981
17	1.1843044	1.288020	1.400241	1.652848	1.947901
18	1.1961474	1.307341	1.428246	1.702433	2.025817
19	1.2081089	1.326951	1.456811	1.753506	2.106849
20	1.2201900	1.346855	1.485947	1.806111	2.191123

Solution of the above example by means of the tables.

The amount of $1 for 3 yr. at 4% is $1.12486.

The amount of $500 will be 500 × $1.12486 = $562.43.

The example may now be completed by using the tables for simple interest for 4 mo. 15 da., or as on p. 175.

EXERCISE 62

1. Find the compound amount and the compound interest of $2000 for 3 yr. 6 mo. at 4% payable semi-annually.

Note. It is evident that if interest is 4% compounded semiannually for 3 yr. 6 mo., the amount is the same as if the rate is 2% compounded annually for 7 yr.

2. What is the difference between the simple and compound interest on $750 for 2 yr. 7 mo. at 5%?

3. Find the amount of $5000 compounded annually for 4 yr. at 4%.

4. Find the amount of $3500 compounded semiannually for 5 yr. at 3%; at 4%; at 6%.

259. Annual Interest. If a note or other written agreement contains the expression "with annual interest" or "with interest payable annually," the interest is due at the end of each year, and if not then paid, will draw simple interest until paid. Such a note or agreement is said to bear **annual interest.** The same principle is applied when the interest is payable semiannually, quarterly, etc.

As in the case of compound interest, in most states annual interest cannot be collected by law, but does not constitute usury.

Ex. George Reed borrowed $1500 at 7%, and agreed to pay interest annually. Having paid no interest, he wishes to settle at the end of 3 yr. 3 mo. 20 da. What is the amount due?

The simple interest on $1500 for 3 yr. 3 mo. 20 da. at 7% = $347.08.
Then, in addition to this, the simple interest on

$105 at 7% for 2 yr. 3 mo. 20 da.
$105 at 7% for 1 yr. 3 mo. 20 da.
$105 at 7% for 3 mo. 20 da.
or on $105 at 7% for 3 yr. 11 mo. = $28.79.

Hence, principal borrowed	=	$1500
Simple interest	=	347.08
Simple interest on interest not paid when due	=	28.79
∴ total amount due at annual interest	=	$1875.87

EXERCISE 63

1. What is the difference between the simple interest and the annual interest in the preceding example? How long is it after the date on which the money is borrowed before the annual interest begins to differ from the simple interest?

2. What is the difference between the compound interest and the annual interest in the preceding example? How long is it before the compound interest begins to differ from the simple interest? from the annual interest?

3. Find what $2500 will amount to in 4 yr. 10 mo. 18 da. at 5% simple interest and at 5% compound interest.

4. Sept. 1, 1896, a man borrows $500 at 6% interest, payable annually. If nothing is paid until Dec. 1, 1901, how much is due?

5. Notes are sometimes given with interest coupons attached. These coupons draw interest, frequently at a higher rate than the note itself, if not paid when due. A coupon note for $2200 is issued July 1, 1896, at 6% interest. Nothing is paid until July 1, 1902. Find the amount at that date, the coupons bearing 7% if not paid when due.

260. Promissory Notes. A **promissory note** is a written promise to pay to a certain person named in the note a specified sum of money on demand, or at a specified time.

261. A promissory note is **negotiable**, *i.e.* can be transferred from one owner to another by indorsement when it is made payable *to the order of* a definite person, *or to bearer.*

The following is a common form of a negotiable promissory note:

> *$1000. Detroit, Mich., March 27, 1904.*
>
> *Sixty days after date I promise to pay to the order of Henry James one thousand and $\frac{no}{100}$ dollars, value received, with interest at 6%.*
>
> *No. 45. Due ______ Andrew Johnson.*

Andrew Johnson is the **maker** of this note, Henry James is the **payee**, and $1000 is the **face**.

262. The above note would be **non negotiable** if the words "the order of" were omitted. In that case the note would be payable to Henry James only.

263. If a note is sold by the payee, he must indorse it by writing his name across the back.

264. There are three common forms of indorsement:

(1) **In blank**, the indorser simply writing his name across the back, thus making the note payable to the bearer.

> INDORSEMENT IN BLANK
>
> *Henry James.*

(2) **In full**, the indorser directing the payment to the order of a definite person.

> INDORSEMENT IN FULL
>
> *Pay to the order of Sibley and Hatch.*
>
> *Henry James.*

> **Qualified Indorsement**
> *Pay to the order of*
> *Sibley and Hatch.*
> *Without recourse.*
> *Henry James.*
> OR SIMPLY
> *Without recourse.*
> *Henry James.*

(3) **Qualified**, the indorser avoiding responsibility by writing the words "without recourse" over his name.

265. By indorsing a note either in blank or in full, the payee becomes responsible for its payment if the maker fails to pay it. The indorsement in full will insure greater safety since in this case the note is made payable to a definite person.

266. A note made payable to Henry James, or bearer, is also negotiable, but does not need indorsement.

267. The custom of allowing three **days of grace** in the payment of a note has been abolished in many states and is rarely used in others.

268. The note on page 179 will **mature** March 27 + 60 days, or May 26, if no grace is allowed. It will mature March 27 + 63 days, or May 29, if grace is allowed. In states where grace is allowed this is indicated by writing in the note, "Due May 26/29."

269. In most states a note falling due on Sunday or a legal holiday must be paid the preceding business day.

EXERCISE 64

1. Write a 30-day promissory note for $500, payable to the order of James Black, bearing the legal rate of interest in your state. By indorsement make the note payable to Henry Wood.

2. What is the maximum contract rate of interest in your state? What is the penalty for usury?

3. Write a 60-day promissory note for $100, with yourself as maker and Charles Jennings as payee, the note bearing the date May 5, 1903. If the note is payable to Charles Jennings, or bearer, in what way may it be transferred? If made payable to Charles Jennings, or order, in what way may it be transferred?

4. Find the date of maturity of the note required in Ex. 3. Add days of grace if they are used in your state.

5. Find the interest on the above note.

6. $250. YPSILANTI, MICH., April 11, 1905.

> Ninety days after date I promise to pay to the order of William Jordan two hundred fifty and $\frac{no}{100}$ dollars, value received, with interest at 5%.
>
> L. M. DAVIS.

When is the above note due? What is the amount at maturity? Who pays the note? Who receives the money? Who receives the note when paid? What indorsement is necessary if the note is sold to John Brown?

270. Partial Payments. Frequently the maker of a note, not being able to pay the whole amount at one time, makes several **partial payments**, which are indorsed on the back of the note with the date of payment.

Ex. April 6, 1902, a man buys a farm for $7500, paying $5000 in cash, and giving a note for the remainder at 6%, with the privilege of paying all or part of it any time within 3 yr. The following payments are made and indorsed on the note by the payee:

Oct. 1, 1903, $ 500
April 1, 1904, $ 50
Oct. 1, 1904, $1000

What amount is due April 6, 1905?

Face of note . . . . = $2500.00

	yr.	mo.	da.
Oct. 1, 1903 =	1903	10	1
April 6, 1902 =	1902	4	6
	1	5	25

The interest on $2500 for 1 yr. 5 mo. 25 da. at 6% = $ 222.92
Amount due Oct. 1, 1903 . . . . . . = 2722.92
Payment . . . . = 500.00
Balance due = new principal . . . . . = $2222.92

	yr	mo	da.
Oct. 1, 1904 =	1904	10	1
Oct. 1, 1903 =	1903	10	1
	1		

The interest on $2222.92 for 1 yr. at 6% . . = $ 133.38
Amount due Oct. 1, 1904 . . . . . . = 2356.30
Payment April 1, 1904 (less than interest due April 1, viz. $66.69) . . . . . . = 50.00
Payment Oct. 1, 1904 . . . . . . = 1000.00
Balance due = new principal . . . . . = $1306.30

	yr.	mo.	da.
April 6, 1905 =	1905	4	6
Oct. 1, 1904 =	1904	10	1
		6	5

The interest on $1306.30 for 6 mo. 5 da. . . = 40.28
∴ the amount due April 6, 1905, is . . . = $1346.58

CHECK DIFFERENCE BETWEEN DATES

	yr.	mo.	da.	Partial Differences yr.	mo.	da.
Date of settlement	1905	4	6	1	5	25
Date of note	1902	4	6	1		
Difference in time =	3				6	5
				2	11	30 = 3 yr.

271. The above example is solved by the **United States Rule of Partial Payments**, which is the legal method in most states. By this method the amount of the note is found to the time when the payment, or the sum of the payments, equals or exceeds the interest due. From this amount the payment, or sum of the payments, is subtracted. This operation is repeated to the time of the next payment and so on.

272. It will be seen that three cases may arise under this rule:

(1) The payment may be exactly equal to the interest due. In this case the payment simply cancels the interest, and the balance due remains the same as the original principal.

(2) The payment may be greater than the interest due. In this case the balance due is diminished by the amount the payment exceeds the interest due.

(3) The payment may be less than the interest due. In this case if the unpaid balance of the interest were added, the principal would be increased, and the debtor would be paying more interest than if he had made no payment at all. Therefore, when the payment is less than the interest due, no change is made at that time in the principal; but the interest is reckoned to the date when the sum of the payments does exceed the interest due, and then the sum of these payments is subtracted.

273. The following method of solving problems in partial payments, called **The Merchants' Rule**, is used among many business men when the note runs for one year or less:

Ex. Sept. 1, 1904, a merchant takes a note for $327.50 from a customer in payment for some goods. The note is to run for 1 yr. at 6%. During the year the following payments are made: Nov. 1, 1904, $75; April 1, 1905, $100; Aug. 1, 1905, $50. Find the amount due Sept. 1, 1905.

The amount of $327.50 for 1 yr. at 6% =		$347.15
The amount of $75 for 10 mo. at 6%	= $ 78.75	
The amount of $100 for 5 mo. at 6%	= 102.50	
The amount of $50 for 1 mo. at 6%	= 50.25	231.50
Balance due Sept. 1, 1905		$115.65

274. By this method the amount of the note is found from the date of the note to the time of settlement. The amount of each payment is also found from its date to the time of settlement. The sum of the amounts of the payments is then subtracted from the amount of the principal.

275. Some states, *e.g.* New Hampshire and Vermont, have rules of their own for solving problems in partial payments. In such states it is left for the teacher to present the rule.

EXERCISE 65

1. Which of the above methods is better for the debtor? Which is better for the creditor?

2. $325 CLEVELAND, OHIO, May 15, 1896.

Three years after date I promise to pay W. W. Johnson, or order, three hundred twenty-five dollars, value received, with interest at 7%.

HENRY GEORGE.

Indorsements: May 15, 1897, $22.75; May 15, 1898, $22.75; June 29, 1900, $100; June 12, 1902, $50. Find the amount due June 12, 1904.

3. Jan. 30, 1897, Arthur Ross borrowed $150; May 10, 1897, $125; and Dec. 10, 1900, $100, all at 6% interest. He paid Oct. 1, 1901, $100; and Dec. 10, 1902, $100. Find the amount due March 10, 1903.

4. March 1, 1897, a man buys a farm for $6000. He pays $3000 in cash and gives a note for the remainder at 6%. He makes a payment of $500 on each of the following dates: March 1, 1898; March 1, 1900; Sept. 1, 1900; and March 1, 1901. Find the amount due March 1, 1902.

5. $3500. YPSILANTI, MICH., Aug. 15, 1899.

Five years after date I promise to pay John Robinson, or order, three thousand five hundred dollars, value received, with interest at 6%.

JAMES ROWE.

Indorsements: Nov. 5, 1902, $300; March 14, 1904, $200; May 14, 1905, $2000. Find the amount due Aug. 14, 1905.

6. A note for $5000, dated March 1, 1903, and payable two years from date, with interest at 5%, has on it the following indorsements: April 1, 1903, $500; June 1, 1903, $500; Sept. 1, 1903, $200; and May 1, 1904, $500. Find the amount due March 1, 1905.

7. Dec. 10, 1903, a merchant takes a note for $260 to run 1 yr. at 7%. During the year the following payments are made: Feb. 1, 1904, $50; June 21, 1904, $25; Oct. 10, 1904, $100. Find the amount due Dec. 10, 1904.

BANKS AND BANKING

276. A **bank** is an institution that deals in money and credit. **Credit** is a promise to pay money in the future. The chief **instruments** of credit are **checks**, **drafts**, and **notes.**

It is a mistake to say that banks deal only in money. Their most profitable business is in credit transactions. Banks, however, have a capital of their own which serves as a guarantee fund. Neither do all bank deposits represent money intrusted to banks by individuals. Most of them represent credit loaned to individuals by banks. Thus, if a bank accepts a promissory note for $5000, it may give in return a deposit credit for $5000 less the discount and will thereby add that sum to its deposits.

277. There are several kinds of banks, among which are **national banks,** organized under the National Banking Act of 1863 and the amendments that have been made thereto; **state banks,** organized under the laws of the state in which they are situated; **savings banks**, and **private banks.**

278. National banks are subject to rigorous supervision by federal authorities. All banks organized under state laws are subject to similar supervision by state authorities.

279. The chief functions of banks are to receive deposits, to lend money on promissory notes, bonds, and mortgages, to discount merchants' notes before they are due, and to buy, sell, and collect drafts or bills of exchange. National

banks also issue bank notes which circulate as a medium of exchange. Savings banks and a few others allow interest on deposits.

280. On opening an account with a bank, the customer is usually given a pass book in which the dates and amounts of all deposits are entered on the credit side. If the customer wishes to draw money from the bank, or to pay a debt, he fills out a check similar to the following, and the dates and amounts of all such checks are entered on the debit side of his pass book:

No. ------ *New York,* ------------------ *190* ---

Fourth National Bank of New York

Pay to Ralph M. Brown ------ *or order $500.00* ---

Five hundred ------------------------ $\frac{no}{100}$ *Dollars.*

-------- *George E. Fox.* -------

George E. Fox is the **drawer** or **maker** of this check, and Ralph M. Brown is the **payee**

281. As in the case of promissory notes, checks may be made payable to **payee or order**, or to **payee or bearer**. The same rules of indorsement apply. When the depositor wishes to draw money at the bank, the check is made payable to **self.**

282. Banks also issue certificates of deposit:

CERTIFICATE OF DEPOSIT

With interest at the rate of 2 per cent per annum if left three months, 3 per cent per annum if left six months. Interest hereon will cease one year from date. No interest paid for fractional part of a month. This certificate not subject to check.

*No.*____ *Ypsilanti, Mich.*_____________*190*___

_____________________________*ha*__*deposited in the*

First National Bank of Ypsilanti,

____*Dollars,*

*payable to the order of*___________________________

subject to the rules of this Bank, on the return of this Certificate, properly endorsed.

____________________ ____________________

Cashier. *Teller.*

The money deposited on such a certificate is not subject to check and can be withdrawn only upon the presentation of the certificate properly indorsed.

EXERCISE 66

1. Write a check for \$35.75 in each of the forms indicated above, with yourself as drawer and Robert Lyons as payee. If necessary, indorse the check as when presented for payment.

2. July 22, 1903, a man deposits \$125 in a savings bank. The bank pays 3% on money left on deposit 3 mo., and $3\frac{1}{2}$% if left 6 mo. or longer. Money must be deposited the first of the month to draw interest for that calendar month. If the money is drawn out Nov. 1 and interest paid for full months, how much does the man receive? if drawn out Dec. 22? if drawn out March 1, 1904?

3. A man owns a certificate of deposit for \$500 dated Aug. 1, 1903. Feb. 1, 1904, he presents the certificate and draws \$200. What is the face of the new certificate issued?

4. Show that the following statement of the resources and liabilities of a savings bank will balance.

RESOURCES

Loans and discounts		\$946,542.72
Bonds, mortgages and securities		504,171.35
Premiums paid on bonds		1,218.75
Overdrafts		1,471.70
Furniture and fixtures		13,801.50
Other real estate		85,337.01
Items in transit		13,409.02
Due from banks in reserve cities	\$214,193.14	
Exchanges for clearing house	20,742.41	
U. S. and National bank currency	70,470.00	
Gold coin	68,200.00	
Silver coin	2,365.00	
Nickels and cents	83.28	

LIABILITIES

Capital stock paid in		\$200,000.00
Surplus fund		36,000.00
Undivided profits, net		19,781.58
Commercial deposits	\$609,991.33	
Certificates of deposit	94,897.63	
Due to banks and bankers	192,674.51	
Certified checks	12,124.52	
Cashier's checks	10,455.21	
Savings deposits	696,425.79	
Savings certificates	69,655.31	

283. If his financial standing is high, a person may borrow money from a bank by giving his individual note. The bank may, however, ask for **security**. In this case the borrower must have some responsible person indorse the note, or he must deposit **collateral security** in the form of stocks, bonds, etc.

284. The following is a common form of a bank note:

> *\$200 $\frac{no}{100}$* *Detroit, Mich., Sept. 6, 1904*
> *Three months after date, I... promise to pay*
> *to the order of..........*
> *The Commercial National Bank..........*
> *Two hundred and $\frac{no}{100}$..........Dollars*
> *at The Commercial Natl. Bank of Detroit, Mich.,*
> *with interest at 6 per cent per annum until due,*
> *and seven per cent per annum thereafter until paid.*
> *Value received.*
> *Jacob H. Rowe.*

If Mr. Rowe wishes to borrow \$200 he takes the above note to the bank and, if necessary, either furnishes an indorser or collateral security, such as bonds, etc., which he assigns to the bank. If the bank authorities are satisfied, he receives \$200 – \$3 = \$197, the interest for 3 mo. being deducted. This interest is called **bank discount**. Days of grace are now rarely used by bankers.

285. In discounting notes, banks count forward by days or months as stated in the note and usually reckon 360 days in the year. Thus, a note dated July 22 at 60 days will mature July 22 + 60 days, or Sept. 20. A note dated July 22 at 2 mo. will mature Sept. 22.

I forgot my checks.

EXERCISE 67

Each of the following notes is discounted on the date of issue. Find the date of maturity and the discount.

	Date of Note	Time	Face	Rate
1.	Jan. 2, 1905	60 da.	\$1000	6%
2.	Aug. 14, 1905	3 mo.	\$525	5%
3.	Aug. 1, 1905	90 da.	\$387.50	6%
4.	April 20, 1905	30 da.	\$500	6%
5.	June 27, 1905	2 mo.	\$325	7%

286. Business men frequently take notes due at some future date from their customers, and in case money is needed before the notes are due, sell them to a bank. (Such a note is shown on page 179.) The seller must give satisfactory security. The bank pays the sum due at maturity less the discount from the date of discount to the date of maturity. The sum paid by the bank is called the **proceeds**. These notes may or may not bear interest. The following examples will illustrate both cases:

Ex. 1. A note for \$527.30, dated Aug. 31, 1904, due in 90 da., without interest, was discounted at the bank Oct. 10, at 6%. Find the proceeds.

Solution.

Face of note	=	\$527.30
Discount for 50 da.	=	4.39
Proceeds	=	\$522.91

Ex. 2. A note for \$378.50, dated Aug. 1, 1904, due in 4 mo. at 6%, was discounted at the bank Oct. 1, 1904, at 6%. Find the proceeds.

Solution.

Face of note	=	\$378.50
Interest for 4 mo.	=	7.57
Amount discounted	=	\$386.07
Discount for 2 mo.	=	3.86
Proceeds	=	\$382.21

EXERCISE 68

Find the discount and proceeds of each of the following non-interest bearing notes:

	Face	Date	Time	Date of Discount	Rate of Discount
1.	$500	July 1	30 da.	July 10	7%
2.	$225.75	April 10	2 mo.	May 1	6%
3.	$253.30	Dec. 14	90 da.	Jan. 2	6%
4.	$150.40	Aug. 12	60 da.	Sept. 5	5%
5.	$1250	Nov. 1	3 mo.	Dec. 1	6%

Find the discount and proceeds of each of the following interest-bearing notes:

	Face	Date	Time	Rate of Interest	Date of Discount	Rate of Discount
6.	$1500	Aug. 10	90 da.	6%	Sept. 1	7%
7.	$97.30	Oct. 2	60 da.	7%	Nov. 1	6%
8.	$152.20	Sept. 4	4 mo.	5%	Oct. 20	6%
9.	$750.50	Jan. 4	30 da.	4%	Feb. 1	7%
10.	$431.40	June 17	2 mo.	6%	July 10	6%

11. A merchant's bank account is overdrawn $2150.75, and he presents to the bank the following notes, which are discounted Dec. 5 at 6% and placed to his credit. What is his balance?

Face	Date	Time	Rate of Interest
$500	Nov. 12	60 da.	5%
$1250.25	Sept. 30	90 da.	4%
$727.40	Oct. 25	3 mo.	no interest

12. For how much must I give my note at the bank, discounted at 6% and due in 60 da., to realize $1500?

Suggestion. Find the proceeds of $1 discounted at 6% for 60 da. and divide $1500 by the result.

EXCHANGE

287. The subject of exchange treats of methods of canceling indebtedness between distant places without the actual transfer of money. This may be accomplished in any one of the following ways:

(1) **By Postal Money Order.** Money orders may be sent for any amount from 1 ct. to $100. The rates are:

For orders for sums not exceeding	$2.50	3 cents.
If over $2.50 and not exceeding	$5.00	5 cents.
If over $5.00 and not exceeding	$10.00	8 cents.
If over $10.00 and not exceeding	$20.00	10 cents.
If over $20.00 and not exceeding	$30.00	12 cents.
If over $30.00 and not exceeding	$40.00	15 cents.
If over $40.00 and not exceeding	$50.00	18 cents.
If over $50.00 and not exceeding	$60.00	20 cents.
If over $60.00 and not exceeding	$75.00	25 cents.
If over $75.00 and not exceeding	$100.00	30 cents.

(2) **By Express Money Order.** The rates are the same as for postal orders. The company is responsible for the payment to wrong persons.

(3) **By Telegraphic Money Order.** In addition to the regular telegraphic charge for a 15-word message between the two places, telegraph companies make transfers of money between their offices subject to the following charges:

For orders for sums not exceeding	$25	25 cents.
Over $25 and not exceeding	$50	35 cents.
Over $50 and not exceeding	$75	60 cents.
Over $75 and not exceeding	$100	85 cents.

Add 20¢ for each $100 or fractional part thereof after the first $100 up to and including $3000.

(4) **By Check.** A personal check on a home bank where the sender has money deposited can be sent. This check, when properly indorsed by the payee and presented at his bank, will probably be cashed without charge. The bank may charge a small fee, called **exchange**, for collecting.

(5) **By Bank Draft.** The ordinary form of bank draft is as follows:

> No. 42,786
>
> **Central Savings Bank**
>
> $\$135\frac{75}{100}$ *Detroit, Mich., Sept. 6, 1905*
>
> *Pay to the order of James H. Kastle*
>
> *One hundred thirty-five and $\frac{75}{100}$ Dollars.*
>
> *To The Fourth National Bank of the City of New York.*
>
> *George M. Case*
> *Cashier.*

The draft differs from the check in that it is drawn by one bank on another, while the check is drawn by an individual on a bank where he has money deposited.

288. Most banks keep money on deposit in some bank called a **correspondence bank**, in a large commercial center like New York or Chicago.

If a customer of a local bank in the West wishes to pay a debt in the East, he buys a draft, signed by the cashier of the local bank, and drawn against the correspondence bank in New York City. This draft will pass as cash at any bank. The above draft is drawn by the Central Savings Bank of Detroit, and the correspondence bank is the Fourth National Bank of the City of New York.

289. The following is another form of a draft, called the **commercial draft**.

> $\$1000\frac{no}{100}$ *New York, Aug. 15, 1903.*
>
> *At sight pay to the order of* First National Bank
>
> *One thousand and $\frac{no}{100}$ Dollars.*
>
> *Value received, and charge the same to the account of*
>
> *To Hawkes & Co., Detroit, Mich.* *American Book Co.*

Hawkes & Co. owe the American Book Co. $1000 past due. The American Book Co. draws the above sight draft payable to the order of the First National Bank of New York, and deposits it for collection. The First National Bank sends it to a Detroit bank to collect. The Detroit bank demands payment of Hawkes & Co. If payment is made, the money is remitted to New York. If payment is refused, the draft is returned to the New York bank, and the American Book Co. is notified. Some other means of collecting must then be employed.

290. If the account against Hawkes & Co. were not due for 60 days, the draft would read as follows:

> $\$1000\frac{no}{100}$ *New York, Aug. 15, 1903.*
>
> *At sixty days sight pay to the order of* First National Bank *One thousand and $\frac{no}{100}$ Dollars.*
>
> *Value received, and charge the same to the account of*
>
> *To Hawkes & Co. Detroit, Mich.* *American Book Co.*

This draft is called a **time draft** and would be taken to Hawkes & Co. as before, who, if they intended to pay it, would write across the face in red ink:

Accepted Aug. 21, 1903.
Hawkes & Co.

After writing these words across the draft, Hawkes & Co. have agreed to pay $1000, and the draft becomes the same as a promissory note.

291. Fluctuations of Exchange. If the banks of San Francisco have sold drafts for a larger sum on the New York banks than they have on deposit in New York, it will be necessary to send money enough to New York to balance the account. The money is usually sent by express at some expense, which must be borne by the purchaser of the drafts. In this case a draft on New York would cost slightly in excess of its face. This excess is called a **premium**. A draft sold at less than its face is said to be sold at a **discount**.

Premiums and discounts are usually quoted as a per cent of the face of the draft. Thus, a quotation of $\frac{1}{10}\%$ premium means that a draft for $100 may be purchased for $100.10. Sometimes the quotation is a certain amount per $1000. Thus, if New York exchange is quoted at $1.50 premium at New Orleans, a draft for $1000 will cost $1001.50.

The above quotations refer to sight drafts. Time drafts are discounted by banks in the same manner as promissory notes.

New York City is the greatest financial center of the United States, and so much business is transacted through the New York banks that New York exchange is generally at a premium. Consequently banks are always willing to cash such checks at par value. People in New York usually pay their indebtedness outside of the city by checks or drafts on New York banks, which find a ready sale at any bank.

292. The Clearing House is an institution organized by the banks of every large city to facilitate settlement of claims against one another.

Clerks from each bank bring daily to the clearing house the checks, etc., due them from all other associated banks, each bank being represented by a separate package. Balances are struck between the credits and debits of each bank against all others, and the manager certifies the amount which each bank owes to the associated banks or is entitled to receive from them. The banks whose debits exceed their credits pay in the balance to the clearing house, which issues clearing house certificates to the banks whose credits exceed their debits. In the New York Clearing House, which is the largest in America, nearly sixty billion dollars of clearings were made in 1904, with only three billions of dollars of balances paid in money.

EXERCISE 69

1. A quotation of $2.50 premium is equivalent to what per cent?
2. What is the cost in Kansas City of a draft on New York for $67.50 at $\frac{1}{4}\%$ premium?
3. What is the cost in Galveston of a draft on New York for $4380.50 at $2.50 premium?
4. In Ex. 2 and Ex. 3 which city is owing the other money?
5. Find the cost of a draft for $500 payable in 60 days after sight, exchange being $\frac{1}{4}\%$ premium, interest 6%.
6. Find the cost of sending $67.50 by telegraphic money order if a 15-word message costs 50 ct.
7. How much would it cost to send the same amount by postal money order? by express money order?
8. A draft on New York for $10000 costs $9980 in Chicago. Is exchange at a premium or a discount? What is the rate of exchange? The balance of trade is in favor of which city?
9. A merchant has a 60-day note for $1200 discounted at the bank at 6% and purchases a draft with the proceeds,

exchange \$1.00. He sends the draft to a creditor to apply on account. How much is placed to his credit?

10. A Chicago banker discounts a draft for \$2500, payable at St. Louis 60 days after sight. What are the proceeds, exchange at $\frac{1}{2}$% discount, interest 6%?

FOREIGN EXCHANGE

293. Foreign exchange is the same in all essential features as domestic exchange, the difference being that exchange takes place between cities in different countries.

294. A draft on a foreign country, usually called a **bill of exchange**, is payable in the currency of the country on which it is drawn.

295. Foreign bills of exchange are generally written in duplicate, called a **set of exchanges**, and are of the following form:

New York, Aug. 19, 1903.

Exchange for £100.

Ten days after sight of this first of exchange (second of same date and tenor unpaid) pay to the order of ---------------- E. H. Mensel ----------------

---------- one hundred pounds sterling, ----------

and charge the same, without further advice, to

To Baring Brothers, *George E. Fox.*
London.

No. 173645.

The duplicate substitutes "second of exchange" for "first of exchange" and "first of same date" for "second of same date," in the original. Either one being paid, the other becomes void.

296. The **par of exchange** between two countries is the value of the monetary unit of one expressed in that of the other. Thus, the gold in the English pound is worth \$4.8665. Exchange on Paris and other countries using the French monetary system is usually quoted at so many francs to the dollar, sometimes at so many cents to the franc. The par of exchange is about $5.18\frac{1}{8}$ francs to the dollar, or 19.3 cents to the franc. Exchange on Germany is quoted at so many cents on 4 marks. The par of exchange is 95.2; quoted in cents per mark it is 23.8.

EXERCISE 70

1. What is the cost of a draft on London for £150, exchange \$4.925?

2. What is the cost of a draft on Paris for 1200 francs, exchange 5.20?

3. In either Ex. 1 or 2 is the balance of trade in favor of the United States?

4. A tourist purchased a letter of credit and drew £80 at London, 1500 francs at Paris, 750 marks at Berlin. How much did the letter cost him if exchange is $\frac{3}{4}$% premium at London, $\frac{1}{2}$% premium at Paris, and $\frac{1}{4}$% premium at Berlin?

5. What is the cost of a draft on Leipsic for 525 marks, exchange 96? exchange 24?

6. What is the cost of a draft on London for £75, exchange \$4.857?

STOCKS AND BONDS

297. A **corporation** or **stock company** is an association of individuals under the laws of a state for the purpose of transacting business as one person. Large-scale production is now usually conducted by corporations.

A corporation is managed by officers elected by a board of directors who are chosen by the stockholders, each stockholder having as many votes as he owns shares of stock. The **capital stock** is divided into a certain number of **shares**, the **par value** of which is determined by the number of shares into which the stock is divided. Thus, a capital stock of $50000 may be divided into 500 shares of $100 each, or 2000 shares of $25 each, etc. Stockholders may own any number of shares and participate in the profits according to the number of shares they own.

298. If a company is prosperous and makes more than its expenses, a **dividend** is paid to the stockholders. The dividend is usually a certain per cent of the par value of the stock, or sometimes so many dollars per share. If the rate of dividend is higher than the current rate of interest, there usually will be a demand for the stock and it will sell at a **premium**. If the rate of dividend is lower, the demand will be slight and the stock will sell at a **discount**.

299. Companies frequently issue two kinds of stock, **preferred** and **common**. The holders of preferred stock are entitled to first share in the net earnings of the corporation up to a certain amount, usually from 5% to 7% of the par value. The holders of common stock are entitled to a share, or all of what is left after the dividend on the preferred stock is paid.

Stock is sometimes issued to the stockholders of a corporation without a corresponding increase in the value of the property. Such stock is called **watered stock**. A corporation may be prohibited by its charter, or by law, from paying dividends in excess of a certain amount. Thus, if a corporation with a capital stock of $100000 makes $16000 and wishes to pay this amount in dividends, but is prohibited from paying more than 8%, watered stock, equal in amount to the capitalization of the corporation, may be issued to the stockholders and an 8% dividend (= $16000) may be declared upon this new basis.

300. Since it is difficult for individuals to buy and sell stock personally, the business is usually transacted through a **stock broker**, who charges a small per cent, called brokerage (usually $\frac{1}{8}$%), of the par value of the stock bought or sold. The stock broker generally belongs to an organization called a **stock exchange**. The New York Stock Exchange is the most important in America.

301. Generally each stockholder is responsible only to the extent of the par value of the stock he owns. In the case of national banks, however, a stockholder is liable to the amount of the par value of his stock in addition to the amount paid for the purchase of the stock.

302. Investors often buy stocks and hold them for the dividend they yield. Speculators buy them to sell at a profit. Speculators usually buy on a **margin**, that is, they pay only a part of the purchase price and borrow the rest by depositing the certificate of stock as collateral. A man who buys stock on a 20-point margin pays down 20% of the par value and borrows the rest. A "bull" is a buyer of stocks which he hopes to sell at a profit. He acts on the belief that prices will go *up*. A "bear" is a seller of stocks which he does not possess, but borrows on the belief that prices will go *down*. Thus, if a stock is quoted at 50, a "bear," thinking it will go down to 45, may sell at 50, and deliver borrowed stock to the purchaser. If the stock goes down to 45, he will purchase it and return it to the owner, thus realizing a profit of 5 points. This is called selling stocks "**short**." Bears are said to be "short" of stock and bulls "long."

303. When for any reason a stock company finds the amount of money paid in by stockholders insufficient, it generally borrows money and issues **bonds**, secured by a mortgage on the property of the company. These bonds are written agreements to pay a certain sum of money within a stated time and at a fixed rate of interest. Bonds have a prior claim over any kind of stock.

304. National governments, states, counties, and cities often issue bonds, but without mortgages, the credit of such organization being considered good.

Registered bonds are issued in the name of the owner, and are made payable to him or his assignee. Interest, when due, is sent direct to the owner.

Coupon bonds are payable to bearer, and have small interest coupons attached, which are cut off when due, and the amount of interest is collected either personally, or through a bank. There is a coupon for each interest period.

Bonds are usually named from the rate of interest they bear, or from the date at which they are payable. Thus, Union Pacific 4's means Union Pacific bonds bearing 4% interest. U.S. 4's reg. 1907 means United States registered bonds bearing 4% interest and due in 1907. Western Union 7's coup. 1900 means Western Union coupon bonds bearing 7% interest and due in 1900.

305. The following quotations show the prices paid for stocks and bonds on a certain day. The daily newspaper will furnish the best source for quotations.

Stocks		Bonds	
Amalgamated Copper	$72\frac{3}{4}$	U.S. New 4's reg.	$135\frac{1}{4}$
A. T. and S. F.	$83\frac{3}{8}$	U.S. New 4's coup.	$136\frac{1}{4}$
A. T. and S. F. preferred	$98\frac{3}{8}$	U.S. 3's reg.	$107\frac{1}{4}$
Canadian Pacific	131	U.S. 3's coup.	108
National Biscuit Co.	$98\frac{3}{8}$	Atchison 4's	$102\frac{7}{8}$
National Biscuit Co. preferred	$105\frac{1}{2}$	N.Y. Central $3\frac{1}{2}$'s	$103\frac{5}{8}$
N. Y. Central	143	C. B. and Q. 4's	$93\frac{1}{2}$
		C. R. I. and P. 4's	105

Stocks		Bonds	
Railway Steel Spring	$33\frac{1}{2}$	Southern Ry. 5's	116
Railway Steel Spring preferred	$87\frac{1}{2}$	Detroit Gas Co. 5's	$106\frac{1}{2}$
U.S. Steel	$37\frac{1}{8}$	Chicago and Alton $3\frac{1}{2}$'s	$76\frac{1}{2}$
U.S. Steel preferred	$37\frac{1}{2}$	Hocking Valley $4\frac{1}{2}$'s	$106\frac{1}{2}$
Wabash	$29\frac{1}{8}$	B. and O. 4's	$100\frac{1}{8}$
Wabash preferred	$50\frac{1}{2}$	U.S. Steel 5's	80
Western Union preferred	$88\frac{3}{4}$		

306. Quotations are usually made at a certain per cent of the par value of the stock or bond. Thus, the quotation of $72\frac{3}{4}$ for Amalgamated Copper means $72\frac{3}{4}\%$ of the par value of one share. The purchaser must pay his broker $72\frac{3}{4} + \frac{1}{8} = 72\frac{7}{8}$, and the seller will receive from his broker $72\frac{3}{4} - \frac{1}{8} = 72\frac{5}{8}$.

307. In the following examples the par value of a share will be taken as \$100 unless otherwise stated. Brokerage at $\frac{1}{8}\%$ must be taken into account in each case where not otherwise stated.

Ex. **1.** A person buys 100 shares of A. T. and S. F. as quoted above, and sells 6 mo. later for $85\frac{1}{8}$, having received a dividend of 2%. Does he gain or lose, and how much, money being worth 6% per annum?

Solution. Each share costs $83\frac{3}{8} + \frac{1}{8} = 83\frac{1}{2}$,
and is sold for $85\frac{1}{8} - \frac{1}{8} = 85$.
∴ the gain on each share is $85 - 83\frac{1}{2} = 1\frac{1}{2}$, or \$1.50.
∴ the gain on 100 shares is $100 \times \$1.50 = \$150$.
The dividend received = 2% of \$10000 = \$200.
∴ the total gain is \$150 + \$200 = \$350.
The amount invested is $100 \times \$83\frac{1}{2} = \$8350$.
The interest for 6 mo. is $\frac{1}{2}$ of 6% of \$8350 = \$250.50.
∴ \$350 − \$250.50 = \$99.50 = the net gain.

Ex. **2.** A man sells short 100 shares of Canadian Pacific at 131 and three days later "covers" (that is, buys the stock) at $128\frac{6}{8}$. What is his net profit?

Solution. Each share sold yields $131 - \frac{1}{8} = \$130\frac{7}{8}$
Each share is bought for $128\frac{6}{8} + \frac{1}{8} = 128\frac{7}{8}$
Therefore the gain on each share is $\$2$
Therefore the net gain on 100 shares is $100 \times \$2 = \$200$.

EXERCISE 71

1. The capital stock of a company is $1000000, $\frac{1}{4}$ of which is preferred, entitled to a 7% dividend, and the rest common. If $47500 is distributed in dividends, what rate of dividend is paid on the common stock?

2. A person buys 302 shares of stock, par value $10, for $7 a share, paying 5 ct. a share brokerage. 6 mo. later, after having received a 5% dividend, he sells for $9.75 a share. How much does he make, money being worth 6%?

3. If, in Ex. 1, $77500 is distributed in dividends, which is the better stock to own, common or preferred?

4. Which is the better property to own, $1000 stock in a company at 6%, or one of its $1000 bonds at 4%?

5. Which is the safer against loss by theft, a coupon bond, or a registered bond? Which is the more readily transferred?

6. Why are U.S. 4's registered quoted at $135\frac{1}{4}$, while U.S. 4's coupon are quoted at $136\frac{1}{4}$?

7. How much will 50 shares of Amalgamated Copper cost?

8. How much will 75 Atchinson 4's cost?

9. How much will 100 shares of New York Central cost?

10. How much will 100 New York Central $3\frac{1}{2}$% bonds cost?

11. Which should you prefer to own, the 100 shares of stocks or the 100 bonds mentioned in Ex. 9 and 10?

12. What is the greatest number of Canadian Pacific shares that can be purchased for $5000?

13. Which is the better investment, a 4% mortgage or Southern 5's as quoted?

14. Which is the better investment, B and O. 4's or U.S. 5's as quoted?

15. What sum must be invested in Atchison 4's at $102\frac{7}{8}$ to secure an annual income of $4120?

16. What rate of income will U.S. 3's registered yield?

17. If I pay $3762.50 for U.S. Steel preferred, how many shares do I buy?

18. How much must I pay for B. and O. 4's to yield an income of 5% on my investment? of 6%?

19. What income will a man receive from an investment of $21625 in U.S. 3's coup.?

20. What dividend can a company declare on a capital stock of $50000 whose net earnings are $7500?

21. A certain bank pays a semiannual dividend of $3\frac{1}{2}$% on its stock; what is the annual dividend on 25 shares?

22. How much must I pay for 5% bonds that the investment may yield 6% income? for 4% bonds? for 3% bonds?

23. A man owns 100 shares of Amalgamated Copper stock. If the company declares a dividend of 5% payable in stock, how much stock will he then own

24. My broker, after selling for me 200 shares of Wabash preferred, remitted to me \$9975. At what price did he sell the stock?

25. How much must be invested in U.S. 3's coup. to bring an annual income of \$500?

26. A bank with a capital stock of \$150000, declares a semiannual dividend of 3%. What is the amount of the dividend, and how much will a person receive who owns 25 shares?

27. A gas company declares a 6% dividend and distributes \$120000 among its stockholders. What is its capital stock?

28. A cement company divides \$80000 among its stockholders. What is the rate of dividend, the capital stock being \$1000000? How much is paid to a person who owns 902 shares of \$10 each?

29. A broker bought for a customer 500 shares of copper stock, par value \$25, at a total cost of \$18015.63. Find the market quotation and brokerage.

30. A man bought 200 shares of New York Central at 143. The market price declined till it reached 139 and then rallied to $141\frac{1}{2}$. Believing that another decline was coming, he sold 500 shares (300 of them short) at $141\frac{1}{2}$. The price continued to rally, however, and he covered by buying 300 shares at $142\frac{1}{4}$. What was the net loss on the whole transaction, making no allowance for interest, but allowing $\frac{1}{8}$% brokerage for each sale and purchase?

INSURANCE

308. There are two general classes of **insurance**: insurance on the **person** in the form of life, endowment, accident, and health insurance, and insurance on **property** in the form of fire, marine, live stock, tornado, plate glass, **boiler insurance**, insurance against bad debts, etc.

PROPERTY INSURANCE

309. The principal kinds of property insurance are **fire** insurance, or insurance against loss by fire; **marine** insurance, or insurance against loss of vessels at sea, or property on board of vessels at sea; **tornado** insurance, or insurance against loss by storms, etc.

310. The written agreement between the company and the person insured is called the **policy**, and the sum to be paid by the company in case of loss, the **face** of the policy. The person insured is called the **insured**, and the amount paid by the insured to the company for insurance, the **premium**.

311. The premium is usually computed as a certain per cent of the face of the policy, or as a certain sum on each \$100 of insurance. In either case it is called the **rate of insurance**.

Ex. A house valued at \$5000 is insured for $\frac{4}{5}$ of its value at 1.1% per annum. What is the annual premium?

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Ex. A house valued at $5000 is insured for $\frac{4}{5}$ of its value at 1.1% per annum. What is the annual premium?

How much would the owner lose if the house were burned after seven premiums had been paid? How much would the company lose?

Solution. Valuation of house is $\frac{4}{5}$ of \$5000 = \$4000.

Premium = 1.1% of \$4000 = \$44.

Loss of owner = \$5000 − \$4000 + 7 × \$44 = \$1308.

Loss of company = \$4000 − 7 × \$44 = \$3692.

The above rate of insurance might have been stated as \$1.10 on each \$100 insured.

EXERCISE 72

1. A house valued at \$6000 is insured for $\frac{2}{3}$ of its value at $\frac{3}{4}$% per annum. What is the annual premium? How much does the owner lose if the house is burned after 10 premiums have been paid? How much does the company lose?

2. How much would the owner lose in case the house were damaged by fire to the extent of \$1500 after 3 premiums had been paid?

3. How much would the owner lose if the house were damaged by fire to the extent of \$350 after 9 premiums had been paid?

4. A residence valued at \$3500 is insured for $\frac{3}{4}$ of its full value at $\frac{4}{5}$% per annum. The company will insure the house for 3 yr. on the payment of $2\frac{1}{2}$ times the annual premium in advance. How much will it cost to insure the house for 3 yr.? They will insure for 5 yr. on the payment of 4 times the annual premium in advance. How much will it cost to insure the house for 5 yr.?

5. How much will it cost to insure a manufacturing plant valued at \$65000 at $\frac{3}{4}$% and the machinery valued at \$30000 at $\frac{9}{10}$%?

6. The insurance in Example 5 is placed in four companies, as follows: building, \$25000, \$20000, \$15000, \$5000; machinery, \$12000, \$8000, \$6000, \$4000. What is the annual premium paid each company?

7. A farmer takes the following insurance on his property: house valued at \$2500 at $1\frac{1}{4}$%; barn valued at \$1800 at $1\frac{1}{2}$%; live stock valued at \$2600 at $\frac{4}{5}$%; grain valued at \$1800 at $\frac{2}{3}$%; he also takes tornado insurance for \$3000 and pays 40 ct. per \$100 for 5 yr. He pays 4 times the annual premium for fire insurance for 5 yr. and 3 times for live stock insurance. What is his total premium for 5 yr.?

8. A dealer in Buffalo ordered his Chicago agent to buy 4000 bu. of wheat at 72 ct., 2500 bu. of oats at 26 ct., 7200 bu. of corn at $37\frac{1}{2}$ ct., paying 2% commission for buying. The grain was shipped by boat, and a policy at $1\frac{1}{8}$% taken to cover the cost of grain and all charges. What was the amount of the policy and what was the premium?

9. In a town where the regular police force consists of 20 or more patrolmen a company will insure a bank against burglary for 1 yr. for 50 ct. per \$100 up to \$3000, and 25 ct. per \$100 above that amount. How much will it cost to insure a bank for \$50000 against burglary in such a town?

10. The annual premium for insuring a plate glass window 6 ft. by 10 ft. is \$3.30. How much will it cost a merchant to insure two such windows for 5 yr.?

LIFE AND ACCIDENT INSURANCE

312. Life insurance is an agreement to pay to the heirs of a person a specified sum upon his death.

313. Endowment insurance is an agreement to pay a specified sum to the person insured if living at the end of a definite period of years, or to his heirs in case of death within that period.

314. Accident insurance is an agreement to pay the person insured a weekly indemnity for loss of time while incapacitated from accident, or a fixed amount in case of permanent injury, such as the loss of both hands, both feet, the entire sight of the eyes, etc., or a fixed amount to his heirs in case of death by accident.

315. Health insurance is an agreement to pay a weekly sum in case of sickness from specified diseases. In addition to the weekly indemnity, health insurance sometimes guarantees the payment of all doctor's fees and special amounts to cover cost of surgical operations.

316. The following tables show the annual rates per $1000 charged by one of the leading life insurance companies doing business in the United States. These rates are for life and endowment policies. Insurance companies also issue rates payable semiannually or quarterly. Such rates are slightly in advance of the annual rate, due to the fact that interest is charged on the amounts not paid at the time when the whole premium is due.

WHOLE LIFE POLICIES

PARTICIPATING

Age	Payments for Life	20 Payments	15 Payments	10 Payments	5 Payments	Single Payment
18		$28 05	$33 75	$45 37	$80 70	$364 89
19		28 47	34 24	46 03	81 84	369 96
20		28 90	34 76	46 71	83 02	375 19
21	$19 50	29 35	35 29	47 41	84 24	380 58
22	19 93	29 82	35 84	48 13	85 50	386 15
23	20 38	30 30	36 41	48 88	86 80	391 89
24	20 86	30 81	37 00	49 65	88 14	397 82
25	21 35	31 33	37 61	50 45	89 52	403 93
26	21 87	31 87	38 24	51 28	90 95	410 24
27	22 42	32 43	38 90	52 14	92 43	416 74
28	22 99	33 01	39 57	53 02	93 96	423 45
29	23 59	33 61	40 28	53 94	95 53	430 37
30	24 22	34 24	41 01	54 89	97 16	437 50
31	24 89	34 89	41 77	55 87	98 84	444 86
32	25 59	35 58	42 55	56 89	100 58	452 44
33	26 33	36 29	43 37	57 94	102 38	460 25
34	27 11	37 03	44 21	59 03	104 23	468 30
35	27 93	37 80	45 10	60 16	106 14	476 58
36	28 80	38 61	46 01	61 33	108 11	485 12
37	29 72	39 45	46 97	62 54	110 15	493 91
38	30 69	40 34	47 96	63 80	112 26	502 95
39	31 71	41 26	48 99	65 10	114 43	512 24
40	32 80	42 24	50 07	66 45	116 67	521 80
41	33 95	43 26	51 20	67 85	118 98	531 62
42	35 17	44 34	52 38	69 30	121 37	541 71
43	36 47	45 48	53 62	70 82	123 83	552 07
44	37 84	46 68	54 92	72 40	126 38	562 70
45	39 31	47 95	56 28	74 04	129 01	573 59
46	40 86	49 30	57 72	75 75	131 72	584 76
47	42 52	50 73	59 23	77 54	134 52	596 18
48	44 29	52 25	60 82	79 40	137 42	607 85
49	46 17	53 87	62 49	81 35	140 41	619 76
50	48 17	55 59	64 26	83 38	143 48	631 89
51	50 31	57 43	66 13	85 50	146 65	644 22
52	52 58	59 38	68 10	87 72	149 92	656 74
53	55 00	61 47	70 19	90 03	153 28	669 43
54	57 59	63 71	72 40	92 46	156 74	682 28
55	60 34	66 10	74 75	94 99	160 30	695 27
56	63 28	68 66	77 24	97 66	163 97	708 38
57	66 42	71 41	79 90	100 45	167 75	721 60
58	69 78	74 37	82 74	103 39	171 65	734 91
59	73 37	77 55	85 77	106 50	175 68	748 28
60	77 20	80 97	89 02	109 77	179 84	761 71

ENDOWMENT POLICIES

PAYMENTS FOR FULL TERM. PARTICIPATING

Age	Due in 10 Years	Due in 15 Years	Due in 20 Years	Due in 25 Years	Due in 30 Years	Due in 35 Years	Due in 40 Years
18	$102 37	$66 25	$48 55	$38 22	$31 58	$27 08	$23 93
19	102 45	66 34	48 65	38 32	31 70	27 21	24 09
20	102 54	66 44	48 75	38 43	31 82	27 35	24 26
21	102 64	66 54	48 86	38 55	31 96	27 51	24 44
22	102 73	66 64	48 97	38 68	32 10	27 68	24 65
23	102 83	66 75	49 09	38 81	32 26	27 86	24 86
24	102 94	66 87	49 22	38 96	32 42	28 06	25 10
25	103 06	66 99	49 36	39 11	32 60	28 27	25 36
26	103 18	67 13	49 50	39 28	32 79	28 50	25 64
27	103 30	67 27	49 66	39 46	33 00	28 75	25 95
28	103 44	67 42	49 83	39 65	33 23	29 03	26 28
29	103 58	67 58	50 00	39 85	33 48	29 33	26 65
30	103 74	67 75	50 20	40 08	33 75	29 66	27 05
31	103 90	67 93	50 41	40 32	34 04	30 02	27 48
32	104 08	68 12	50 63	40 59	34 37	30 41	27 96
33	104 26	68 34	50 87	40 88	34 72	30 84	28 48
34	104 46	68 56	51 14	41 20	35 10	31 31	29 04
35	104 68	68 81	51 43	41 55	35 53	31 83	29 66
36	104 91	69 07	51 74	41 93	35 99	32 39	30 33
37	105 16	69 36	52 09	42 35	36 50	33 01	31 06
38	105 43	69 68	52 47	42 81	37 07	33 69	31 85
39	105 72	70 02	52 88	43 31	37 68	34 43	32 71
40	106 04	70 40	53 34	43 87	38 36	35 24	33 64
41	106 38	70 82	53 84	44 49	39 10	36 12	
42	106 76	71 27	54 40	45 16	39 92	37 09	
43	107 18	71 78	55 01	45 91	40 82	38 14	
44	107 64	72 33	55 69	46 73	41 81	39 29	
45	108 14	72 95	56 44	47 64	42 90	40 54	
46	108 70	73 63	57 27	48 64	44 09		
47	109 32	74 39	58 18	49 75	45 39		
48	110 00	75 22	59 20	50 97	46 82		
49	110 76	76 14	60 31	52 31	48 38		
50	111 59	77 15	61 54	53 78	50 07		
51	112 51	78 27	62 89	55 38			
52	113 51	79 49	64 38	57 15			
53	114 61	80 84	66 01	59 07			
54	115 82	82 33	67 81	61 18			
55	117 16	83 96	69 78	63 47			
56	118 62	85 76	71 94				
57	120 22	87 73	74 31				
58	121 98	89 91	76 91				
59	123 92	92 30	79 75				
60	126 05	94 93	82 85				

The premiums on life policies are to be paid during the entire life of the insured, or during the period indicated in the preceding table. The face of the policy is to be paid at the death of the insured. The endowment policy provides for the payment of the face of the policy in 10, 15, 20, 25, 30, 35, or 40 yr. from the date of issue, or at the death of the insured if it occurs before the close of the stated period.

317. The premiums charged by the life insurance companies are determined by three considerations: (1) the probability that the insured will live as long as a healthy person of his age may be expected to live; (2) the rate of interest the company can earn on the premiums paid in; (3) the necessary expense of managing the company.

In order to secure safety of the policy contract, premiums are made higher than the above considerations render necessary. The portion of the premium remaining unused at the end of any year may be returned to the policy holder in the form of an **annual dividend**, or it may be allowed to accumulate for a term of years, called the **accumulation period**. The period is usually 10, 15, or 20 years. In the latter case, no dividend is paid unless the policy is kept in force to the end of the accumulation period.

The excess of assets over liabilities due to accumulated dividends, interest earned, etc., forms the **surplus** of the company. The **reserve** of a company is the amount held to meet the payment of policies when due.

Great care is taken by life insurance companies to protect the insured against forfeiture through nonpayment of premiums.

318. The following tables illustrate the **loan value**, or the amount the company agrees to loan the insured if the policy is assigned to the company as security; the **cash value**, or the amount the company agrees to pay the insured on surrender of the policy; the **paid-up policy**, or the face of a paid-up policy the company agrees to exchange for the original policy if surrendered; the **extended insurance**, or the time the company will continue the full amount of insurance without further payment. These privileges are granted in consideration of the premiums already paid.

20-Payment Life

Year	Loan	Cash Value	Paid up Policy	Extended Insurance Years	Extended Insurance Days
	Age 35.		Premium: A. $37.80, S. A. 19.60, Q. 9.98		
3	$47	$53	$131	6	194
4	70	78	183	8	276
5	94	105	235	10	317
6	118	132	287	12	284
7	144	160	339	14	167
8	170	189	391	15	332
9	196	218	442	17	55
10	224	249	493	18	75
11	252	281	544	19	41
12	281	313	595	19	324
13	312	347	646	20	205
14	343	382	696	21	56
15	376	418	746	21	251
16	408	454	797	22	69
17	441	491	847	22	250
18	476	529	898	23	76
19	511	568	948	23	287
20	548	609	Policy full paid.		
25	599	666			
30	650	723			

20-Year Endowment

Year	Loan	Cash Value	Paid up Policy	Extended Insurance Years	Extended Insurance Days	Endowment
	Age 35.			Premium: A. $51.43, S. A. 26.67, Q. 13.57		
3	$82	$92	$147	10	196	
4	118	132	203	13	348	
5	155	173	259	15	0	$41
6	193	215	314	14	0	121
7	233	259	368	13	0	198
8	274	305	421	12	0	272
9	316	352	474	11	0	343
10	360	401	525	10	0	411
11	405	451	576	9	0	477
12	453	504	626	8	0	541
13	502	558	675	7	0	602
14	553	615	724	6	0	660
15	606	674	771	5	0	716
16	659	733	818	4	0	781
17	716	796	865	3	0	842
18	774	861	910	2	0	898
19	835	928	955	1	0	951
20		1000				

EXERCISE 73

From the tables find the annual premium required for:

1. A life policy for $2500, age 24.
2. A ten-payment life policy for $4000, age 29.
3. A ten-year endowment policy for $5000, age 40.
4. A twenty-payment life policy for $3000, age 37.
5. A twenty-year endowment policy for $6000, age 37.

6. At age 24 Mr. Robbins takes out a life policy for $5000; if he dies at the age of 41, how much does the face of the policy exceed the premiums paid?

7. If money is worth 6% per annum, what do the premiums paid in Ex. 6 amount to? How much does the face of the policy exceed the amount?

8. At age 35 Mr. Andrews takes out a $5000 twenty-payment life policy; what is the face of the paid-up life policy that will be given to him if he stops paying premiums and surrenders his policy at age 46? What is the guaranteed cash value of the policy at age 45?

9. At age 31 a man took out a $2500 life policy and at age 36 a $1500 twenty-five-year endowment policy and a $1000 twenty-year endowment policy. How much does the insurance exceed the premiums paid if he dies at the age of 43?

10. If the annual dividends on a twenty-payment life policy, age 35, average 21% of the premiums, how much has a $1000 policy cost at the end of 20 years, money being worth 5%?

11. If dividends are not paid annually, but are allowed to accumulate for a period of twenty years on the above twenty-payment life policy, the insured would be privileged to withdraw the accumulated surplus in cash and still retain a full-paid policy for $1000 payable at death. Should the accumulated surplus amount to $391.78 at the end of twenty years, how much does the policy cost, money being worth 5%?

12. Mr. Young takes out a $5000 fifteen-payment life policy Nov. 19, 1887, at age 40. In 1902, instead of continuing the insurance, he surrenders for a cash value of $4036.75, which includes the accumulated dividends. Allowing $15 per annum per $1000 for protection afforded, what rate of interest has his money earned in the 15 years?

TAXES AND DUTIES

319. The expenses of the **United States government** for pensions, army and navy, salaries of the President, congressmen, and other officials, etc., amount to something over $1000000 a day. The **state** must have money for the care of the insane, blind, deaf and dumb, criminals; for educational purposes, salaries of state officials, etc. The **county** needs money for public buildings, bridges, salaries, educational purposes, etc. The **city** and **village** must have public improvements, fire protection, police, schools, etc. These expenses are met by **taxes.**

TAXES

320. The expenses for the support of the state, county, city, etc., are paid by taxes on real estate and personal property. In addition to the property tax most states collect a **poll** tax of from $1 to $3 from each male citizen over 21 years of age and under 50.

321. The rate of taxation is usually expressed as a certain number of mills on each dollar, or as a certain number of cents on each $100 of valuation.

EXERCISE 74

1. The valuation of the property of a certain county is $7500000. If the general state tax and the general county tax are each 60 ct. on each $100 and in addition the

bridge tax is 40 ct. and the school tax 30 ct., what is the total tax of the county and what is the amount set aside for each of the above purposes?

2. What are the taxes of a man who owns 160 acres of land in the above county worth $60 an acre and assessed at $\frac{2}{5}$ of its value, and personal property amounting to $1850?

3. The total assessed value of property in Michigan in 1901 was $1578100000. What amount did the State University receive in 1903 from a $\frac{1}{4}$ of a mill tax?

4. How much of this tax did a farmer have to pay who owns 200 acres of land valued at $75 an acre and assessed at $\frac{3}{5}$ of its value?

5. A certain city is bonded for $6000; its taxable property is valued at $7500000. How much of the above bonded indebtedness does a man worth $10000 pay?

6. Suppose the above city wishes to build a high school building valued at $50000, what will be the tax on each $100?

7. The taxable property of a certain county is $125000000. What will be the tax on each $100 to build a courthouse worth $90000?

8. The Michigan State Normal College received from the state, in 1903, $103200. How much of this did a man pay who owns $7500 worth of taxable property, the state having property listed at $1578100000?

9. The assessed value of a town is $250000 and the amount of tax to be raised is $3500; what is the rate of taxation?

DUTIES

322. The income for the support of the national government is derived largely from **custom revenue** (tariff or duty on imports, collected at **customhouses** established by the government at ports of entry), and **internal revenue** (taxes on spirits, tobacco, etc.).

323. Merchandise brought into the country is subject to **ad valorem duty** (a certain per cent of the cost of the goods), **specific duty** (a certain amount of weight, number, or measure, without regard to value), or both **ad valorem** and **specific duty**. Some goods are admitted duty free.

Illustrations. Cut glass and laces pay an ad valorem duty of 60%. Machinery pays 45%. Tin plates pay a specific duty of $1\frac{1}{2}$ ct. per pound, horses valued at \$150 or less pay \$30, and wheat pays 25 ct. per bushel. Cigars pay a duty of \$4.50 per pound and 25%, and lead pencils pay 45 ct. per gross and 25%. Books published in foreign languages are admitted duty free.

EXERCISE 75

1. What will be the duty on 1 T. 4 cwt. of tin plate?
2. What will be the duty on 20 gross of lead pencils?
3. What is the cost per gross of lead pencils on which the two rates of duty are equal?
4. The duty on ready-made clothing is 50%. What is the duty on \$6000 worth?
5. If the duty on linen collars and cuffs is 40 ct. per dozen and 20%, what is the duty on 10 doz. collars at 75 ct. a dozen and 10 pairs of cuffs at 25 ct. a pair?
6. What is the duty at 50% on 500 doz. kid gloves at 75 francs a dozen?
7. Find the duty on an importation of £750 8*s.* 4*d.* worth of English crockery at 40%.

THE PROGRESSIONS

324. By a **series** is meant a succession of terms formed according to some common law.

325. An **arithmetical progression** (A. P.) is a series in which each term differs from the preceding by a constant quantity called the **common difference.**

Thus, 2, 5, 8, 11, ⋯, and 15, 10, 5, 0, − 5, − 10, ⋯, are arithmetical progressions. In the first, 3 is the common difference and is added to each term to form the next; in the second, − 5 is the common difference and is added to each term to form the next.

326. A **geometrical progression** (G. P.) is a series in which each term after the first is derived by multiplying the preceding term by a constant multiplier called the **ratio.**

Thus, 2, 4, 8, 16, ⋯, and 18, − 6, 2, $-\frac{2}{3}$, $\frac{2}{9}$, ⋯, are geometrical progressions, the ratios being respectively 2 and $-\frac{1}{3}$.

327. Last Term. If a is the first term, l the last term, d the common difference, r the ratio, and n the number of terms, we have from the definitions, —

	1st	2d	3d		nth
A. P.	a	$(a+d)$	$(a+2d)$	⋯	$a+(n-1)d$
G. P.	a	ar	ar^2	⋯	ar^{n-1}

∴ the formulas for the last of n terms are:

A. P. $l = a + (n-1)d.$

G. P. $l = ar^{n-1}.$

Ex. Find the last term in an A. P. in which the first term is 10, the common difference 4, and the number of terms 12.

Solution. $l = a + (n-1)d = 10 + (12-1) \times 4 = 54.$

Ex. Find the last term in a G. P. in which the first term is 2, the common ratio 3, and the number of terms 5.

Solution. $l = ar^{n-1} = 2 \times 3^4 = 162.$

328. Sum of Series.

A. P. Take the series 3, 5, 7, 9, 11, in which $a=3$, $d=2$, $l=11$, and the sum $(S)=35$.

Then $S = 3 + (3+2) + (3+4) + (3+6) + (3+8),$

and in reverse order

$$S = 11 + (11-2) + (11-4) + (11-6) + (11-8).$$

Adding and canceling the 2, 4, 6, and 8,

$$2S = (3+11) + (3+11) + (3+11) + (3+11) + (3+11)$$
$$= 5(3+11).$$
$$\therefore S = \tfrac{5}{2}(3+11) = 35,$$

or *the sum of the series equals one half of the number of terms times the sum of the first and last terms.*

Take the general series $a, a+d, a+2d, \cdots, a+(n-1)d$. In this series it will be noticed that each term is formed by adding to the first term the common difference multiplied by the number of the term less one.

Then $S = a + (a+d) + (a+2d) + \cdots (l-d) + l,$

and in reverse order

$$S = l + (l-d) + (l-2d) + \cdots (a+d) + a.$$

Adding and canceling the d's.

$$2S = a + l + a + l + a + l + \cdots a + l + a + l = n(a+l).$$
$$\therefore S = \frac{n}{2}(a+l).$$

G. P. Take the series 2, 6, 18, 54, 162, in which $a=2$, $r=3$, $l=162$, $n=5$, and $S=242$.

Then $S = 2 + 2\times 3 + 2\times 3^2 + 2\times 3^3 + 2\times 3^4.$

Multiplying by 3, $3S = 2\times 3 + 2\times 3^2 + 2\times 3^3 + 2\times 3^4 + 2\times 3^5.$

Subtracting and canceling common terms,

$$S - 3S = 2 - 2\times 3^5.$$
$$\therefore S = \frac{2 - 2\times 3^5}{1-3} = 242,$$

or *the sum of the series equals the first term minus the first term times the ratio raised to a power equal to the number of terms divided by one minus the ratio.*

Take the general series $a, ar, ar^2, \cdots, ar^{n-1}$. In this series it will be noticed that each term is formed by multiplying the first term by the ratio raised to a power one less than the number of the term.

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}.$$

Multiplying by r, $rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n.$

Subtracting, $S - rS = a - ar^n.$

$$\therefore S = \frac{a - ar^n}{1-r}.$$

Ex. Find the sum of the first 8 terms in an A. P. when the first term is 5 and the common difference is 3.

Solution. Since $S = \dfrac{n(a+l)}{2}$ and $l = a + (n-1)d$,

$$\therefore S = \frac{n}{2}[2a + (n-1)d] = 4(10 + 7\times 3) = 124.$$

Ex. Sum to 6 terms the series $2 + 6 + 18 + \cdots$.

Solution. $S = \dfrac{a - ar^n}{1-r} = \dfrac{2 - 2\times 3^6}{1-3} = 728.$

329. Infinite Series. Writing the formula

$$S=\frac{a-ar^n}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r},$$

we see that when r is a proper fraction and n becomes large, ar^n becomes small. If we make n sufficiently large, ar^n and hence $\frac{ar^n}{1-r}$ will approach as near to zero as we please, and hence, when n becomes infinite, $S=\frac{a}{1-r}$.

Ex. Sum to infinity the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$.

Solution. $S=\frac{1}{1-\frac{1}{2}}=2.$

330. Circulating Decimals. $\frac{1}{3}=0.3333\cdots$ and $\frac{7}{37}=0.189189\cdots$. In the first case 3 is repeated indefinitely, and in the second case the digits 189 are repeated indefinitely in the same order. Such decimals are called **circulating decimals**. The repeating figures are called the **repetend**. A circulating decimal is expressed by writing the repetend once and placing a dot over the first and the last figure of the part repeated.

Thus, $0.333\cdots=0.\dot{3}$ and $0.189189\cdots=0.\dot{1}8\dot{9}$.

Ex. Reduce $0.\dot{3}$ to an equivalent common fraction.

Solution. $0.3=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots=$ a G. P. with the first term $=\frac{3}{10}$, and the ratio $=\frac{1}{10}$. $\therefore S=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{1}{3}$.

EXERCISE 76

1. Find the 12th term of the series 5, 7, 9, ⋯.
2. Find the 7th term of the series $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \cdots$.
3. Find the sum of 9 terms of $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\cdots$.

4. If a body falls $16\frac{1}{12}$ ft. the first second, 3 times as far the next second, 5 times as far the third second, and so on, how far will it fall in the twelfth second? How far will it fall in 12 sec.?
5. Find the 8th term in the series $1, \frac{1}{3}, \frac{1}{9}, \cdots$.
6. Find the sum of $1+\frac{1}{3}+\frac{1}{9}+\cdots$ to infinity.
7. Find the 7th term in the series $4, -2, 1, \cdots$.
8. Find the value of $0.4\dot{2}\dot{3}$.
9. Find the 5th and 9th terms of the series 3, 6, 12, ⋯.
10. Find the 9th term of the series $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}, \cdots$.
11. Sum to 5 terms the series $9, -6, 4, \cdots$.
12. Find the value of $0.\dot{2}$; $0.\dot{2}\dot{3}$; $0.2\dot{4}$; $1.71\dot{4}\dot{5}$.
13. Find the sum of $3+0.3+0.03+\cdots$ to infinity.
14. Find the sum of the first 25 odd numbers; the first 25 even numbers.
15. What is the distance passed through by a ball before it comes to rest, if it falls from a height of 40 ft. and rebounds half the distance at each fall?

LOGARITHMS

331. Early in the seventeenth century, John Napier, a Scotchman, invented **logarithms**, by the use of which the arithmetical processes of multiplication, division, evolution and involution are greatly abridged.

1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
2048	11
4096	12
8192	13
16384	14
32768	15
65536	16
131072	17
262144	18
524288	19
1048576	20

332. Many simple arithmetical operations can be performed by the use of two columns of numbers, as given in the annexed table.

The left-hand column is formed by writing unity at the top and doubling each number to get the next. The right-hand column is formed by writing opposite each power of 2, the index of the power. Thus $512 = 2^9$, the number opposite 512 indicating the power of 2 used to produce 512.

Ex. 1. Multiply 4096 by 64.

From the table $4096 = 2^{12}$ and $64 = 2^6$.

$\therefore 4096 \times 64 = 2^{12} \times 2^6 = 2^{12+6} = 2^{18} = 262144$ (from table).

The student should notice that the simple operation of addition is substituted for multiplication, the product being found in the left-hand column opposite 18, the sum of 12 and 6.

Ex. 2. Divide 1048576 by 2048.

$1048576 \div 2048 = 2^{20} \div 2^{11} = 2^{20-11} = 2^9 = 512$ (subtraction takes the place of division).

Ex. 3. Find $\sqrt[5]{32768}$.

$\sqrt[5]{32768} = \sqrt[5]{2^{15}} = 2^{\frac{15}{5}} = 2^3 = 8$ (division takes the place of evolution).

In the preceding table the numbers in the right-hand column are called the **logarithms** of the corresponding numbers in the left-hand column. 2 is called the **base** of this system. Therefore, *the logarithm of a number is the exponent by which the base is affected to produce the number.*

333. Any other base than 2 might have been used and columns similar to the above formed. In practice 10 is always taken as the base and the logarithms are called **common logarithms** in distinction from the **natural logarithm**, of which the base is 2.71828. *Common logarithms are indices, positive or negative, of the power of* 10.

From the definition of common logarithms, it follows that since

$10^0 = 1$,	$\log 1 = 0$.	$10^{-1} = 0.1$,	$\log 0.1 = -1$.
$10^1 = 10$,	$\log 10 = 1$.	$10^{-2} = 0.01$,	$\log 0.01 = -2$.
$10^2 = 100$,	$\log 100 = 2$.	$10^{-3} = 0.001$,	$\log 0.001 = -3$.
$10^3 = 1000$,	$\log 1000 = 3$.	$10^{-4} = 0.0001$,	$\log 0.0001 = -4$.
etc.		etc.	

334. Since most numbers are not exact powers of 10, logarithms will in general consist of an integral and decimal part. Thus, since $\log 100 = 2$ and $\log 1000 = 3$, the logarithms of numbers between 100 and 1000 will lie between 2 and 3, or will be 2+ *a fraction.* Also since $\log 0.01 = -2$ and $\log 0.001 = -3$, the logarithms of all numbers between 0.01 and 0.001 will lie between -2 and -3 or will be $-3 +$ *a fraction.* The integral part of the logarithm is called the **characteristic** and the decimal part the **mantissa**.

335. The characteristic of the logarithm of a number is independent of the digits composing the number, but depends on the position of the decimal point. Characteristics, therefore, are not given in the tables. Thus, since 246 lies between 100 and 1000, log 246 will lie between 2 and 3, or will be $2 +$ a fraction. Again since 0.0024 lies between 0.001 and 0.01, its logarithm lies between -3 and -2, or $\log 0.0024 = -3 +$ a fraction.

336. From the above illustrations it readily appears that the *characteristic of the logarithm of a number, partly or wholly integral, is zero or positive and one less than the number of figures in the integral part.*

337. *The characteristic of the logarithm of a pure decimal is negative and one more than the number of zeros preceding the first significant figure.*

EXERCISE 77

1. Determine the characteristic of the logarithm of 2; 526; 75.34; 0.0005; 300.002; 0.05743.

2. If $\log 787 = 2.8960$, what are the logarithms of 7.87, 0.0787, 78700, 78.7?

338. The mantissa of the logarithm of a number is independent of the position of the decimal point, but depends on the digits composing the number. Mantissas are always positive and are found in the tables, for moving the decimal point is equivalent to multiplying the number by some integral power of 10, and therefore adds to or subtracts from the logarithm an integer.

Thus,
$$\log 76.42 = \log 76.42,$$
$$\log 764.2 = \log 76.42 \times 10 = \log 76.42 + 1,$$
$$\log 7642 = \log 76.42 \times 10^2 = \log 76.42 + 2,$$
$$\log 7.642 = \log 76.42 \times 10^{-1} = \log 76.42 + (-1).$$

So that the mantissas of all numbers composed of the digits 7642 in that order are the same, since moving the decimal point affects the characteristic alone.

Log 0.0063 is never written $-3 + 7993$, but $\bar{3}.7993$. The minus sign is written above to indicate that the characteristic alone is negative. To avoid negative characteristics 10 is added and subtracted. Thus, $\bar{3}.7993 = 7.7993 - 10$.

339. The principles used in working with logarithms are as follows:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

III. *The logarithm of a power equals the index of the power times the logarithm of the number.*

IV. *The logarithm of a root equals the logarithm of the number divided by the index of the root.*

For let $10^x = n$ and $10^y = m$,

then $\log n = x$ and $\log m = y$.

Therefore, since $mn = 10^{x+y}$,

$$\log mn = x + y = \log n + \log m;$$

and $n \div m = 10^{x-y}$,

then $\log \frac{n}{m} = x - y = \log n - \log m.$

Also $\qquad n^r = (10^x)^r = 10^{rx},$

then $\qquad \log n^r = rx = r \log n.$

Finally $\qquad \sqrt[r]{n} = \sqrt[r]{10^x} = 10^{\frac{x}{r}},$

then $\qquad \log \sqrt[r]{n} = \frac{x}{r} = \frac{1}{r} \log n.$

EXERCISE 78

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, find:

1. $\log 6$. **3.** $\log 5^2$. **5.** $\log 0.18$. **7.** $\log \frac{6}{5}$.

2. $\log 15$. **4.** $\log \sqrt{18}$. **6.** $\log 7.5$. **8.** $\log 3^2 \times 5^3$.

9. Find the number of digits in 30^{25}; in 25^{30}.

USE OF TABLES

340. In the tables here given the mantissas are found correct to but four decimal places. By using these tables results can generally be relied upon as correct to 3 figures and usually to 4. If a greater degree of accuracy is required, five-place or even seven-place tables must be used.

341. To find the logarithm of a given number.

Write the characteristic before looking in the tables for the mantissa.

Find the mantissa in the tables.

(1) *When the number consists of not more than three figures.*

In the column N, at the left-hand side of the page, find the first two figures of the number. In the row N, at the top or bottom of the page, as convenient, find the third figure. The mantissa of the number will be found at the intersection of the row containing the first two figures and the column containing the third figure.

Ex. Find log 384.

The characteristic is 2 (Why?). In the column N find 38 and in row N find 4. The mantissa 5843 will be found at the intersection of the row 38 and column 4.

$\therefore \log 384 = 2.5843.$

What is log 3.84? log 38.4? log 0.0384?

(2) *When the number consists of more than three figures.*

Find as above the mantissa of the logarithm of the number consisting of the first three figures. To correct for the remaining figures *interpolate by assuming that, for differences small as compared with the numbers, the differences between numbers are proportional to the differences between their logarithms.* This statement is only approximately true, but its use leads to results accurate enough for ordinary computations.

Ex. Find log 3847.

$$\begin{array}{ll} \text{Mantissa of } \log 3850 = & 5855. \\ \text{Mantissa of } \log \underline{3840} = & \underline{5843.} \\ \qquad\qquad\qquad 10 & 0.0012. \end{array}$$

$$\text{Mantissa of } \log 3847 = 5843 + \tfrac{7}{10} \text{ of } 0.0012 = 5851.$$

The difference between 3840 and 3850 is 10, the difference between the mantissas of their logarithms (5855 − 5843) is 0.0012. Assuming that each increase of 1 unit between 3840 and 3850 produces an increase of 1 tenth of the difference in the mantissas, the addition for 3847 will be 7 tenths of 0.0012 or 0.00084. 5843 + 0.00084 = 5851. Therefore, the mantissa of log 3847 = 5851.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

EXERCISE 79

1. Find log 1845.
2. Find log 6.897.
3. Find log 0.04253.

342. To find the number corresponding to a given logarithm.

The number corresponding to a logarithm is called the **antilogarithm**.

The characteristic determines the position of the decimal point.

(1) *If the mantissa is found in the tables*, the number is found at once.

Ex. 1. Find antilog 3.5877.

The mantissa is found at the intersection of column 7 and row 38.
$\therefore$ antilog 3.5877 = 3870.

(2) *If the exact mantissa is not found in the tables*, the first three figures of the corresponding number can be found and to them can be annexed figures found by interpolation.

Ex. 2. Find antilog 3.5882.

log 3880 = 3.5888		log required number = 3.5882	
log 3870 = 3.5877		log 3870 = 3.5877	
10	0.0011	log req. no. – log 3870 = 0.0005	

$$3870 + \left(\frac{5}{11} \text{ of } 10\right) = 3874.54+.$$

The two mantissas in the table nearest to the given mantissa are 5888 and 5877 differing by 0.0011. Their corresponding numbers, since the characteristic is 3, are 3880 and 3870, differing by 10. The difference between the smaller mantissa 5877 and the required mantissa 5882 is 0.0005. Since an increase of 11 ten thousandths in mantissas corresponds to an increase of 10 in the numbers, an increase of 5 ten thousandths in mantissas may be assumed to correspond to an increase of $\frac{5}{11}$ of 10 in the numbers. Therefore the number is $3870 + (\frac{5}{11} \text{ of } 10) = 3874.54+$.

EXERCISE 80

1. Find antilog 2.9445; antilog $\bar{2}.4065$.
2. Find antilog $\bar{1}.6527$; antilog 3.7779.
3. Find antilog 1.9994; antilog 0.7320.

343. The **cologarithm** of a number is the logarithm of its reciprocal. The cologarithm of 100 equals the logarithm of $\frac{1}{100}$, *i.e.* -2. As the cologarithm of a number equals the logarithm with its sign changed, adding the cologarithm will give the same result as subtracting the logarithm. The former is sometimes more convenient.

Since $\quad \log 1 = 0, \therefore \log \frac{1}{n} = \log 1 - \log n = 0 - \log n,$

therefore $\quad \text{colog } n = -\log n.$

To avoid negative results it is often more convenient to add and subtract 10.

Then $\quad \text{colog } n = 10 - \log n - 10.$

Ex. 1. Find colog 47.3.

$$\log 1 = 10.0000 - 10$$
$$\log 47.3 = 1.6749$$
$$\text{colog } 47.3 = 8.3251 - 10$$

In subtracting 1.6749 or any other logarithm from 10, the result may be obtained mentally by subtracting the right hand figure from 10 and all the others from 9.

Ex. **2.** Find the value of $\frac{452 \times 23}{5371 \times 29}$.

$$\log \frac{452 \times 23}{5371 \times 29} = \log 452 + \log 23 - \log 5371 - \log 29$$
$$= \log 452 + \log 23 + \text{colog } 5371 + \text{colog } 29$$

$$\log 452 = 2.6551$$
$$\log 23 = 1.3617$$
$$\text{colog } 5371 = 6.2699 - 10$$
$$\text{colog } 29 = 8.5376 - 10$$
$$\log 0.066728+ = 8.8243 - 10$$

Therefore $\frac{452 \times 23}{5371 \times 29} = 0.066728+.$

Ex. **3.** Find $\log 50^{\frac{3}{4}}$.

$$\log 50^{\frac{3}{4}} = \tfrac{3}{4} \log 50$$
$$\log 50 = 1.6990$$
$$\tfrac{3}{4} \log 50 = \tfrac{3}{4} \text{ of } 1.6990 = 1.2742$$
$$1.2742 = \log 18.8$$
$$\therefore 50^{\frac{3}{4}} = 18.8.$$

EXERCISE 81

Find the value of :

1. $(5 \times 4 \div 7)^{\frac{1}{2}}$.
2. $\frac{1}{225}$.
3. $\sqrt[3]{\frac{23 \times 30}{72}}$.
4. $\frac{3.14 \times 56.7}{29}$.
5. $(0.625)^{\frac{1}{100}}$.
6. $0.0625 \div 0.25$.
7. $\frac{31 \times 47 \times 53}{29 \times 43 \times 50}$.
8. $\sqrt{\frac{621 \times 4325}{729}}$.
9. $\sqrt{\pi} \times 10.16$.
10. $\pi^2;\ \frac{1}{\pi}$.

EXERCISES FOR REVIEW

In connection with each exercise the student should review all principles involved. The following list will then furnish a complete review of the book.

1. What are the various names given to the symbol 0?
2. Read the numbers 200, 0.02; 100.045, 0.145.
3. Solve $4672 - 2134 + 7635 + 2377 - 8432$ by adding the proper arithmetical complements and subtracting the proper powers of 10.
4. Multiply 5280 by 25; by $16\frac{2}{3}$.
5. Multiply 1760 by 9; by 11; by 81; by 16.
6. Multiply 4763 by 998.
7. Multiply 4634 by 4168.
8. Multiply 746 by 18.
9. Show that to multiply a number by 1.5 is the same as to add $\frac{1}{2}$ of the number to the multiplicand.
10. Show that to divide a number by $112\frac{1}{2}$ is the same as to move the decimal point two places to the left and subtract $\frac{1}{9}$ of the number.
11. Form a table of multiples of the multiplier and multiply (*a*) 7461, (*b*) 3465, (*c*) 761, (*d*) 98723, (*e*) 1846, each by 3762. Also find each product by using logarithms.
12. Form a table of multiples of the divisor and divide (*a*) 7346, (*b*) 5280, (*c*) 8976, (*d*) 4284, each by 361. Also find each quotient by using logarithms.

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$$\log \frac{452 \times 23}{5371 \times 29} = \log 452 + \log 23 - \log 5371 - \log 29$$
$$= \log 452 + \log 23 + \text{colog } 5371 + \text{colog } 29$$

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12. Form a table of multiples of the divisor and divide (*a*) 7346, (*b*) 5280, (*c*) 8976, (*d*) 4284, each by 361. Also find each quotient by using logarithms.

13. Show that every number divisible by 4 is the sum of two consecutive odd numbers.

14. Show that the sum and difference of two odd numbers are always even.

15. Prove that the difference between a number and the number formed by writing its digits in reverse order is divisible by 9.

16. Perform the following operations and check by casting out the 9's: 86942×763; $46342 \div 216$; 842×21.34; $987.4 \div 3.1416$.*

17. Find the quotient of 764321 divided by 2136 correct to four significant figures.

18. Find the quotient of 76.421 divided by 3.1416 correct to 0.01.

19. Multiply 5276 by $12\frac{1}{2}$ and divide the result by $33\frac{1}{3}$.

20. Prove that a number is divisible by 4 if the units' digit minus twice the tens' digit is divisible by 4.

21. When it is 10 P.M. Sunday, Feb. 15, at Greenwich, what time and date is it at 165° W.?

22. Suppose a transport returns troops from Manila starting July 4, reaching San Francisco 35 da. later; what is the date?

23. 27 is composed of 16 and 11; write all of the other two numbers that make up 27.

24. Reduce 43132_5 to the decimal scale.

25. What methods did the ancient Babylonians, Egyptians, Greeks and Romans adopt to represent numbers? Were these characters ever employed as instruments of calculation?

* Perform also by logarithms.

26. From what source was the decimal system of notation with its 9 digits derived?

27. Explain clearly the difference between the intrinsic value and the local value of the 9 digits.

28. In the decimal scale explain why the number of characters used cannot be more nor less than 10.

29. What is the difference between the sum of 4623, 256, 145231, 7649, and a million?

30. Find the excess of 864213 over 634795 by means of arithmetical complements.

31. Multiply 37635 by 648, using but two partial products.

32. Prove that any number composed of three consecutive figures is divisible by 3.

33. Find the sum and difference of 6523 and 5436 in the scale of 8.

34. Multiply 529t by 1903 in the scale of 12.

35. Divide 4234 by 213 in the scale of 5.

36. What weights must be selected out of 1, 3, 9, 27, 81, etc., pounds to weigh 1907 lb.?

37. A carriage wheel revolves 2 times in going 25 ft.; how many times will it revolve in going a mile?

38. How much will it cost to build a cement walk 6 ft. wide around a block 500 ft. square at $10\frac{1}{2}$ ct. per square foot?

39. If a tight board fence 6 ft. high is built around the same block 2 ft. inside of the walk, how will its area compare with that of the walk?

40. What must be the depth of a cistern 6 ft. in diameter which shall contain 600 gal., if a gallon of water weighs 10 lb. and a cubic foot of water weighs 1000 oz.?

41. If the pressure of the atmosphere at the surface of the earth, when the barometer stands at 30 in., is about 15 lb. to the square inch, what is the pressure on the human body if its surface is 16 sq. ft.? What would be the difference in pressure if the barometer stood at 29 in.?

42. How many grains of gold are there in 6 lb. 4 oz. 5 pwt.?

43. If employed 6 da. in the week and 8 hr. daily, how many weeks would it take to count $50000000 at the rate of $100 a minute?

44. If sound travels at the rate of 1100 ft. per second, and the report of a gun is heard 10 sec. after the appearance of the smoke, how far distant is the observer?

45. What number between 300 and 400 is exactly divisible by 2, 3, 4, 5?

46. If 4 cu. in. of iron weigh a pound, find the weight of a rectangular vessel an inch and a half thick without a top, the vessel being $10\frac{1}{2}$ ft. by $8\frac{1}{2}$ ft. by $5\frac{1}{4}$ ft. outside measure.

47. A cubic foot of copper weighs $556\frac{1}{4}$ lb., and can be drawn into a wire 1 mi. 125 rd. long. Find the weight of copper necessary for a wire 60 mi. long and also the area of a cross section of the wire.

48. How long is an iron bar containing a cubic foot of iron if its dimensions are $\frac{3}{4}$ of an inch by $\frac{1}{2}$ of an inch?

49. If a cubic foot of water weighs 1000 oz., find the number of grains in a cubic inch.

50. Explain whether 0.023 or 0.024 is more nearly equal to 0.02349 and state in words the error in excess or defect in each case.

51. Divide 0.34827 by 0.23 correct to 0.01.

52. Multiply 3.1459 by 16.325 correct to 0.1.

53. If the meter is 39.3708 in., what part of a meter is a yard?

54. If the average length of a degree of latitude is 365000 ft., find the length of a meter in feet and inches.

55. If water expands 10% when it freezes, how much does ice contract when it turns into water?

56. Find the discount of $1000 for 90 da. at 6%. Show that the interest on this discount for the same time is equal to the difference between the interest and the discount of $1000.

57. Show that the interest on the discount of $1000 for one year at 6% is the same as the discount on the interest at the same rate for the same time.

58. If a person saves $300 a year, and invests his savings at 4% compound interest for 10 yr., what amount does he accumulate?

59. Which is the better investment, bonds bought at 112 yielding 6% interest, or stocks bought at 85 yielding 4% dividends?

60. A person owns 302 $10 shares of Wolverine Portland Cement Stock, paying a semiannual dividend of 5%; 20 shares of bank stock of $100 each, paying a semiannual dividend of $2\frac{1}{2}$%; 30 Mexican Plantation Bonds of $300 each, paying 7% interest. What is his total annual income from these sources?

61. A merchant adds $33\frac{1}{3}$% to the cost price of his goods, and gives his customers a discount of 10%; what profit does he make?

62. If a ship sails from San Francisco Oct. 15 and reaches Japan after 20 da., what is the date of her arrival?

63. When it is 2 P.M. Sunday, Feb. 15, at Greenwich, what time and date is it at longitude 165° W.?

64. The Canadian Pacific Railway uses twenty-four-hour clocks (hours from noon to midnight are 12 to 24 o'clock) at Port Arthur and west. When it is 20 o'clock, standard time, at Winnipeg, what time is it at Toronto?

65. A street 40 ft. wide is to be paved for a distance of 1680 ft. If it costs 32 ct. a cubic yard for excavating to a depth of 2 ft., 4 ct. a square yard for sand cushion, $1.17 a square yard for crushed stone filling, and $48\frac{2}{3}$ ct. a square yard for concrete, what is the cost of the paving?

66. Dec. 28, 1886, Mr. Harvey insured his life for $3000 on the fifteen-payment life plan, paying a quarterly (*i.e.* four times a year) premium of $44.10. Instead of continuing the insurance at the end of the 15 yr., he accepts a cash settlement of $2942.20. Allowing $15 a year per $1000 for protection afforded, what rate of interest has his money earned?

67. In 1903 Michigan levied a tax of $397525 for the support of the State University at the rate of $\frac{1}{4}$ of a mill. What was the valuation of the state property?

68. How many tons of coal will a bin 10 ft. by $6\frac{1}{2}$ ft. by $7\frac{2}{3}$ ft. hold if one ton occupies 36 cu. ft.?

69. Simplify $\frac{2}{3}$ of $\frac{8}{9} \div 2\frac{6}{7} + 5\frac{1}{2} \times \frac{9}{17}$.

70. Find the least fraction that added to $\frac{2}{7}$, $\frac{8}{21}$ and $\frac{8}{35}$ will make the result an integer.

71. Simplify $\frac{4.561}{0.015} \times \frac{0.0075}{21.05}$.

72. A person's income is $2500 a year. He spends on an average $27.75 a week. If he deposits his savings in a bank every 3 mo., how much will he accumulate in 10 yr. if the bank pays 3% compound interest?

73. How many miles are there in 10000 ft. and 1000000 in.?

74. Use short methods in finding the product of 14×76, 369×81, 4728×998, 85×85, 67×73.

75. Find by factors the square root of 44100, 1352, 225. Find the square root of these numbers by logarithms.

76. The distance between two places on a map is 207^{mm}. What is the distance in kilometers if the scale of the map is 1 to 10000?

77. A copper wire 2 yd. 1.23 ft. long is cut into pieces 0.022 of a foot long. How many pieces will there be, and what length will be left over?

78. How many rolls of paper 20 in. wide and 12 yd. long will be required to paper a room 16 ft. long, 12 ft. wide, and 9 ft. high, allowing 96 sq. ft. for windows and doors?

79. Find the specific gravity of a substance that weighs 12^g in air and 7^g in water.

80. A pound Troy is what per cent of a pound avoirdupois?

81. What are the proceeds of a note for $1250 at 5%, dated Oct. 17, 1905, at 3 mo., and discounted Dec. 1 at 6%?

82. If a liter of air weighs 1.29^g, find the weight of air in a room 40 ft. by 30 ft. by 12 ft.

83. If sound travels at the rate of 1090 ft. per second, how far distant is a thundercloud when the sound of the thunder follows the flash of lightning after 6 sec.?

84. A merchant sold some goods for $125 and took in payment a 90-da. note at 5%, dated July 10, 1905. Aug. 5 he discounted the note at the bank at 6%. What were the proceeds of the note?

85. Twenty-five loads of gravel are spread uniformly over a path 200 ft. long and 5 ft. wide. What is the depth of the gravel, a load being 1 cu. yd.?

86. If a half of a liter of a given substance weighs 1500^g, what is the specific gravity of the substance?

87. Find the exact interest on $500 from July 3 to Sept. 10 at 6%.

88. A wholesale dealer sold goods at a discount of 25%, 10% and 3% for cash. He received in payment $3269.75. What was the list price of the goods?

89. When U.S. 3's can be bought at 108 (brokerage $\frac{1}{8}$), how many bonds can be bought for $4325?

90. The nearest fixed star is estimated to be 23000000-000000 mi. distant. How many years does it take light to travel this distance at the rate of 186000 mi. a second?

91. On a note for $5000, dated Jan. 4, 1904, due in 1 yr. with interest at 6%, payments of $100 had been made on the 4th of each month for 11 mo. in succession. What amount was due Jan. 4, 1905?

92. What must be the face of a note at 90 da. so that the borrower shall receive $1000, the discount being at the rate of 7% per annum?

93. A note for $1000, due in 1 yr. at 5%, has an indorsement of $250 made 5 mo. after date. What is the amount due at the end of the year?

94. A note for $500, dated March 1, 1903, and payable 2 yr. from date, with interest at 6% per annum, has on it the following indorsements: April 1, 1903, $50; June 1, 1903, $50; Sept. 1, 1903, $20; and May 1, 1904, $50. What amount is due March 1, 1905?

95. A note for $2000, dated May 15, 1903, at 5% per annum, has the following indorsements: July 1, 1903, $60; Aug. 1, 1903, $10; Oct. 1, 1903, $20; Jan. 2, 1904, $100; May 15, 1904, $100; Sept. 1, 1904, $20; Nov. 1, 1904, $20; May 15, 1905, $200. What amount is due Jan. 2, 1906?

96. If bank stock pays a 7% annual dividend, at what price must it be bought to yield a 5% income on the investment?

97. A traveler bought in New York a bill of exchange on London for £500, exchange being at 4.87. How much did he pay the banker?

98. The number of thousands of people who emigrated annually from Ireland between and including 1876 and 1885 were as follows: 37.5, 38.5, 41.1, 47, 95.5, 78.4, 89.1, 108.7, 75.8, 62. Illustrate graphically.

99. The annual premiums charged by one of the leading life insurance companies at certain ages to insure the payment of $1000 at death are as follows:

Age	21	24	27	30	35	40	45	50
Premium	$19.53	$20.86	$22.40	$24.18	$27.88	$32.76	$39.36	$48.39

Illustrate graphically and determine the probable premiums at ages 25, 33, and 48.

NEW YORK STATE REGENTS' EXAMINATIONS

The following exercises are taken from the Regents' examination questions in advanced arithmetic for the state of New York:

1. Columbus discovered America Oct. 12, 1492. Explain why we celebrated the 400th anniversary Oct. 23, 1892.

2. Find the prime factors of each of the following numbers: 42, 48, 126, 144. Indicate the combination of factors necessary to produce (*a*) the greatest common divisor of these numbers, (*b*) their least common multiple.

3. Find the number of square yards in the four walls and ceiling of a room $16\frac{1}{2}$ ft. long, $13\frac{1}{2}$ ft. wide, and 9 ft. high, making no allowance for openings.

4. Make a receipted bill of the following: William Stone buys this day, of Flagg Brothers, 2 bbl. of flour at \$5.50, 20 lb. sugar at $5\frac{1}{2}$ ct., 4 lb. coffee at 35 ct., 5 lb. butter at 28 ct., 2 bu. potatoes at 45 ct.

5. Simplify $\dfrac{1.75 \times 0.5 + 325 - 0.33\frac{1}{3} \times 21}{0.25 + 0.049 + 0.014}$ and express the result both as a common fraction and as a decimal fraction.

6. A man walks $8\frac{3}{4}$ mi. in 2 hr. 20 min. How long will it take him to walk $11\frac{1}{8}$ mi.? (Solve both by analysis and by proportion.)

7. How many liters of water will be contained in a vessel whose base is 1^{m} square and whose depth is 6^{dm}?

8. A merchant sold goods for \$1225; half he sold at an advance of 25% on the cost, two fifths at an advance of $12\frac{1}{2}\%$ and the remainder at $\frac{1}{2}$ the cost. How much did he pay for the goods?

9. Two successive discounts of 15% and 10% reduced a bill to \$489.60. What was the original bill?

10. Find the proceeds of a note for \$500, payable in 90 da., with interest at 6%, if discounted at a bank at 6%, 40 da. after date.

11. A house and lot cost \$5000; the insurance is \$25, taxes are \$50 and repairs \$75 annually. What rent must be received in order to realize 6% on the investment?

12. At what price must 5% bonds be bought so as to realize $7\frac{1}{2}\%$ on the investment?

13. Find the square root of 243.121 correct to three decimal places.

14. Three families, consisting of 3, 4, and 5 persons respectively, camped out during the summer months, agreeing that the expenses should be divided in the ratio of the number of persons in each family. The expenses amounted to \$606. What number of dollars should each family pay?

15. The diagonal of a square field is 40 rd. How many acres does the field contain?

16. A schoolhouse costing \$9500 is to be built in a district whose property is valued at \$1920000. Find (*a*) the rate of taxation, (*b*) the amount of tax to be paid by a man whose property is valued at \$6500.

17. A sight draft on New York was sold in St. Louis for \$3542, exchange being $\frac{1}{5}\%$ premium. Required the face of the draft.

18. Which would be the better, to invest \$4356.25 in industrial 4's at 87, brokerage $\frac{1}{8}$, or, with the same sum, to purchase real estate which yields an annual rental of \$300?

19. On a note for \$700, dated Oct. 15, 1898, due in one year, with interest at 5%, the following payments have been made: March 9, 1899, \$300; June 1, 1899, \$250. Find the amount due at maturity.

20. A house worth \$12000 was insured for $\frac{7}{8}$ of its value by three companies; the first took $\frac{1}{3}$ of the risk at $\frac{1}{5}$%, the second $\frac{1}{5}$ of the risk at $\frac{1}{4}$%, and the third the remainder at $\frac{3}{8}$%. What was the whole premium paid?

21. Find the trade discount on a bill of goods amounting at list price to \$360, but sold 30%, 8% and 5% off.

22. (*a*) $22\frac{1}{2}$ is what per cent of $7\frac{1}{2}$? (*b*) What per cent of 5 lb. avoirdupois is $7\frac{1}{2}$ oz.? (*c*) $\frac{3}{11}$ is 225% of what number?

23. The specific gravity of copper is 8.9, of silver 10.5, and in an alloy of these metals the weight of the copper is to the weight of the silver as 5:6. Find the ratio of the bulk of copper in the alloy to that of the silver.

24. How many kilograms of water are required to fill a tank 2^m deep whose base is a regular hexagon 0.4^m on a side?

25. A horse costs three times as much as a buggy, and the harness and robes cost one half as much as the horse. If the total cost was \$330, what was the cost of each? Write an analysis.

26. Reduce the couplet $9\frac{2}{7} : 32\frac{1}{12}$ to the integral form in lowest terms.

27. What is the height of a wall which is $14\frac{1}{2}$ yd. in length and $\frac{7}{10}$ of a yard in thickness, and which cost \$406, it having been paid for at the rate of \$10 per cubic yard?

28. Find the cost, at \$15 per M, of 75 pieces of lumber each 14 ft. by 16 in. by $1\frac{3}{4}$ in.

29. Find the prime factors of 18902.

30. The diameters of the wheels of three bicycles are 24 in., 32 in. and 34 in. respectively. Each has a ribbon tied to the top of the wheel. How far must the bicycles go that the ribbons may be again in the same relative positions?

31. If a boy buys peaches at the rate of 5 for 2 ct., and sells them at the rate of 4 for 3 ct., how many must he buy and sell to make a profit of \$4.20?

32. Give a method of (*a*) proving addition; (*b*) subtraction; (*c*) multiplication; (*d*) division.

33. Express by signs of per cent, by a decimal, and by a common fraction in its lowest terms, each of the following: (*a*) $\frac{3}{16}$ per cent; (*b*) $4\frac{2}{7}$%; (*c*) five sixty-fourths; (*d*) three thousand one hundred fifteen thousandths.

34. Write a number that shall be at the same time simple, composite, abstract and even. State why it fills each of these requirements.

35. Add together 15262986957 and 3879, and multiply the 19th part of the sum by 76.

36. In trying numbers for factors, why is it unnecessary to try one larger than the square root of the number?

37. Find the cost, at 25 ct. a rod, of building a fence round a square 10-acre field.

38. How many cords of wood can be stored in a shed 16 ft. long, 12 ft. wide and 6 ft. high?

39. Find the sum of $1\frac{1}{8}$, $\frac{2}{5} \times 1\frac{1}{2}$, 3, $\frac{7}{16}$. Express the result as a decimal.

40. If I sell $\frac{2}{3}$ of a farm for what $\frac{4}{5}$ of it cost, what is my per cent of gain?

41. I sell goods at 15% below the market price and still make a profit of 10%. What per cent above cost was the market price?

42. How was the principal unit of the metric system determined? Explain the relation between this unit and the metric units of capacity and weights.

43. Find the cube root of 4.080659192.

44. Prove that the product of any three consecutive numbers is divisible by 6 or by 24. Determine when it is divisible by 6; when it is divisible by 24.

45. The diameters of four spheres are 3.75, 5, 6.25 and 7.5. Prove that the volume of one of them is equal to the volume of the remaining three.

46. A merchant buys goods to the amount of $4000; in order to pay for them he gets his note for 60 da. discounted at a bank. If the face of the note is $4033.61, what is the rate of discount?

47. Prove that the exact interest of any sum for a given number of days is equal to the interest of the same sum for the same number of days (as usually computed) diminished by $\frac{1}{73}$ of itself.

48. A sells a certain amount of 5% stock at 86 and invests in 6% stock at 103; by so doing his income is changed $1. What amount of stock did he sell? Was his income increased or diminished?

49. Divide $\frac{3}{4}$ by $\frac{5}{7}$ and demonstrate the correctness of the work.

50. Multiply 42.35 by 3.14159, using the contracted method and finding the result correct to two decimal places. Prove the work by division, using the contracted method.

51. A man borrows $4500, and agrees to pay principal and interest in four equal annual installments. If the rate of interest is 6%, what will be the amount of each annual payment?

52. When it is Monday, 7 A.M., at San Francisco, longitude 122° 24′ 15″ W., what day and time of day is it at Berlin, longitude 13° 23′ 55″ E.?

53. When exchange is at 5.18, find the gain on 100^m of silk bought in Paris at 2 francs a meter and sold in New York at 89 ct. a yard, the duty being 6% ad valorem.

54. Find the face of a sight draft that can be bought for $585.80 when exchange is at a premium of $\frac{5}{8}$%.

55. Divide 0.8487432 by 0.075637 and multiply the quotient by 0.835642. Find the result correct to three decimal places, using the contracted methods of division and multiplication of decimals.

56. Express in words each of the following: 600.035, 0.635, $600\frac{30}{5000}$, $\frac{630}{5000}$, $\frac{635}{1000}$.

57. A body on the surface of the earth weighs 27 lb. Assuming that the radius of the earth is 4000 mi., find the weight of the same body 2000 mi. above the surface. (The weight of a body above the surface of the earth varies inversely as the square of the distance from the center of the earth.)

58. Washington is 77° 3′ W. longitude and Pekin 116° 29′ E. longitude. When it is 9.30 P.M., Tuesday, Dec. 31, 1901, at Washington, what is the time of the day, the day of the week, and the date at Pekin?

59. Find the exact interest on \$590 from Sept. 18, 1893, to March 1, 1894, at $4\frac{1}{2}\%$.

60. Is the merchants' rule or the United States rule for computing partial payments more favorable to the debtor? Give reasons.

61. A locomotive runs $\frac{3}{4}$ of a mile in $\frac{4}{5}$ of a minute. At what rate an hour does it run? (Give analysis in full.)

62. The edges of a rectangular parallelopiped are in the proportion of 3, 4 and 6; its volume is 720 cu. in. Find its entire surface.

63. A note for \$250, due in 1 yr., with interest at 6%, is dated Jan. 1, 1892. What is the true value of this note Oct. 1, 1892?

64. At 10 A.M. Jan. 5 a watch is 5 min. too slow; at 2 P.M. of Jan. 9 it is 3 min. 20 sec. too fast. When did it mark correct time?

65. A gallon contains 231 cu. in.; a cubic foot of water weighs 62.5 lb.; mercury is 13.5 times as heavy as water. How many gallons of mercury will weigh a ton?

66. Find the face of a note that will yield \$861.44 proceeds when discounted for 90 da. at 6%.

67. A merchant buys goods listed at \$2500, getting successive trade discounts of 20, 10 and 5; he sells his goods at 20% above the cost price, taking in payment a note at 60 da. without interest; he then gets the note discounted at 6% and pays his bill. Find his entire gain.

68. A person deposits \$100 a year in a savings bank that pays 4% interest, compounded annually. How much money stands to his credit immediately after the fifth deposit?

69. Change 200332 in the quinary scale to an equivalent number in the decimal scale, and prove the work.

70. A New York merchant remitted to London through his broker £12000 18*s*. 9*d*. Find the cost of the draft if exchange is at $4.89\frac{1}{4}$ and brokerage is $\frac{1}{4}\%$.

71. In extracting the cube root state and explain the process of (*a*) separating into periods, (*b*) forming the trial divisor, (*c*) completing the divisor.

72. A merchant buys goods at a list price of \$800, getting discounts of 10, 20 and 5 with 60 da. credit, or a further discount of 5% for cash. How much will he gain by borrowing at 6% to pay the bill?

73. At a certain election 510 votes were cast for two candidates; $\frac{2}{3}$ of those cast for one equaled $\frac{3}{4}$ of those cast for the other. How many votes were cast for each candidate?

74. If the cost of an article had been 8% less, the gain would have been 10% more. What was the per cent gain?

75. Prove that the excess of 9's in the product of two numbers is equal to the excess in the product of the excesses in the two factors.

76. Derive a rule for marking goods so that a given reduction may be made from the marked price and a given profit still made on the cost.

77. The greatest common divisor and the least common multiple of two numbers between 100 and 200 are respectively 6 and 3150. Find the numbers.

78. How much will the product of two numbers be increased by increasing each of the numbers by 1? Give proof.

79. The longer sides of an oblong rectangle are 15 ft. and the diagonal is 20 ft. Find its area.

80. Find the fourth term of the following proportion and demonstrate the principle on which the operation is based: $8 : 12 = 10 : x$.

81. Demonstrate the following: If the greater of two numbers is divided by the less, and the less is divided by the remainder, and this process is continued till there is no remainder, the last divisor will be the greatest common divisor.

82. Find in inches to two places of decimals the diagonal of a cube whose volume is 9 cu. ft.

83. Compare the standard units of money of the United States, England, France, and Germany as to relative value. Find the value of $100 in each of the other units.

84. A dealer sent a margin of $1500 to his broker, April 16, 1905, and ordered him to buy 100 shares of American Sugar stock. The broker filled the order at $131\frac{7}{8}$ and sold the stock May 1 at $126\frac{1}{2}$, charging $\frac{1}{8}\%$ brokerage each way and 6% interest. How much money should be returned to the dealer?

85. A four months' note for $584, without interest, is discounted at a bank at 5% on the day of its issue. Find the proceeds of the note.

86. What is the difference between a discount of 10% and two successive discounts of 5% each on a bill of $832?

87. If I buy cloth at $1.20 a yard, how must I sell it so as to gain 25%?

88. Find the cost of paving a walk 140^{cm} wide and $\frac{3}{5}$ of a kilometer long at $1.25 a square meter.

89. Indicate the factors which, multiplied together, equal the square root of 441.

90. A newsboy buys 144 daily papers at 20 ct. a dozen, and sells them at 3 ct. each. At the end of 6 da. he has 18 papers on hand. How much has he made during the week?

91. The diameters of two concentric circles are 20 ft. and 30 ft. Find the area of the ring.

92. What yearly income will $2267.50 produce when invested in U. S. 4's at $113\frac{1}{4}$, brokerage $\frac{1}{8}\%$?

93. Find the amount of $486.50 for 1 yr. 5 mo. and 17 da. at $5\frac{1}{2}\%$ simple interest.

94. I buy stocks at 4% discount and sell at 4% premium; what per cent profit do I make on the investment?

95. A merchant buys goods to the amount of $1575 on 9 months' credit; he sells them for $1800 cash. Money being worth 6%, how much does he gain?

96. Find the cost, at 60 ct. a yard, of carpeting a room 16 ft. 4 in. wide and 21 ft. 6 in. long with carpet 27 in. wide, if the strips of carpet run lengthwise.

97. Find the cost at 45 ct. a roll of papering the walls of a room $16\frac{1}{2}$ ft. long, 15 ft. wide, and 12 ft. high, making no allowance for openings.

98. Find the cost of plastering the four walls and the ceiling of a room 15 ft. long, 12 ft. wide and 9 ft. high at 15 ct. a sq. yd., allowing 6 sq. yd. for openings.