FRACTIONS

66. Historically the **fraction** is very old. A manuscript on arithmetic, entitled "Directions for obtaining a Knowledge of All Dark Things," written by Ahmes, an Egyptian priest, about 1700 B.C., begins with fractions. In this manuscript all fractions are reduced to fractions with unity as the numerator. Thus, the first exercise is $\frac{2}{5} = \frac{1}{6} + \frac{1}{18}$.

67. While the Egyptians reduced all fractions to those with constant numerators, the Babylonians used them with a constant denominator of 60. Only the numerator was written, with a special mark to denote the denominator. This method of writing fractions lacked only the symbol for zero and the substitution of the base 10 for 60 to become the modern decimal fraction. Sexagesimal fractions are still used in the measurement of angles and time.

68. The Romans used duodecimal fractions exclusively. They had special names and symbols for $\frac{1}{12}$, $\frac{2}{12}$, $\cdots$, $\frac{11}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, etc. To the Romans, fractions were concrete things. They never advanced beyond expressing them in terms of **uncia** ($\frac{1}{12}$), **silicus** ($\frac{1}{4}$ uncia), **scrupulum** ($\frac{1}{24}$ uncia), etc., all subdivisions of the **as**, a copper coin weighing one pound.

69. The sexagesimal and duodecimal fractions prepared the way for the decimal fraction, which appeared in the latter part of the sixteenth century. In 1585 Simon Stevin of Bruges published a work in which he used the notations $7\ 4'\ 6''\ 5'''\ 9''''$, or 7⓪ 4① 6② 5③ 9④ for 7.4659. During the early stage of its development the decimal fraction was written in various other forms, among which are found the following: $7\ \overset{\text{I}}{4}\ \overset{\text{II}}{6}\ \overset{\text{III}}{5}\ \overset{\text{IV}}{9}$, $7\ \overset{1}{4}\ \overset{2}{6}\ \overset{3}{5}\ \overset{4}{9}$, 7|4659, $7\lfloor\underline{4659}$, 7,4659. The decimal point was first used, in 1612, by Pitiscus in his trigonometrical tables; but the decimal fraction was not generally used before the beginning of the eighteenth century.

70. The **primary conception** of a fraction is *one or several of the equal parts of a unit.* Thus, the fraction $\frac{4}{5}$ indicates that 4 of the 5 equal parts of a unit are taken.

71. The term *naming* the number of parts into which the unit is divided is called the **denominator.** The term *numbering* the parts is called the **numerator.**

72. A **proper fraction** is less than unity; an **improper fraction** is equal to or greater than unity.

73. A number consisting of an integer and a fraction is called a **mixed number.**

74. Our conception of a fraction must, however, be enlarged as we proceed, and be made to include such expressions as $\frac{2.5}{3.25}$, $\frac{3}{-7}$, 3.14159 ·, $\frac{\frac{3}{4}}{\frac{5}{7}}$, etc. The more **general conception** of a fraction is that it is *an indicated operation in division* where the numerator represents the dividend and the denominator the divisor.

75. Decimal fractions are included in the above definition, as 0.5 means $\frac{5}{10}$ or $\frac{1}{2}$. Using the decimal point (0.5) is simply another way of writing $\frac{5}{10}$.

76. General Principles. *Multiplying the numerator or dividing the denominator of a fraction by a number multiplies the fraction by that number.*

Let $\frac{5}{7}$ be any fraction where 7 represents the number of parts into which unity is divided, and 5 the number of these parts taken.

(1) $\frac{3 \times 5}{7} = 3 \times \frac{5}{7}$, since there are three times as many of the 7 parts of unity as before.

(2) $\frac{5}{7 \div 3} = 3 \times \frac{5}{7}$, since dividing the denominator by 3 divides by 3 the number of equal parts into which unity is divided and therefore the fraction is 3 times as large as before.

77. *Dividing the numerator or multiplying the denominator of a fraction by a number divides the fraction by that number.*

(1) $\frac{7 \div 4}{9} = \frac{7}{9} \div 4$, since dividing the numerator by 4 divides by 4 the number of parts taken without changing the value of the parts.

(2) $\frac{7}{9 \times 4} = \frac{7}{9} \div 4$, since multiplying the denominator by 4 multiplies by 4 the number of parts into which unity is divided and therefore the fraction is $\frac{1}{4}$ as large as before.

78. *Multiplying or dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction.*

(1) $\frac{5 \times 2}{5 \times 3} = \frac{2}{3}$. Multiplying both numerator and denominator by 5 both multiplies and divides the value of the fraction by 5. The value of the fraction therefore remains unchanged.

(2) $\frac{2 \div 5}{3 \div 5} = \frac{2}{3}$. Dividing both numerator and denominator by 5 both divides and multiplies the value of the fraction by 5. The value of the fraction therefore remains unchanged.

79. *A mixed number may be reduced to an improper fraction and an improper fraction may be reduced to a mixed number or an integer.*

Thus, $5\frac{3}{4} = \frac{5 \times 4 + 3}{4}$. Since 5×4 = the number of 4ths in 5 and $5 \times 4 + 3$ = the number of 4ths in $5\frac{3}{4}$, $\therefore 5\frac{3}{4} = \frac{23}{4}$.

Reversing the process,

$$\frac{23}{4} = 23 \div 4 = 5 + \frac{3}{4} = 5\frac{3}{4}.$$

80. *When the numerator and denominator of a fraction are prime to each other, the fraction is said to be in its lowest terms.*

Ex. Express $\frac{42}{70}$ in its lowest terms.

$$\frac{42}{70} = \frac{3 \times 14}{5 \times 14} = \frac{3}{5}.$$

81. *Two or more fractions may be reduced to equivalent fractions having a common denominator.*

Ex. 1. Reduce $\frac{3}{4}$, $\frac{5}{9}$, $\frac{1}{12}$, to equivalent fractions having a common denominator.

The l. c. m. of 4, 9, 12, is **36**.

$$\frac{3}{4} = \frac{3 \times 9}{4 \times 9} = \frac{27}{36}.$$

$$\frac{5}{9} = \frac{5 \times 4}{9 \times 4} = \frac{20}{36}.$$

$$\frac{1}{12} = \frac{1 \times 3}{12 \times 3} = \frac{3}{36}.$$

$\therefore \frac{27}{36}$, $\frac{20}{36}$, $\frac{3}{36}$ are fractions having a common denominator, equivalent to $\frac{3}{4}$, $\frac{5}{9}$, $\frac{1}{12}$. Since 36 is the l. c. m. of 4, 9, 12, it is called the **least common denominator.**

82. Sometimes, instead of finding the l.c.m., it is more convenient to take as the common denominator the product of all the denominators and multiply each numerator by the product of all the denominators except its own.

Ex. 2. Reduce $\frac{3}{4}$, $\frac{1}{6}$ and $\frac{2}{3}$ to fractions having a common denominator.

$$\frac{3}{4} = \frac{3 \times 6 \times 3}{4 \times 6 \times 3} = \frac{54}{72}.$$

$$\frac{1}{6} = \frac{1 \times 4 \times 3}{4 \times 6 \times 3} = \frac{12}{72}.$$

$$\frac{2}{3} = \frac{2 \times 4 \times 6}{4 \times 6 \times 3} = \frac{48}{72}.$$

Since the common denominator is $4 \times 6 \times 3$, 4 is contained in it 6×3 times and the first numerator will be $3 \times 6 \times 3$, 6 is contained in the common denominator 4×3 times and the second numerator will be $1 \times 4 \times 3$, 3 is contained in the common denominator 4×6 times and the third numerator will be $2 \times 4 \times 6$.

83. Addition and Subtraction of Fractions. Since only the same kinds of units, or the same parts of units, can be added to or subtracted from one another, it is necessary to reduce fractions to a common denominator before performing the operations of addition or subtraction.

Ex. 1. Add $\frac{5}{12}$, $\frac{7}{36}$ and $\frac{1}{84}$.

The l. c. m. of 12, 36 and 84 is 252.

$$\frac{5}{12}=\frac{5\times 21}{12\times 21}=\frac{105}{252},\quad \frac{7}{36}=\frac{7\times 7}{36\times 7}=\frac{49}{252},\quad \frac{1}{84}=\frac{1\times 3}{84\times 3}=\frac{3}{252}.$$

$$\frac{5}{12}+\frac{7}{36}+\frac{1}{84}=\frac{105}{252}+\frac{49}{252}+\frac{3}{252}=\frac{157}{252}.$$

Ex. 2. Add $2\frac{3}{4}$, $1\frac{5}{9}$ and $3\frac{13}{48}$.

$$2\tfrac{3}{4}+1\tfrac{5}{9}+3\tfrac{13}{48}=2+1+3+\tfrac{3}{4}+\tfrac{5}{9}+\tfrac{13}{48}=6+\tfrac{108}{144}+\tfrac{80}{144}+\tfrac{39}{144}=7\tfrac{83}{144}.$$

After a little practice the student should be able to abbreviate the work very much. *Ex.* 1 might be worked briefly, thus:

$$\frac{5}{12}+\frac{7}{36}+\frac{1}{84}=\frac{105+49+3}{252}=\frac{157}{252}.$$

Ex. 2, thus:

$$2\tfrac{3}{4}+1\tfrac{5}{9}+3\tfrac{13}{48}=6+\frac{108+80+39}{144}=7\tfrac{83}{144}.$$

Ex. 3. From $22\frac{5}{42}$ subtract $18\frac{13}{42}$.

$$22\tfrac{5}{42}=21\tfrac{47}{42}.$$

$$21\tfrac{47}{42}-18\tfrac{13}{42}=3\tfrac{34}{42}=3\tfrac{17}{21}.$$

84. Multiplication of Fractions. *The product of two numbers may be found by performing the same operation on one of them as is performed on unity to produce the other.*

Thus, in $3\times 4=12$, unity is taken three times to produce the multiplier 3, hence 4 is taken three times to produce the product 12. Again, in $\frac{2}{3}\times\frac{5}{7}=\frac{10}{21}$, unity is divided into 3 parts and 2 of them are

taken to produce the multiplier $\frac{2}{3}$, hence, $\frac{5}{7}$ is divided into 3 parts, each of which is $\frac{5}{3\times 7}$ (Why?), and 2 of them are taken to produce the product $\frac{2\times 5}{3\times 7}=\frac{10}{21}$.

When the multiplier is a common fraction, the sign ($\times$) should be read "of." Thus, $\frac{2}{3}\times \$5$ means $\frac{2}{3}$ of $5.

Ex. 1. Multiply $\frac{23}{42}$ by $\frac{21}{69}$.

$$\frac{23}{42}\times\frac{21}{69}=\frac{\overset{1}{\cancel{23}}\times\overset{1}{\cancel{21}}}{\underset{2}{\cancel{42}}\times\underset{3}{\cancel{69}}}=\frac{1}{6}.$$

85. *The student should use cancellation whenever possible. He should never multiply or divide until all possible factors have been removed by cancellation.*

NOTE. Although a knowledge of the principles of multiplication and division of decimals has been assumed in examples given before, it is well to review these principles at this point to make sure that they are thoroughly understood.

Ex. 2. Multiply 0.234 by 0.16.

Solution.

$$0.234\times 0.16=\frac{234}{1000}\times\frac{16}{100}=\frac{234\times 16}{1000\times 100}=\frac{3744}{100000}=0.03744.$$

```
 0.234
 0.16
 -----
  234
 1404
-------
0.03744
```

The number of decimal places in the product is the same as the number of zeros in the denominator of the product, that is, it equals the number of decimal places in the multiplicand plus the number in the multiplier. The decimal point in (0.03744) simply provides a convenient way of writing $\frac{3744}{100000}$. It is better, however, to determine the position of the decimal point before beginning the multiplication. This can be done by considering only the last figure at the right of the multiplier and multiplicand. Thus, we see that $0.004\times 0.06=0.00024$. Hence, the order of the product will be hundred thousandths

86. The following simple truths, or **axioms**, are frequently used in arithmetic.

(1) *Numbers that are equal to the same number are equal to each other.*

Thus, if $x = 5$ and $y = 5$, then $x = y$.

(2) *If equals are added to equals, the sums are equal.*

Thus, if $x = 5$, $x + 3 = 5 + 3$.

(3) *If equals are subtracted from equals, the remainders are equal.*

Thus, if $x = 4$, then $x - 2 = 4 - 2$.

(4) *If equals are multiplied by equals, the products are equal.*

Thus, if $\frac{x}{2} = 3$, then $x = 6$.

(5) *If equals are divided by equals, the quotients are equal.*

Thus, if $3x = 6$, then $x = 2$.

87. Division of Fractions. Division may be regarded as the inverse of multiplication. The problem is, therefore, to find one of two factors when the product and the other factor are given.

Thus, $3 \times 4 = 12$, $\therefore 12 \div 3 = 4$, and $12 \div 4 = 3$. Axiom 5.

Again, $\frac{5}{6} \times \frac{7}{8} = \frac{35}{48}$, $\therefore \frac{35}{48} \div \frac{5}{6} = \frac{7}{8}$, and $\frac{35}{48} \div \frac{7}{8} = \frac{5}{6}$.

To divide one fraction by another.

Solution. Let $\frac{35}{48} \div \frac{5}{6} = q$ (a quotient).

Then $\frac{35}{48} = \frac{5}{6} \times q$ (multiplying both members of the equation by $\frac{5}{6}$), and $\frac{35}{48} \times \frac{6}{5} = q$ (multiplying both members of the equation by $\frac{6}{5}$).

$\therefore$ *the quotient is obtained by multiplying the dividend by the reciprocal of the divisor.*

If a number of factors are connected by the signs $\times$ and $\div$, the operations are to be performed from left to right.

Thus, $8 \times 3 \div 4 \times 2 \div 6 = 2$.

The operation indicated by the word "of" following a fraction is to be performed before the operations indicated by $\times$ and $\div$.

Thus, $\frac{1}{2} + 4 \times \frac{1}{3}$ of $6 \div \frac{2}{5} - \frac{4}{9} \div \frac{3}{4}$ of $\frac{2}{3} = 19\frac{11}{18}$.

Observe that the operations $\frac{1}{3}$ of 6 and $\frac{3}{4}$ of $\frac{2}{3}$ are performed first, followed by multiplication and division from left to right, and finally by addition and subtraction.

88. A fraction of the form $\frac{\frac{3}{4}}{\frac{5}{6}}$ is called a **complex fraction** and may be considered as equivalent to $\frac{3}{4} \div \frac{5}{6}$ and treated as a problem in division. In general, however, a complex fraction may be more readily simplified by multiplying both terms by the l. c. m. of the denominators of the two fractions in the numerator and denominator.

Ex. 1. $\frac{\frac{5}{9}}{\frac{6}{7}} = \frac{63 \times \frac{5}{9}}{63 \times \frac{6}{7}} = \frac{35}{54}$.

Ex. 2. Divide 38.272 by 7.36.

```
        5.2
7.36)38.272
     36.80
     -----
      1.472
      1.472
      -----
```

Solution. $38.272 \div 7.36 = \frac{38272}{1000} \div \frac{736}{100}$
$= \frac{38272}{1000} \times \frac{100}{736} = \frac{38272}{736} \times \frac{100}{1000} = \frac{52}{1} \times \frac{1}{10} = 5.2$

The number of decimal places in the quotient will equal the number of zeros in the denominator of the last product. This will be the same as the number of zeros in the denominator of the dividend minus the number of zeros in the denominator of the divisor, or, what is the same thing, the number of decimal places in the dividend minus the number of decimal places in the divisor.

If the number of decimal places in the dividend is less than the number of decimal places in the divisor, we may annex zeros to the dividend till the number of decimal places is the same in both dividend and divisor. The quotient up to this point in the division will be an integer, and, in case it is necessary to carry the division farther, more zeros may be annexed to the dividend. The remaining figures of the quotient will be decimals.

Ex. 3. Divide 52.36 by 3.764.

```
              13.9
3.764)52.360|0
      37.64
      14.720
      11.292
       3.4280
       3.3876
          404
```

EXERCISE 16

1. Change $\frac{2}{3}$ to 9ths; $\frac{8}{21}$ to 168ths.

2. Reduce to lowest terms each of the following fractions: $\frac{9}{27}$, $\frac{111}{370}$, $\frac{72}{999}$, $\frac{1728}{8640}$.

3. Explain the reduction of $7\frac{3}{5}$ to an improper fraction.

4. Explain the reduction of $\frac{1895}{125}$ to a mixed number.

5. Simplify $\frac{4\frac{2}{3}}{2\frac{5}{6}}$, $\frac{\frac{5}{9}}{\frac{2}{3}}$, $\frac{0.5}{\frac{1}{2}}$, $\frac{0.75}{\frac{1}{3}}$.

6. Add $\frac{3663}{10989}$, $\frac{1221}{13431}$ and $\frac{5}{1221}$.

7. From $75\frac{5}{42}$ take $12\frac{3}{14}$.

8. Multiply $2\frac{1}{2} \div \frac{3}{4}$ by $\frac{1}{2}$ of $\frac{3}{4} \times \frac{5}{6}$.

9. Find the value of $\frac{5}{9}$ of $\frac{3}{13} \div \frac{2}{3}$ of $\frac{6}{11}$ of $\frac{22}{23}$.

10. Find the value of $\frac{1}{5} \div \frac{3}{4}$ of $\frac{2}{3} + 16 \times \frac{1}{2}$ of $4 \div \frac{2}{3}$ of $\frac{3}{4}$.

11. Find the value of $5\frac{3}{4} - 0.9$ of $2.7 + 25\frac{1}{6} \times 0.02 - \frac{2.7}{15}$.

12. Find the value of $\frac{3\frac{2}{17} \text{ of } 8\frac{1}{6}}{4\frac{2}{13} \text{ of } 2\frac{1}{16}}$.

13. By what must $\frac{3}{4}$ be multiplied to produce $3\frac{3}{4}$?

14. What number divided by $\frac{18}{5}$ of $\frac{5}{4}$ will give $4\frac{1}{8}$ as a quotient?

15. Simplify: $\frac{\frac{1}{2} + \frac{3}{13} \text{ of } \frac{1}{6} - \frac{3}{4} \times \frac{4}{7}}{1 - \frac{1}{5} \text{ of } \frac{10}{13} + \frac{3}{4} \times \frac{4}{7}}$.

16. What fraction added to the sum of $\frac{1}{2}$, $\frac{1}{4}$, and 5.25 will make 6.42?

17. Simplify: $\frac{2 - \frac{1}{2} \text{ of } (\frac{3}{4} - \frac{1}{20})}{5 + 0.5 \text{ of } (1 - 0.9)}$

18. How is the value of a proper fraction affected by adding the same number to both numerator and denominator? How is the value of an improper fraction affected?

19. A merchant bought a stock of goods for $2475.50 and sold $\frac{1}{2}$ of it at an advance of $\frac{1}{3}$ of the cost, $\frac{1}{4}$ of it at an advance of $\frac{1}{4}$ of the cost, and the remainder at a loss of $\frac{1}{10}$ of the cost. Did he gain or lose and how much?

20. A ship is worth $90,000 and a person who owns $\frac{5}{12}$ of it sells $\frac{1}{4}$ of his share. What is the value of the part he has left?

21. If 1 is added to both numerator and denominator of $\frac{3}{2}$, by how much is its value diminished?

22. If 1 is added to both numerator and denominator of $\frac{2}{3}$, by how much is its value increased?

89. Cancellation. Much time may be saved in solving problems by writing down a complete statement of the condition given and then canceling common factors if any are present. The student should do this at every stage in the solution of a problem, always factoring and canceling whenever possible, and *never multiplying or dividing till all possible factors have been removed by cancellation.*

Ex. 1. If $\frac{6}{25}$ of a business block is worth $6252.66, what is the value of $\frac{11}{15}$ of it?

Solution. $\frac{6}{25}$ is worth $6252.66.
$\frac{1}{25}$ is worth $\frac{1}{6}$ of $6252.66.
∴ the whole is worth $\frac{25}{6}$ of $6252.66.
∴ $\frac{11}{15}$ is worth $\frac{11}{15}$ of $\frac{25}{6}$ of $6252.66.

$$= \frac{11 \times \overset{5}{\cancel{25}} \times \$\overset{\overset{347.37}{\cancel{1042.11}}}{\cancel{6252.66}}}{\underset{3}{\cancel{15}} \times \cancel{6}} = \$19105.35.$$

Ex. 2. How much must be paid for 59,400 lb. of coal at $4 per ton of 2000 lb.?

Statement. $\frac{59400 \times \$4}{2000} = \$118.80.$

EXERCISE 17

Find the value of:

1. $\frac{27 \times 72 \times 80}{36 \times 45 \times 30}$.

2. $\frac{144 \times 1728 \times 999}{96 \times 270 \times 33}$.

3. $\frac{1320 \times 432 \times 660}{4400 \times 297 \times 288}$.

4. $\frac{1760 \times 9 \times 125}{55 \times 360}$.

5. How much must be paid for shipping 1200 bbl. of apples at $35 per hundred barrels?

6. How many bushels of potatoes at 50 ct. a bushel will pay for 500 lb. of sugar at 4 ct. a pound?

7. A merchant bought 12 carloads of apples of 212 bbl. each, 3 bu. in each barrel at 45 ct. per bushel. He paid for them in cloth at 25 ct. per yard. How many bales of 500 yd. did he deliver?

8. How many bushels of potatoes at 55 ct. per bushel must be given in exchange for 22 sacks of corn, each containing 2 bu., at 60 ct. a bushel?

APPROXIMATE RESULTS

90. In scientific investigations exact results are rarely possible, since the numbers used are obtained by observation or by experiments in which, however fine the instrument, the results are only approximate, and there is a degree of accuracy beyond which it is impossible to go.

91. On the other hand, the approximate value of such incommensurable quantities as $\sqrt{2} = 1.414+$, $\pi = 3.14159+$ can be obtained to any required degree of accuracy. The value of π has been computed to 707 decimal places, but no such accuracy is necessary or desirable. The student should always bear in mind that *it is a waste of time to carry out results to a greater degree of accuracy than the data on which they are founded.*

92. It is frequently necessary to determine the value of a decimal fraction correct to a definite number of decimal places. The value of $\frac{23}{24} = 0.95833+$ correct to four decimal places is 0.9583. 0.958 and 0.96 are the values correct to three and two places. The real value of this fraction correct to four places lies between 0.9583 and 0.9584. 0.9583 is 0.00003+ less than the true value, while 0.9584 is 0.00006+ greater. Therefore 0.9583 is nearer the correct value, and is said to be the value correct to four decimal places. Similarly, 0.96 is the value correct to two places.

93. If 5 is the first rejected digit, the result will apparently be equally correct whether the last digit is increased by unity or left unchanged. The value of 0.4235 correct to three places may be either 0.423 or 0.424. However, as the 5 itself is usually an approximation, it can readily be determined which course to pursue by noticing whether the 5 is in excess or defect of the correct value. 0.23649 correct to four places is 0.2365, but correct to three places 0.236 is nearer the true value than 0.237.

94. Addition. *Ex.* Add 0.234673, 0.322135, 0.114342, 0.563217, each fraction being correct to six decimal places.

Solution. It is clear that the last digit in this sum is not correct, since each of the four numbers added may be either greater or less than the correct value by a fraction less than 0.0000005. Hence, the total error in the sum cannot be greater than 0.000002. The required sum must therefore lie between 1.234369 and 1.234365, and in either case the result correct to five places is 1.23437.

```
 0.234673
 0.322135
 0.114342
 0.563217
 --------
 1.234367
```

95. The next to the last digit in the sum may be incorrect, as shown in the following example:

Ex. Add 0.131242, 0.276171, 0.113225, 0.342247, each fraction being correct to six decimal places.

Solution. In this case the sum lies between 0.862887 and 0.862883. Hence, it is uncertain whether 0.86289 or 0.86288 is the value correct to five places. 0.8629 is, however, the value correct to four places.

```
 0.131242
 0.276171
 0.113225
 0.342247
 --------
 0.862885
```

96. The third digit from the last may be left in doubt, as in the following example:

Ex. Add 5.866314, 3.715918, 0.568286, 4.342233, each fraction being correct to six places.

Solution. Here the true value of the sum lies between 14.492753 and 14.492749. Hence, it is uncertain whether the value correct to four places is 14.4928 or 14.4927. The value correct to three places is 14.493.

```
  5.866314
  3.715918
  0.568286
  4.342233
 ---------
 14.492751
```


97. Subtraction. *Ex.* Subtract 0.238647 from 0.329528, each fraction being correct to six decimal places.

Solution. Since each fraction cannot differ from the true value by a fraction as large as 0.0000005, the difference cannot be greater or less than the correct value by a fraction as large as 0.000001. Hence, the difference must lie between 0.090882 and 0.090880, and the value correct to five places is 0.09088.

```
 0.329528
 0.238647
 --------
 0.090881
```

98. Cases will arise where the second and third digits from the last are in doubt, as in addition. The student should determine how far the result may be relied upon in the following examples:

(1) Subtract 0.371492 from 0.764237.
(2) Subtract 0.11132 from 0.23597.
(3) Subtract 15.93133 from 43.71288.

99. Multiplication. From the examples in addition given above the student will notice that it will be sufficient in most cases to carry out the partial products correct to two places more than the required result.

Ex. Find the square of 3.14159 correct to four decimal places.

Solution. The multiplication in full and the contracted form are as follows:

```
 3.14159            3.14159
 3.14159            3.14159
 -------            -------
 9.42477            9.42477
  .314159            .314159
  .1256636           .125664
  .00314159             3142
  .001570795            1571
  .0002827431            283
 ------------       -------
 9.8695877281       9.8696
```

After pointing off the first partial product we proceed as indicated in the above contracted form until the multiplication by 3 and 1 are completed. Multiplication by 4 would give a figure in the seventh place. Instead of writing down the figures we add the nearest 10 to the next column. Thus, 4 times 9, 36, add 4 to the next column since 3.6 = 4 approximately. 4 times 5, 20 and 4, 24. 4 times 1, 4 and 2, 6, etc.

In multiplying by the next 1 it is not necessary to take the 9 in the multiplicand into account. So, also, in multiplying by the 5, the 5 and 9 in the multiplicand may both be ignored. And so on until the multiplication is completed.

100. Division. *Ex.* Divide 9376245 by 3724 correct to the units' place.

Solution. The division in full and the contracted form are as follows:

2517	2517
3724)9376245	3724)9376245
7448	7448
19282	19282
18620	18620
6624	662
3724	372
29005	290
26068	260
2937	30

The first two digits in the quotient are 2 and 5 and the second remainder is 662. It is not necessary to bring down any more figures to have a result correct to units since tens divided by thousands will give hundredths. The divisor may also be contracted at this stage of the work. Thus, cutting off the 4, 372 is contained once in the second remainder, 662. Cutting off the 2, 37 is contained 7 times in the next remainder, 290. This gives the units' figure of the quotient.

It will be noticed that the next figure of the quotient is greater than 0.5, therefore the result correct to units is 2518.

The work may be further abridged by omitting the partial products and writing down the remainders only.

```
     2517
3724)9376245
     19282
       662
       290
        30
```


101. *Ex.* Divide 62.473 by 419.6789.*

Solution. First shift the decimal point four places in each so as to have an integral divisor, and then work as follows: The 1 and 4 are obtained without abbreviating and the 8, 8, 5, 9, 0 by cutting off 9, 8, 7, 6, 9 in succession from the divisor.

```
           0.1488590
4196789)624730.0
        20505110
         3717954
          360523
           24780
            3796
              19
```

Ex. Divide 0.0167 by 423.74.*

```
       0.00003941
42374)1.67000
        39878
         1741
           46
            4
```

*From Langley's "Treatise on Computation," p. 68.

EXERCISE 18

1. Divide 100 by 3.14159 correct to 0.01.

2. Find the quotient of 67459633 divided by 4327 correct to five significant figures.

3. Determine without dividing by what number less than 13, 339295680 is exactly divisible.

Determine by casting out the 9's whether the following are correct:

4. 959 × 959 = 919681. 5. 954 × 954 × 954 = 868250664.

6. 33920568 ÷ 729 = 42829. 7. 1019 × 1019 = 1036324.

8. 6234751 ÷ 43265 = 14.41+ a remainder 2645.

9. Find the sum of 23.45617, 937.34212, 42.31759, 532.23346, 141.423798 correct to two decimal places.

10. Subtract 987.642 from 993.624 correct to tenths.

11. Find the product of 32.4736 × 24.7955 correct to five significant figures.

12. Divide 47632 by $3\frac{1}{3}$. 13. Multiply 23793 by $12\frac{1}{2}$.