ADVANCED ARITHMETIC

NOTATION AND NUMERATION

1. Our remote ancestors doubtless did their counting by the aid of the ten fingers. Hence, in numeration it became natural to divide numbers into *groups of tens.* This accounts for the almost universal adoption of the *decimal scale* of notation.

2. It is uncertain what the first number symbols were. They were, probably, fingers held up, groups of pebbles, notches on a stick, etc. Quite early, however, groups of strokes I, II, III, IIII, ⋯, were used to represent numbers.

3. The earliest written symbols of the Babylonians were *cuneiform* or *wedge-shaped* symbols. The vertical wedge (𒁹) was used to represent unity, the horizontal wedge (𒀸) to represent ten, and the two together (𒁹𒀸) to represent one hundred. Other numbers were formed from these symbols by writing them adjacent to each other. Thus,

𒁹𒁹𒁹 $= 1 + 1 + 1 = 3,$

𒀸𒀸𒁹𒁹 $= 10 + 10 + 1 + 1 = 22,$

𒀸𒁹𒀸 $= 10 \times 100 = 1000,$

𒐊𒁹𒀸𒀸𒁹𒁹 $= 5 \times 100 + 10 + 2 = 512.$

To form numbers less than 100 the symbols were placed adjacent to each other and the numbers they represented were added. To form numbers greater than 100 the symbols representing the number of hundreds were placed at the left of the symbol for one hundred and used as a multiplier.

4. The Egyptians used *hieroglyphics*, pictures of objects, or animals that in some way suggested the idea of the number they wished to represent. Thus, one was represented by a vertical staff (ı), ten by a symbol shaped like a horseshoe (∩), one hundred by a short spiral (ℓ), one hundred thousand by the picture of a frog, and one million by the picture of a man with outstretched hands in the attitude of astonishment. They placed the symbols adjacent to each other and added their values to form other numbers. Thus, ℓ∩ı = 100 + 10 + 1 = 111. The Egyptians had other symbols also.

5. The Greeks used the *letters* of their *alphabet* for number symbols, and to form other numbers combined their symbols much as the Babylonians did their wedge-shaped symbols.

6. The Romans used *letters* for number symbols, as follows:

1	5	10	50	100	500	1000
I	V	X	L	C	D	M

Numbers are represented by combinations of these symbols according to the following principles:

(1) The *repetition* of a symbol *repeats* the value of the number represented by that symbol; as, III = 3, XX = 20.

(2) The value of a number is *diminished* by placing a symbol of less value *before* one of greater value; as, IV = 4, XL = 40, XC = 90. The less number is subtracted from the greater.

(3) The value of a number is *increased* by placing a symbol of less value *after* one of greater value, as XI = 11, CX = 110. The less number is added to the greater number.

(4) The value of a number is *multiplied* by 1000 by placing a bar over it, as $\bar{C}$ = 100,000, $\bar{X}$ = 10,000.

7. Among the ancients we do not find the characteristic features of the Arabic, or Hindu system where each symbol has two values, its *intrinsic value* and its *local value*, *i.e.* the value due to the position it occupies. Thus, in the number 513 the intrinsic value of the symbol 5 is *five*, its local value is *five hundred*. Written in Roman notation 513 = DXIII. In the Roman notation each symbol has its intrinsic value only.

8. The ancients lacked also the symbol for *zero*, or the absence of quantity. The introduction of this symbol made place value possible.

9. With such cumbersome symbols of notation the ancients found arithmetical computation very difficult. Indeed, their symbols were of little use except to record numbers. The Roman symbols are still used to number the chapters of books, on clock faces, etc.

10. The Arabs brought the present system, including the symbol for zero and place value, to Europe soon after the conquest of Spain. This is the reason that the numerals used to-day are called the *Arabic* numerals. The Arabs, however, did not invent the system. They received it and its figures from the Hindus.

11. The origin of each of the number symbols 4, 5, 6, 7, 9, and probably 8 is, according to Ball, the initial letter of the corresponding numeral word in the Indo-Bactrian alphabet in use in the north of India about 150 B.C. 2 and 3 were formed by two and three parallel strokes written cursively, and 1 by a single stroke. Just when the zero was introduced is uncertain, but it probably appeared about the close of the fifth century A.D. The Arabs called the sign 0, sifr (sifra = empty). This became the English *cipher* (Cajori, "History of Elementary Mathematics").

12. The Hindu system of notation is capable of unlimited extension, but it is rarely necessary to use numbers greater than billions.

13. In the development of any series of number symbols into a complete system, it is necessary to select some number to serve as a base. In the Arabic, or Hindu system *ten* is used as a base; *i.e.* numbers are written up to 10, then to 20, then to 30, and so on. In this system 9 digits and 0 are necessary. If five is selected as the base, but 4 digits and 0 are necessary. If twelve is selected, 11 digits and 0 are necessary.

The following table shows the relations of numbers in the scales of 10, 5, and 12. (t and e are taken to represent ten and eleven in the scale of 12.)

BASE															
10	1	2	3	4	5	6	7	8	9	10	11	12	21	48	100
5	1	2	3	4	10	11	12	13	14	20	21	22	41	143	400
12	1	2	3	4	5	6	7	8	9	*t*	*e*	10	19	40	84

Ex. **1.** Reduce 431_5 to the decimal scale.

Note. 431_5 means 431 in the scale of 5.

Solution. 4 represents $4 \times 5 \times 5 = 100$
3 represents $3 \times 5 = 15$
1 represents $1 = 1$
$\therefore 431_5 = 116_{10}$

Ex. **2.** Reduce 4632_{10} to the scale of 8.

Solution.

8 | 4632
579 0 = 579 units of the second order and none of the first order.
72 3 = 72 units of the third order and 3 of the second order.
9 0 = 9 units of the fourth order and none of the third order.
1 1 = 1 unit of the fifth order and 1 of the fourth order.

$\therefore 4632_{10} = 11030_8.$

EXERCISE 1

1. What number symbols are needed for the scale of 2? of 8? of 6? of 11? Write 12 and 20 in the scale of 2.

2. Reduce 234_5 and 546_7 to the decimal scale.

3. Reduce 7649_{10} to the scale of 4.

4. Compare the local values of the two 9's in 78,940,590,634. What is the use of the zero? Why is the number grouped into periods of three figures each? Read it.

5. If 4 is annexed to the right of 376, how is the value of each of the digits 3, 7, 6 affected? if 4 is annexed to the left? if 4 is inserted between 3 and 7?

6. What is the local value of each figure in 76,345? What would be the local value of the next figure to the right of 5? of the next figure to the right of this?

7. For what purpose is the decimal point used?

8. Read 100.004 and 0.104; 0.0002; 0.0125 and 100.0025.

ADDITION

14. If the arrangement is left to the computer, numbers to be added should be written in columns with units of like order under one another.

15. In adding a column of given numbers, the computer should think of results and not of the numbers.

He should not say three and two are five and one are six and four are ten and nine are nineteen, but simply five, six, ten, *nineteen*, writing down the 9 as he names the last number. The remaining columns should be added as follows: three, seven, nine, fifteen, *seventeen*, writing down the 7; nine, fifteen, seventeen, twenty-four, *twenty-seven*, writing down the 7; nine, *eighteen*, writing down the 18. Time in looking for errors may be saved by writing the numbers to be carried underneath the sum as in the exercise.

329
764
221
9642
7823
18779
211

16. Checks. If the columns of figures have been added **upward**, check by adding **downward**. If the two results agree, the work is probably correct.

Another good check for adding, often used by accountants, is to add beginning with the left-hand column.

Thus, the sum of the thousands is 16 thousands, of the hundreds 26 hundreds, of the tens 16 tens, and of the units 19 units.

16000	or	16
2600		26
160		16
19		19
18779		18779

EXERCISE 2

1. What is meant by the order of a digit? Define *addend, sum*.

2. Why should digits of like order be placed in the same column? State the general principle involved.

3. Why should the columns be added from right to left? Could the columns be added from left to right and a correct result be secured? What is the advantage in beginning at the right?

4. In the above exercise, why is 1 added ("carried") to the second column? 1 to the third column? 2 to the fourth column?

17. Accuracy and **rapidity** in computing should be required from the first. Accuracy can be attained by *acquiring the habit of always checking results*. Rapidity comes with *much practice*.

18. The 45 simple combinations formed by adding consecutively each of the numbers less than 10 to itself and to every other number less than 10 should be practiced till the student can announce the sum at sight. These combinations should be arranged for practice in irregular order similar to the following:

1	2	2	5	9	8	1	4	5	7	6	4	2	3	4
1	2	3	1	3	8	9	7	6	8	6	9	4	6	4
6	6	1	1	3	4	9	5	2	7	1	2	5	3	5
9	7	2	8	7	5	9	8	7	9	6	8	5	4	7
5	4	2	2	6	4	3	1	2	1	8	3	1	3	7
9	8	5	9	8	6	5	3	6	7	9	3	4	8	7

19. Rapid counting by ones, twos, threes, etc., up to nines is very helpful in securing both accuracy and rapidity.

Ex. Begin with 4 and add 6's till the result equals 100. Add rapidly, and say simply 4, 10, 16, 22, · · ·, 94, 100.

20. It is helpful also to know combinations, or groups that form certain numbers. Thus,

1	2	3	4	5
9	8	7	6	5

and

8	7	6	6	5	5	4
1	2	2	3	4	3	3
1	1	2	1	1	2	3

, etc., are groups that form 10, and

9	9	9	9	8	8	8	7
9	8	7	6	8	7	6	7
2	3	4	5	4	5	6	6

are groups that form 20.

21. Such groups should be carefully studied and practiced until the student readily recognizes them in his work. He should also familiarize himself with other groups. The nine-groups and the eleven-groups are easy to add, since adding nine to any number diminishes the units' figure by one, and adding eleven increases the units' and the tens' digits each by one.

EXERCISE 3

1. Begin with 8 and add 7's till the result is **50.**
2. Begin with 3 and add 8's till the result is **67.**

Form the following sums till the result exceeds **100:**

3. Begin with 3 and add 7's.
4. Begin with 7 and add 8's.
5. Begin with 5 and add 9's.
6. Begin with 8 and add 5's.
7. Begin with 5 and add 6's.
8. Begin with 6 and add 3's.

Add the following columns, beginning at the bottom, and check the results by adding downward. Form such groups as are convenient and add them as a single number. In the first two exercises groups are indicated.

9.	10.	11.	12.	13.	14.	15.	16.
6	7	5	25.4	2122	275	5427	47.683
3	9	4	76.1	7642	267	6742	72.125
1	1	1	34.59	8321	979	8374	94.467
8	4	6	43.33	9789	231	9763	53.2124
5	5	8	67.27	2432	486	2134	91.576
2	2	4	81.2	5765	523	5666	14.421
9	4	2	28.3	1297	752	3249	32.144
6	7	9	32.99	6423	648	1678	67.6797
5	2	7	16.25	1678	486	2432	19.045
3	8	4	53.11	3212	529	5469	54.091
9	7	3	91.5	7679	926	8761	86.2459
8	4	2	85.4	2144	842	2332	27.654
2	4	1	74.1	1576	236	5467	98.346
1	5	5	22.22	4467	574	1023	84.6211

In commercial operations it is sometimes convenient to add numbers written in a line across the page. If totals are required at the right-hand side of the page, add from left to right and *check* by adding from right to left.

Add:

17. 23, 42, 31, 76, 94, 11, 13, 27, 83, 62, 93.
18. 728, 936, 342, 529, 638, 577, 123, 328, 654.
19. 1421, 2752, 7846, 5526, 3425, 1166, 7531, 8642.
20. 46, 72, 88, 44, 39, 37, 93, 46, 64, 73, 47.
21. 1728, 3567, 2468, 5432, 4567, 2143, 9876, 6789.

Find the sum of the following numbers by adding the columns and then adding the results horizontally. Check by adding the rows horizontally and then adding the columns of results.

22.	7642	6241	5331	
	3124	4724	8246	
	9372	3623	2793	
				51096

23.	793	864	927
	531	642	876
	927	426	459

24.	7942	8349	2275	3673
	9527	2136	3411	4212
	6524	7641	5675	7987
	3171	1234	2892	6425

25.	26	72	126	467	354	987	54	86
	13	34	45	56	67	67	87	43
	98	87	765	453	342	465	783	5
	21	5	43	350	9	11	321	24
	8	25	196	961	649	378	452	36
	77	66	555	444	888	999	111	222

Exercises for further practice in addition can be readily supplied by the teacher. The student should be drilled till he can add accurately and rapidly. Accuracy, however, should never be sacrificed to attain rapidity.

Expert accountants, by systems of grouping and much practice, acquire facility in adding two or even three columns of figures at a time. Elaborate calculating machines have also been invented, and are much used in banks and counting offices. By means of these machines, columns of numbers can be tabulated and the sum printed by simply turning a lever.

SUBTRACTION

22. In **subtraction** it is important that the student should be able to see at once what number added to the smaller of two numbers of one figure each will produce the larger. Thus, if the difference between 5 and 9 is desired, the student should at once think of 4, the number which added to 5 produces 9.

23. Again, if the second number is the smaller, as in 7 from 5, the student should think of 8, the number which added to 7 produces 15, the next number greater than 7 which ends in 5.

24. The complete process of subtraction is shown in the following exercise:

8534	7 and 7 are 14, carry 1. **(Why carry 1?)**
5627	3 and 0 are 3.
2907	6 and 9 are 15, carry 1.
	6 and 2 are 8.

25. The student should think "What number added to 5627 will produce 8534?" After a little practice, it is unnecessary to say more than 7 and 7, 3 and 0, 6 and 9, 6 and 2, writing down the underscored digit just as it is named.

26. Check. To check, add the remainder and the subtrahend upward, since in working the exercise the numbers were added downward.

27. The above method of subtraction is important not only because it can be performed rapidly, but because it is very useful in long division. It is also the method of "making change" used in stores.

28. There are two other methods of subtraction in common use. The processes are shown in the following exercises:

(1)
$$\begin{array}{l} 643 = 600 + 40 + 3 = 500 + 130 + 13 \\ \underline{456 = 400 + 50 + 6 = 400 + 50 + 6} \\ 187 = \qquad\qquad\quad 100 + 80 + 7 \end{array}$$

6 from 13, 7; 5 from 13, 8; 4 from 5, 1.

(2)
$$\begin{array}{l} 643 = 600 + 40 + 3, \ 600 + 140 + 13 \\ \underline{456 = 400 + 50 + 6, \ 500 + 60 + 6} \\ 187 = \qquad\qquad\ 100 + 80 + 7 \end{array}$$

6 from 13, 7; 6 from 14, 8; 5 from 6, 1.

EXERCISE 4

1. Define the terms *subtrahend*, *minuend*, *difference*.

2. How should the terms be arranged in subtraction? Where do we begin to subtract? Why?

3. Is the difference affected by adding the same number to both subtrahend and minuend? Is this principle used in either (1) or (2)?

4. If a digit in the minuend is less than a digit of the corresponding order in the subtrahend, explain how the subtraction is performed in both (1) and (2).

29. Arithmetical Complement. The arithmetical complement of a number is the difference between the number and the next higher power of 10. Thus, the arithmetical complement of 642 is 358, since $358 + 642 = 1000$. The arithmetical complement of 0.34 is 0.66, since $0.66 + 0.34 = 1$.

EXERCISE 5

1. Name rapidly the complements of the following numbers: 75, 64, 32, 12, 90, 33, 25, 0.25, 0.16, 125, 500, 5000, 1250, 625.

2. Name the amount of change a clerk must return if he receives a five-dollar bill in payment of each of the following amounts: $1.25, $3.75, $2.34, $3.67, $0.25, $0.88, $4.91, $1.85.

3. Name the amount of change returned if the clerk receives a ten-dollar bill in payment of each of the following amounts: $7.34, $3.42, $9.67, $5.25, $2.67, $6.45, $4.87, $0.68, $3.34.

Determine in each of the following exercises what number added to the smaller number will produce the larger. The student will notice that in some cases the subtrahend is placed over the minuend. It is often convenient in business to perform work in this way.

4.	5.	6.	7.	8.	9.
9	36	75	246	8937	5280
5	42	31	167	9325	3455

10.	11.	12.	13.
7621	2339	9654327	4680215
6042	5267	6098715	9753142

14. Show that to subtract 73854 from 100000 it is necessary only to take 4 from 10 and each of the remaining digits from 9.

15. Subtract 76495 from 100000, and 397.82 from 1000, as in Ex. 14.

16. Show that to subtract 3642 from 5623 is the same as to add the arithmetical complement of 3642 and subtract 10000 from the sum.

17. From 8757 take the sum of 1236, 2273 and 3346.

8757	
1236	6, 9, 15 and 2; 17.
2273	5, 12, 15 and 0; 15.
3346	4, 6, 8 and 9; 17.
1902	4, 6, 7 and 1; 8.

18. From 53479 take the sum of 23, 1876 and 41253.

19. From 7654 take the sum of 3121, 126 and 2349.

20. From 764295 take the sum of 45635, 67843, 125960 and 213075.

21. A clerk receives a twenty-dollar bill in payment of the following items: $2.25, $11.50, $0.13, $0.75. How much change does he return?

22. Find the value of 2674 + 1782 − 1236 + 8420 − 4536 by adding the proper arithmetical complements and subtracting the proper powers of ten.

30. To find the balance of an account.

Dr. FIRST NATIONAL BANK, YPSILANTI, *in acct. with* JOHN SMITH *Cr.*

1904					1904				
Aug. 3	Balance	1	486	87	Aug. 4	By check		500	00
Aug. 22	To deposit		290	00	Aug. 10	By check		57	30
Sept. 30	To deposit		198	75	Sept. 1	By check		235	75
Oct. 24	To deposit		773	40	Sept. 21	By check		11	80
Nov. 20	To deposit		110		Oct. 15	By check		97	30
					Nov. 3	By check	1	000	00
					Nov. 25	Balance		956	87
		2	859	02			2	859	02
Nov. 25	Balance		956	87					

The preceding form represents the account of John Smith with the First National Bank from Aug. 3 till Nov. 25. The items at the left of the central dividing line are the amounts that the bank owes Mr. Smith. This side is called the **debit** side of the account. The items at the right represent the amounts withdrawn by Mr. Smith. This side is called the **credit** side of the account. The difference between the sums of the credits and the debits is called the **balance** of the account.

It is evident that the debit side of the above account is greater than the credit side. Therefore, to balance the account, add the debit side first, and then subtract the sum of the credit side from the result, as in Ex. 17 above. The difference will be the balance, or the amount left in the bank to the credit of Mr. Smith. The work can be checked by adding the balance to the credit column. The result should equal the sum of the debit column.

EXERCISE 6

Find the balance of each of the following accounts:

1.

Dr.		*Cr.*	
234	50	246	84
798	34	125	00
500	00	450	00
212	60	55	30
351	00	97	30
100	00	60	00
75	00		
198	30		

2.

Dr.		*Cr.*	
1250	00	527	30
888	80	131	60
210	60	927	50
1100	00	500	00
2681	50	975	25
69	00	659	75
10	46	50	00
100	00		

3.

Dr.		*Cr.*	
235	67	564	90
1000	00	75	00
750	25	34	68
104	69	100	00
566	66	1200	00
195	75	275	80
302	00	625	30
259	00		

4. On May 1 R. F. Joy had a balance of $1376.24 to his account in the bank. He deposited on May 1, $189; June 27, $166; July 28, $75; Aug. 5, $190.60; Aug. 10, $192.22. He withdrew by check the following amounts: June 1, $153; June 10, $300; July 3, $25; July 27, $575.50. What was his balance Aug. 15?

MULTIPLICATION

31. The multiplication table should be so well known that the factors will at once suggest the product. Thus, 7 × 6, or 6 × 7, should at once suggest 42.

32. The student should also be able to see at once what number added to the product of two numbers will produce a given number. Thus, the number added to 4 × 9 to produce 41 is <u>5</u>, or 4 × 9 and <u>5</u> are 41.

It is a common practice in multiplication to write the multiplier first as 2 × \$5 = \$10. In this case the sign (×) is read "times." If the multiplier is written after the multiplicand, as in \$5 × 2 = \$10, the sign (×) is read "multiplied by." The multiplier is always an abstract quantity (Why?), but the multiplicand may be either abstract or concrete.

33. The following examples show the complete process of multiplication:

Ex. 1. Multiply 2743 by 356.

Solution. In multiplying one number by another it is not necessary to begin with the units' digit of the multiplier. We may begin with either the units' digit or the digit of the highest order. In fact, it is frequently of decided advantage to begin with the digit of highest order, especially in multiplying decimals; but care should be taken in placing the right-hand figure of the first partial product. Since 3 hundred times 3 units = 9 hundred, the 9 must be put in the third or hundreds' place, etc.

2743	2743
356	356
16458	8229
13715	13715
8229	16458
976508	976508

Ex. 2. Multiply 3.1416 by 26.34.

Solution. In beginning the multiplication we see that 20 × 0.0006 = 0.012. Hence the 2 is written in the thousandths' place. The work is then completed as indicated in the annexed example. It will readily be seen that the rest follows after pointing off the first partial product correctly.

The advantage of beginning with the digit of the highest order is seen in approximations (see p. 59), where considerable work is thereby saved.

3.1416
26.34
62.832
18.8496
.94248
.125664
82.749744

34. Check. Multiplication may be checked by using the multiplicand as the multiplier and performing the multiplication again. However, the check by "casting out the nines" (p. 41), is more convenient.

EXERCISE 7

1. Define *multiplier*, *multiplicand*, *product*.

2. Explain why multiplication is but an abridged method of addition.

3. Can the multiplier ever be a concrete number? Explain.

4. How should the terms be arranged in multiplication? Does it make any difference in what order we multiply by the digits of the multiplier? Might we begin to multiply with the 5 in *Ex.* 1 and with the 6 in *Ex.* 2?

5. How is the order of the right-hand figure of each partial product determined?

6. How does the presence of a zero in the multiplier affect the work?

7. In multiplying 3.1416 by 26.34, can we tell at once how many integral places there will be in the product? Can we tell the number of decimal places?

8. How many decimal places will there be in each of the following products: 21.34 × 5.9? 98.65 × 76.43? 321.1 × 987.543? 1.438 × 42.345?

35. The following short methods are useful:

1. *To multiply any number by* 5, 25, $16\frac{2}{3}$, $33\frac{1}{3}$, 125.

Since $5 = \frac{10}{2}$, to annex a cipher and divide by 2 is the same as to multiply by 5. The student in a similar manner should explain short processes of multiplying by 25, $16\frac{2}{3}$, $33\frac{1}{3}$, 125.

2. *To multiply any number by* 9.

Since 9 = 10 − 1, it is sufficient to annex a cipher to the number and subtract the original number.

Ex. Multiply 432 by 9.

$$\begin{aligned} 432 \times 10 &= 4320 \\ 432 \times 1 &= \underline{\ 432} \\ 432 \times 9 &= 3888 \end{aligned}$$

3. *To multiply any number by* 11.

Since 11 = 10 + 1, it is sufficient to annex a cipher to the number and add the original number.

Ex. Multiply 237 by 11.

$$\begin{aligned} 237 \times 10 &= 2370 \\ 237 \times 1 &= \underline{\ 237} \\ 237 \times 11 &= 2607 \end{aligned}$$

This result can readily be obtained by writing down the right-hand figure first and then the sums of the first and second figures, the second and third, etc., and finally the left-hand figure.

4. *To multiply any number by a number differing but little from some power of* 10.

Annex as many ciphers to the number as there are ciphers in the next higher power of 10, and subtract the product of the number multiplied by the complement of the multiplier.

Ex. Multiply 335 by 996. 996 = 1000 − 4.

$$\begin{aligned} 335 \times 1000 &= 335000 \\ 335 \times 4 &= \underline{\ \ \ 1340} \\ 335 \times 996 &= 333660 \end{aligned}$$

In practice written 335
1340
333660

5. *To multiply any number by a number of two figures ending with* 1.

Multiply by the tens' figure of the multiplier, writing this product under the number one place to the left.

Ex. Multiply 245 by 71.

$$\begin{aligned} 245 \times 1 &= 245 \\ 245 \times 70 &= \underline{17150} \\ 245 \times 71 &= 17395 \end{aligned}$$

6. *To multiply any number by a number between twelve and twenty.*

Multiply by the units' figure of the multiplier, writing the product under the number one place to the right.

Ex. Multiply 427 by 13.

$$\begin{aligned} 427 \times 10 &= 4270 \\ 427 \times 3 &= \underline{1281} \\ 427 \times 13 &= 5551 \end{aligned}$$

7. *To square a number ending in* 5.

$35^2 = 3 \times 400 + 25$, $45^2 = 4 \times 500 + 25$, $55^2 = 5 \times 600 + 25$, etc.

8. *To multiply by a number when the multiplier contains digits which are factors of other parts of the multiplier.*

Ex. Multiply 25631 by 74221.

Since 7 is a factor of 42 and 21, multiply by 7, placing the first figure in the partial product under 7. (Why?) Then multiply this product by 6 (42 = 6 × 7), placing the first figure under 2 in hundreds' place. (Why?) Then multiply the first partial product by 3 (21 = 3 × 7), placing the first figure under 1. (Why?) The sum of these partial products will be the product of the numbers.

```
     25631
     74221
  --------
179417
  1076502
     538251
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1902358451
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EXERCISE 8

Name rapidly the products of the successive pairs of digits in each of the following numbers:

1. 75849374657. **3.** 67452367885.

2. 265374867598. **4.** 98765432345.

5. In each of the following groups of digits add rapidly to the product of the first two the sum of all that follow: 567, 432, 7654, 3456, 9753, 3579, 8642, 2468, 7896, 5436, 3467.

6. Multiply 1264 by 125; by $12\frac{1}{2}$; by $1\frac{1}{4}$.

7. Multiply 76.26 by $16\frac{2}{3}$; by $33\frac{1}{3}$.

8. Multiply 2348 by 25; by $2\frac{1}{2}$; by 50; by 0.5.

9. Multiply 645 by 9; by 11; by 17; by 41.

10. Multiply 8963 by 848.

11. Multiply 37439 by 4832.

12. Show that to multiply a number by 625 is the same as to multiply by 10000 and divide by 16.

13. Subtract 5×12631 from 87642.

The work should be done as follows:

	5 × 1 and 7, 12.
87642	5 × 3 and 1 and 8, **24**.
12631	5 × 6 and 2 and 4, **36**.
24487	5 × 2 and 3 and 4, **17**.
	5 × 1 and 1 and 2, **8**.

14. Subtract 3×2462 from 9126.

15. Subtract 6×42641 from 768345.

16. Subtract 2×86473 from 291872.

When the same number is to be used as a multiplier several times, work may be saved by forming a table of its multiples. Thus,

5764 × 784 =		1	784
3136	(4)	2	1568
4704	(6)	3	2352
5488	(7)	4	3136
3920	(5)	5	3920
4518976		6	4704
		7	5488
		8	6272
		9	7056

The partial products in each case are taken from the table.

17. Use the above table and multiply 5764, 74591, 84327, 23145, each by 784.

18. Form a table of multiples of 6387, and use it to find the product of 7482, 3.1416, 742896, 342312, 67564534, 897867, 65768798, 56024.85, each by 6387.

19. Multiply 2785 by 9998, and 1728 by 997.

20. Multiply 78436 by 25×125.

21. Multiply 32.622 by 0.0125.

22. Multiply 486.72 by 0.25×0.25.

23. Multiply 320.4 by 5×1.25.

24. Multiply 5763 by $16\frac{2}{3} \times 33\frac{1}{3}$.

DIVISION

36. In **division** the student should be able to see at once how many times a given digit is contained in any number of two digits with the remainder. Thus, 7 is contained in 46, 6 times with a remainder 4. The student should think simply 6 and 4 over.

Ex. 6)354279
     59046 remainder 3.

The whole mental process should be 5 and 5, 9 and 0, 0 and 2, 4 and 3, 6 and 3.

Two interpretations arise from considering division as the inverse of multiplication.

Thus, since 4 × \$6 = \$24.

(1) \$24 ÷ 4 = \$6, separation into groups. \$24 has been separated into 4 equal groups.

(2) \$24 ÷ \$6 = 4, involving the idea of measuring, or being *contained in*.

\$6 is contained in \$24, 4 times.

37. The following examples show the complete process of long division.

```
         346
4541)1571186
      13623
      -----
       20888
       18164
       -----
        27246
        27246
        -----
```

It assists in determining the order of the digits in the quotient to write them in their proper places above the dividend.

38. The work in long division may be very much abridged by omitting the partial products and writing down the remainders only. These remainders are obtained by the method used in Ex. 13, p. 26.

Ex. Divide 764.23 by 2.132.

The work will be simplified by multiplying both numbers by 1000 to avoid decimals. The first remainder, 1246, is obtained as follows:

```
        358
2132)764230     3 × 2, 6 and 6, 12.
      12463     3 × 3, 9 and 1, 10 and 4, 14.
       18030    3 × 1, 3 and 1, 4 and 2, 6.
         974    3 × 2, 6 and 1, 7.
```

Then bring down 3 and proceed as before to form the other remainders.

39. Check. Division may be checked by multiplying the quotient by the divisor, the product plus the remainder should equal the dividend. The check by "casting out the nines" (p. 42) may be used.

EXERCISE 9

1. Define *dividend*, *divisor*, *quotient*, *remainder*.

2. Explain the two interpretations arising from considering division as the inverse of multiplication. 5 × \$10 = \$50. Give the two interpretations as applied to this example.

3. How is the order of the right-hand figure in each partial product determined?

4. Explain why the sum of the partial products plus the remainder, if any, must equal the dividend if the work is correct.

5. Explain why the quotient is not affected by multiplying both dividend and divisor by the same number.

40. If the same number is used as a divisor several times, or if the dividend contains a large number of places, work may be saved by forming a table of multiples of the divisor. Thus:

Ex. Divide 786342 by 4147.

1	4147	189	
2	8294	4147)786342	
3	12441	37164	
4	16588	39882	
5	20735	2559	remainder.
6	24882		
7	29029		
8	33176		
9	37323		

EXERCISE 10

1. Divide 987262, 49789 and 314125 each by 4147.
2. Divide 896423, 76425, 9737894 each by 5280.
3. Divide 44.2778 by 63.342.

Find the value of:

4. $32.36 \div 8.9$.
5. $1.25 \div 0.5$ and $12.5 \div 0.05$.
6. $144 \div 1.2$ and $14.4 \div 12$.
7. $625 \div 25$ and $62.5 \div 2.5$.
8. $1125 \div 50$ and $11.25 \div 9.5$.
9. $5280 \div 12.5$ and $580 \div 125$.
10. $750 \div 2.5 \div 0.5$.

41. In addition to the checks on the fundamental processes given above, it is well when possible *to form the habit of estimating results before beginning the solution of a problem*. Thus, in multiplying $19\frac{1}{2}$ by $12\frac{1}{2}$ it is evident that the result will be about $12 \times 20 = 240$.

In using this check the student should form a rough estimate of the result, then solve the problem and compare results. A large error will be at once detected.

EXERCISE 11

Solve the following, first giving approximate answers, then the correct result:

1. Multiply $15.3 \times 3\frac{48}{49}$ (about 15×4).
2. Divide 594 by $\frac{99}{302}$ (about $594 \div \frac{1}{3}$).
3. Divide 32.041 by 0.499 (about $32.041 + \frac{1}{2}$).
4. How much will 21 horses cost at $145 each?
5. Multiply 30.421 by 20.516.
6. At $12\frac{1}{2}$ ct. a dozen, how much will $6\frac{1}{2}$ doz. eggs cost?
7. At $37\frac{1}{2}$ ct. a pound, how much will 11 lb. of coffee cost?
8. How many bushels of potatoes can be bought for $5.25 at 35 ct. a bushel?
9. At 1.12\frac{1}{2}$ a barrel, how many barrels of salt can be bought for $22.50?
10. How far will a train travel in $12\frac{1}{2}$ hr. at the rate of 45 mi. an hour?
11. How much will $8\frac{3}{4}$ T. of coal cost at $7.25 a ton?
12. The net cost of printing a certain book is 49 ct. a copy. How much will an edition of 2500 cost?
13. At the rate of 40 mi. an hour, how long will it take a train to run 285 mi.?