FACTORS AND MULTIPLES

42. A **factor** or **divisor** of a number is any integral number that will exactly divide it.

43. A number that is divisible by 2 is called an **even** number, and one that is not divisible by 2 an **odd** number.

Thus, 24 and 58 are even numbers, while 17 and 83 are odd numbers.

44. A number that has no factors except itself and unity is called a **prime** number.

Thus, 1, 2, 3, 5, 7, etc., are prime numbers.

45. Write down all of the odd numbers less than 100 and greater than 3. Beginning with 3 reject every third number; beginning with 5 reject every fifth number; beginning with 7 reject every seventh number. The numbers remaining will be all of the prime numbers between 3 and 100. (Why?)

46. This method of distinguishing prime numbers is called the **Sieve of Eratosthenes**, from the name of its inventor, Eratosthenes (276–196 B.C.). He wrote the numbers on a parchment and cut out the composite numbers, thus forming a sieve.

47. A number that has other factors besides itself and unity is called a **composite** number.

48. Numbers are said to be **prime to each other** when no number greater than 1 will exactly divide each of them.

Are numbers that are prime to each other necessarily prime numbers?

49. An integral number that will exactly divide two or more numbers is called a **common divisor,** or **a common factor** of these numbers.

Thus, 2 and 3 are common divisors of 12 and 18.

50. The greatest common factor of two or more numbers is called the **greatest common divisor (g. c. d.)** of the numbers.

Thus, 6 is the g. c. d. of 12 and 18.

51. A **common multiple** of two or more numbers is a number that is exactly divisible by each of them.

Thus, 12, 18, 24, and 48 are common multiples of 3 and 6, while 12 is the **least common multiple** (l. c. m.) of 3 and 6.

52. It is of considerable importance in certain arithmetical operations, particularly in cancellation, to be able readily to detect small factors of numbers. In proving the tests of divisibility by such factors, the two following principles are important.

1. *A factor of a number is a factor of any of its multiples.*

Proof. Every multiple of a number contains that number an exact number of times; therefore, it contains every factor of the number.

Thus, 5 is a factor of 25, and hence of 3×25, or 75.

2. *A factor of any two numbers is a factor of the sum or difference of any two multiples of the numbers.*

Proof. Any factor of two numbers is a factor of any of their multiples by Principle 1. Therefore, as each multiple is made up of parts each equal to the given factor, their sum or difference will be made up of parts equal to the given factor, or will be a multiple of the given factor.

Thus, 3 is a factor of 12 and of 15, and hence of $5 \times 12 + 2 \times 15$, or 90. 3 is also a factor of $5 \times 12 - 2 \times 15$, or 30.

53. Tests of Divisibility. 1. *Any number is divisible by* 2 *if the number represented by its last right-hand digit is divisible by* 2.

Proof. Any number may be considered as made up of as many 10's as are represented by the number exclusive of its last digit plus the last digit. Then, since 10 is divisible by 2, the first part, which is a multiple of 10, is divisible by 2. Therefore, if the second part, or the number represented by the last digit, is divisible by 2, the whole number is.

Thus, $634 = 63 \times 10 + 4$ is divisible by 2 since 4 is.

2. *Any number is divisible by* 4 *if the number represented by the last two digits is divisible by* 4.

Proof. Any number may be considered as made up of as many 100's as are represented by the number exclusive of its last two digits plus the number represented by the last two digits. Then, since 100 is divisible by 4, the first part, which is a multiple of 100, is divisible by 4. Therefore, if the number represented by the last two digits is divisible by 4, the whole number is.

Thus, $85648 = 856 \times 100 + 48$ is divisible by 4 since 48 is.

3. *Any number is divisible by* 5 *if the last digit is* 0 *or* 5.

The proof, which is similar to the proof of 1, is left for the student.

Note. 0 is divisible by any number, and the quotient is always 0.

4. *Any number is divisible by* 8 *if the number represented by its last three digits is divisible by* 8.

The proof is left for the student.

5. *Any number is divisible by* 9 *if the sum of its digits is divisible by* 9.

Proof. Since $10 = 9 + 1$, any number of 10's = the same number of 9's + the same number of units; since $100 = 99 + 1$, any number of 100's = the same number of 99's + the same number of units; since $1000 = 999 + 1$, any number of 1000's = the same number of 999's + the same number of units; etc. Therefore, any number is made up of a multiple of 9 + the sum of its digits, and hence is divisible by 9 if the sum of its digits is divisible by 9.

Thus,
$$\begin{aligned} 7362 &= 7 \times 1000 + 3 \times 100 + 6 \times 10 + 2 \\ &= 7(999 + 1) + 3(99 + 1) + 6(9 + 1) + 2 \\ &= 7 \times 999 + 3 \times 99 + 6 \times 9 + 7 + 3 + 6 + 2 \\ &= \text{a multiple of } 9 + \text{the sum of the digits.} \end{aligned}$$

Therefore, the number is divisible by 9 since $7 + 3 + 6 + 2 = 18$ is divisible by 9.

6. *Any number is divisible by* 3 *if the sum of its digits is divisible by* 3.

The proof, which is similar to the proof of Principle 5, is left for the student.

7. *Any even number is divisible by* 6 *if the sum of its digits is divisible by* 3.

The proof is left for the student.

8. *Any number is divisible by* 11 *if the difference between the sums of the odd and even orders of digits, counting from units, is divisible by* 11.

Proof. Since $10 = 11 - 1$, any number of 10's = the same number of 11's − the same number of units; since $100 = 99 + 1$, any number of 100's = the same number of 99's + the same number of units; since $1000 = 1001 - 1$, any number of 1000's = the same number of 1001's − the same number of units; etc. Therefore, any number is made up of a multiple of 11 + the sum of the digits of odd order − the sum of the digits of even order, and hence is divisible by 11 if the sum of the digits of odd order − the sum of the digits of even order is divisible by 11.

Thus, $753346 = 7 \times 100000 + 5 \times 10000 + 3 \times 1000 + 3 \times 100 + 4 \times 10 + 6$

$= 7(100001 - 1) + 5(9999 + 1) + 3(1001 - 1)$

$+ 3(99 + 1) + 4(11 - 1) + 6$

$= 7 \times 100001 + 5 \times 9999 + 3 \times 1001 + 3 \times 99 + 4 \times 11$

$- 7 + 5 - 3 + 3 - 4 + 6$

= a multiple of 11 + the sum of the digits of odd order

− the sum of the digits of even order.

Therefore, the number is divisible by 11 since $5 + 3 + 6 - (7 + 3 + 4) = 0$ is divisible by 11.

9. *The test for divisibility by 7 is too complicated to be useful.*

EXERCISE 12

1. Write three numbers of at least four figures each that are divisible by 4.

2. Write three numbers of six figures each that are divisible by 9.

3. Is 352362257 divisible by 11? by 3?

4. Without actual division, determine what numbers less than 19 (except 7, 13, 14, 17) will divide 586080.

5. Explain short methods of division by 5, 25, $16\frac{2}{3}$, $33\frac{1}{3}$, 125.

6. Divide 3710 by 5; by 25; by 125; by $12\frac{1}{2}$.

7. Divide 2530 by 0.5; by 0.025; by 1.25.

8. Prove that to divide by 625 is the same as to multiply by 16 and divide by 10000.

9. State and prove a test for divisibility by 12; by 15; by 18.

10. If 7647 is divided by 2 or 5, how will the remainder differ from the remainder arising from dividing 7 by 2 or 5? Explain.

11. If 26727 is divided by 4 or 25, how will the remainder differ from the remainder arising from dividing 27 by 4 or 25? Explain.

12. Explain how you can find the remainder arising from dividing 26727 by 8 or 125 in the shortest possible way.

54. Relative Weight of Symbols of Operation. In the use of the symbols of operation (+, −, ×, ÷), it is important that the student should know that the numbers connected by the signs × and ÷ must first be operated upon and then those connected by + and −; for the signs of multiplication and division connect factors, while the signs of addition and subtraction connect terms. Factors must be combined into simple terms before the terms can be added or subtracted.

Thus, $5 + 2 \times 3 - 15 \div 5 + 4 = 12$, the terms 2×3 and $15 \div 5$ being simplified before they are combined by addition and subtraction.

55. The ancients had no convenient symbols of operation. Addition was generally indicated by placing the numbers to be added adjacent to each other. Other operations were written out in words. The symbols + and − were probably first used by Widman in his arithmetic published in Leipzig in 1489. He used them to mark excess or deficiency, but they soon came into use as symbols of operation. × as a symbol of multiplication was used by Oughtred in 1631. The dot (·) for multiplication was used by Harriot in 1631. The Arabs indicated division in the form of a fraction quite early. ÷ as a symbol of division was used by Rahn in his algebra in 1659. Robert Recorde introduced the symbol = for equality in 1557. : was used to indicate division by Leibnitz and Clairaut. In 1631 Harriot used > and < for greater than and less than. Rudolff used √ to denote square root in 1526.

56. Greatest Common Divisor. In many cases the g. c. d. of two or more numbers may readily be found by factoring, as in the following example:

Ex. Find the g. c. d. of 3795, 7095, 30030.

$$3795 = 3 \times 5 \times 11 \times 23,$$
$$7095 = 3 \times 5 \times 11 \times 43,$$
$$30030 = 2 \times 3 \times 5 \times 11 \times 91,$$

and since the g. c. d. is the product of all of the prime factors that are common to the three numbers, it is $3 \times 5 \times 11 = 165$.

57. Euclid, a famous Greek geometer, who lived about 300 B.C., gave the method of finding the g. c. d. by division. This method is useful if the prime factors of the numbers cannot be readily found.

Ex. Find the g. c. d. of 377 and 1479.

		1479
377	3	1131
348	1	348
29	12	348

The g. c. d. cannot be greater than 377, and since 377 is not a factor of 1479, it is not the g. c. d. of the two numbers.

Divide 1479 by 377. Then, since the g. c. d. is a common factor of 377 and 1479, it is a factor of $1479 - 3 \times 377$, or 348 (Principle 2, p. 33).

Therefore, the g. c. d. is not greater than 348. If 348 is a factor of 377 and 1479, it is the g. c. d. sought.

But 348 is not a factor of 377. Therefore, it is not the g. c. d. sought.

Divide 377 by 348. Then, since the g. c. d. is a factor of 377 and 348, it is a factor of $377 - 348$, or 29 (Principle 2, p. 33).

Therefore, the g. c. d. is not greater than 29, and if 29 is a factor of 348, 377, and 1479, it is the g. c. d. sought. (Why?)

29 is a factor of 348. Therefore, it is a factor of 377 and of 1479. (Why?)

Therefore, 29 is the g. c. d. sought.

58. Least Common Multiple. In many cases the l. c. m. of two or more numbers may readily be found by factoring, as in the following example.

Ex. Find the l. c. m. of 414, 408, 3330.

$$414 = 2 \times 3 \times 3 \times 23,$$
$$408 = 2 \times 2 \times 2 \times 3 \times 17,$$
$$3330 = 2 \times 3 \times 3 \times 5 \times 37.$$

The l. c. m. must contain all of the prime factors of 414, 408, 3330, and each factor must occur as often in the l. c. m. as in any one of the numbers. Thus, 3 must occur twice in the l. c. m., 2 must occur three times, and 23, 17, 5, 37 must each occur once.

Therefore, the l. c. m. $= 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 23 \times 17 \times 37 = 5208120$.

59. When the numbers cannot readily be factored, the g. c. d. may be used in finding the l. c. m.

Since the g. c. d. of two numbers contains all of the factors that are common to the numbers, if the numbers are divided by the g. c. d., the quotients will contain all the factors that are not common. The l. c. m. is therefore the product of the quotients and the g. c. d. of the numbers.

Ex. Find the l. c. m. of 14482 and 32721.

The g. c. d. of 14482 and 32721 is 13.

$14482 \div 13 = 1114$. $\therefore$ the l. c. m. of the two numbers is

$$1114 \times 32721 = 36451194.$$

EXERCISE 13

1. Find the l. c. m. and g. c. d. of 384, 2112, 2496.
2. Find the l. c. m. of 3, 5, 9, 12, 14, 16, 96, 128.
3. Find the g. c. d. and l. c. m. of 1836, 1482, 1938, 8398, 11704, 101080, 138945.
4. Prove that the product of the g. c. d. and l. c. m. of two numbers is equal to the product of the numbers.
5. What is the length of the longest tape measure that can be used to measure exactly two distances of 2916 ft. and 3582 ft. respectively?
6. Find the number of miles in the radius of the earth, having given that it is the least number that is divisible by 2, 3, 4, 5, 6, 8, 9, 10, 11, 12.

CASTING OUT NINES

60. The check on arithmetical operations by casting out the nines was used by the Arabs. It is a very useful check, but fails to detect such errors as the addition of 9, the interchange of digits, and all errors not affecting the sum of the digits. (Why?)

The remainder arising from dividing any number by 9 is the same as that arising from dividing the sum of its digits by 9.

Thus, the remainder arising by dividing 75234 by 9 is 3, the same as arises by dividing $7 + 5 + 2 + 3 + 4$ by 9.

The student should adapt the proof of Principle 5, p. 34, to this statement.

61. The most convenient method is to add the digits, dropping or "casting out" the 9 as often as the sum amounts to that number.

Thus, to determine the remainder arising from dividing 645738 by 9, say 10 (reject 9), 1, 6, 13 (reject 9), 4, 7, 15 (reject 9), 6. Therefore, 6 is the remainder. After a little practice the student will easily group the 9's. In the above, 6 and 3, 4 and 5, could be dropped, and the excess in 7 and 8 is seen to be 6 at once.

62. Check on Addition by casting out the 9's.

Ex. Add 56342, 64723, 57849, 23454 and check the work by casting out the 9's.

Since each number is a multiple of 9 plus some remainder, the numbers can be written as indicated in the annexed solution.

$$\begin{array}{rl}
56342 = 9 \times 6260 + & 2 \text{ rem.} \\
64723 = 9 \times 7191 + & 4 \text{ rem.} \\
57849 = 9 \times 6427 + & 6 \text{ rem.} \\
23454 = 9 \times 2606 + & 0 \text{ rem.} \\
\hline
202368 = 9 \times 22484 + & 12 \text{ rem.}
\end{array}$$

But $12 = 9 + 3$.

$$\therefore\ 202368 = 9 \times 22484 + 9 + 3$$
$$= 9 \times 22485 + 3.$$

Thus, the excess of 9's is 3 and the excess in the sum of the excesses, 2, 4, 6, and 0, is 3, therefore the work is probably correct.

63. The proof may be made general by writing the numbers in the form $9x + r$. This can be done since all numbers are multiples of 9 plus a remainder. Hence, by expressing the numbers in this form and adding we have for the sum a multiple of 9 plus the sum of the remainders. Therefore, *the excess of the 9's in the sum is equal to the excess in the sum of the excesses.*

$$\begin{array}{l}
9x + r \\
9x' + r' \\
9x'' + r'' \\
\text{——} \\
\text{——} \\
\hline
9(x + x' + x'' + \cdots) \\
+ r + r' + r'' + \cdots)
\end{array}$$

64. Check on Multiplication by casting out the 9's.

Since any two numbers may be written in the form $9x + r$ and $9x' + r'$, multiplying $9x + r$ by $9x' + r'$, we have $81xx' + 9(x'r + xr') + rr'$. From this it is evident that the excess of 9's in the product arises from the excess in rr'. Therefore, *the excess of 9's in any product is equal to the excess in the product found by multiplying the excesses of the factors together.*

$$\begin{array}{r}
9x + r \\
9x' + r' \\
\hline
9xr' + rr' \\
81xx' + 9x'r \\
\hline
81xx' + 9(x'r + xr') + rr'
\end{array}$$

Ex. Multiply 3764 by 456 and check by casting out the 9's.

$$3764 \times 456 = 1716384.$$

The excess of 9's in 3764 is 2; the excess in 456 is 6; the excess in the product of the excesses is 3 ($2 \times 6 = 12$; $12 - 9 = 3$); the excess in 1716384, the product of the numbers, is 3. Therefore, the work is probably correct.

65. Check on Division by casting out the 9's.

Division being the inverse of multiplication, the dividend is equal to the product of the divisor and quotient plus the remainder. Therefore, *the excess of 9's in the dividend is equal to the excess of 9's in the remainder plus the excess in the product found by multiplying the excess of 9's in the divisor by the excess of 9's in the quotient.*

Ex. Divide 74563 by 428 and check by casting out the 9's.

$$74563 \div 428 = 174 + \tfrac{91}{428},$$

or

$$74563 = 174 \times 428 + 91$$

The excess of 9's in 74563 is 7; in 174, 3; in 428, 5; in 91, 1. Since 7, the excess of 9's in 74563 = the excess in 3 × 5 + 1, or 16, which is the product of the excesses in 174 and 428 plus the excess in 91, the work is probably correct.

EXERCISE 14

1. State and prove the check on subtraction by casting out the 9's.

2. Determine without adding whether 89770 is the sum of 37634 and 52146.

3. Add 74632, 41236, 897321 and 124762, and check by casting out the 9's.

4. Multiply 76428 by 5937, and check by casting out the 9's.

5. Determine without multiplying whether 2718895 is the product of 3785 and 721.

6. Show by casting out 9's that 18149 divided by $56 = 324\tfrac{5}{56}$.

7. Show that results may also be checked by casting out 3's; by casting out 11's.

8. Is 734657 divisible by 9? by 3? by 11?

9. Perform the following operations and check: 91728 × 762; 849631 ÷ 2463; 17 × 3.1416; 78.54 ÷ 3.1416.

10. Does the proof for casting out the 9's hold as well for 4, 6, 8, etc.? May we check by casting out the 8's? Explain.

MISCELLANEOUS EXERCISE 15

1. What is the principle by which the ten symbols, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, are used to represent any number?

2. Why is the value of a number unaltered by annexing zeros to the right of a decimal?

3. How is the value of each of the digits in the number 326 affected by annexing a number, as 4, to the right of it? to the left of it?

4. How is the value of each of the digits of 7642 affected if 5 is inserted between 6 and 4?

5. Write 4 numbers of 4 places each that are divisible by (*a*) 4, (*b*) 2 and 5, (*c*) 6, (*d*) 8, (*e*) 9, (*f*) 11, (*g*) 16, (*h*) 12, (*i*) 15, (*j*) 18, (*k*) 3, (*l*) 50, (*m*) 125, (*n*), both 6 and 9, (*o*) both 8 and 3, (*p*) both 30 and 20.

6. Determine the prime factors of the following numbers: (*a*) 3426, (*b*) 8912, (*c*) 6600, (*d*) 6534, (*e*) 136125, (*f*) 330330, (*g*) 570240.

7. Mr. Long's cash balance in the bank on Feb. 20 is $765.75. He deposits, Feb. 21, $150; Feb. 25, $350.25; Feb. 26, $97.50; and withdraws, Feb. 23, $200; Feb. 24, $123.40 and $112.50; Feb. 28, $321.75. What is his balance March 1?

8. Form a table of multiples of the multiplier and multiply 7642, 93856, 24245, 6420246, each by 463.

9. Form a table of multiples of the divisor and use it in dividing 86420, 97531, 876123, 64208, each by 765.

10. Use a short method to multiply 8426 by $16\frac{2}{3}$; by $33\frac{1}{3}$; by 945; by 432.

11. Find, without dividing, the remainder when 374265 is divided by 3; by 9; by 11.

12. Evaluate $45 + 32 \times 25 - 800 \div 125 + 180 \times 33\frac{1}{3}$.

13. Determine by casting out the 9's whether the following are correct: (*a*) $786 \times 648 = 509328$; (*b*) $24486 \div 192 = 127 + 102$ rem.; (*c*) $415372 \div 267 = 1555 + 187$ rem.; (*d*) $16734 \times 3081 = 52557454$.

14. Perform each of the operations indicated in Ex. 13.

15. Subtract from 784236 the sum of 7834, 5286, 23462 and 345679.

16. What are the arithmetical complements of 12000, 1728, 3.429, 86, 0.1, 125?

17. Light travels at the rate of 186000 mi. per second. Find the distance of the sun from the earth if it takes a ray of light from the sun 8 min. 2 sec. to reach the earth.

18. A cannon is 2 mi. distant from an observer. How long after it is fired does it take the sound to reach the observer if sound travels 1090 ft. per second?

19. Replace the zeros in the number 760530091 by digits so that the number will be divisible by both 9 and 11.

20. Show that every even number may be written in the form $2n$ and every odd number in the form $2n + 1$ where n represents any integer.

21. Show that the product of two consecutive numbers must be even and the sum odd.

22. Show that all numbers under and including 15 are factors of 360360.

23. Find, without dividing, the remainder after 364257 has been divided by 3; by 9; by 11.

24. Evaluate $10 + 144 \times 25 - 2130 \div 15 + 5 \times 3$.

25. Write 4 numbers of 5 places each that are divisible by both 9 and 11.

26. Write 4 numbers of 6 places each that are divisible by both 3 and 6.

27. Write 4 numbers of 4 places each that are divisible by 4, 5, 6.

28. Evaluate $47 \times 68 + 68 \times 53$.

29. Evaluate $346 \times 396.84 - 146 \times 396.84$.

30. Evaluate $27 \times 3.1416 - 41 \times 3.1416 + 49 \times 3.1416 + 65 \times 3.1416$.

31. If lemons are 20 ct. a dozen and oranges are 25 ct., how many oranges are worth as much as $12\frac{1}{2}$ doz. lemons?

32. A farmer received 6 lb. of coffee in exchange for 9 doz. eggs at $12\frac{1}{2}$ ct. a dozen. How much was the coffee worth per pound?

33. Two piles of the same kind of shot weigh respectively 1081 lb. and 598 lb. What is the greatest possible weight of each shot?