EXERCISE 25

1. The sum of two numbers is 31 and their difference is 11. What are the numbers?

2. The difference of two numbers is 60, and if both numbers are increased by 5 the greater becomes four times as large as the smaller. What are the numbers?

3. Find two numbers such that their difference is 95 and the smaller divided by the greater is $\frac{4}{9}$.

4. After spending $\frac{2}{3}$ of his money a man pays bills of \$25, \$40 and \$14, and finds that he has \$132 left. How much money had he at first?

5. Find a number such that its half, third and fourth parts shall exceed its fifth part by 106.

6. A father wishes to divide \$28,000 among his two sons and a daughter so that the elder son shall receive twice as much as the younger, and the younger son twice as much as the daughter. Find the share of each.

POWERS AND ROOTS

155. Archimedes (287–212 B.C.) in his measurements of the circle computed the approximate value of a number of square roots, but nothing is known of his method. A little later, Heron of Alexandria also used the approximation $\sqrt{a^2+b} = a + \frac{b}{2a}$, *e.g.* $\sqrt{85} = \sqrt{9^2+4} = 9 + \frac{4}{18}$.

156. The Hindus included powers and roots among the fundamental processes of arithmetic. As early as 476 A.D. they used the formulæ $(a+b)^2 = a^2 + 2ab + b^2$ and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ in the extraction of square and cube root, and they separated numbers into periods of two and three figures each.

157. The Arabs also extracted roots by the formula for $(a+b)^n$. They introduced a radical sign by placing the initial letter of the word *jird* (root) over the number.

158. The **power** of a number is the product that arises by multiplying the number by itself any number of times. The second power is called the **square** of the number and the number itself is called the **square root** of its second power. The third power is called the **cube** of the number, and the number itself is called the **cube root** of its third power. Thus, 4 is the square of 2, and 2 is the square root of 4. 125 is the cube of 5, and 5 is the cube root of 125.

159. Since the square root of a number is one of the two equal factors of a perfect second power, numbers that are not exact squares have no square roots. However, they are treated as having approximate square roots. These approximate square roots can be found to any required degree of accuracy. Thus, the square root of 91 correct to 0.1 is 9.5, and correct to 0.01 is 9.54.

160. According to the definition, only abstract numbers have square roots.

161. The square, the cube, fourth power, etc., of 2 are expressed by 2^2, 2^3, 2^4, etc. The small figure which denotes the power is called the **exponent**.

162. The square root is denoted by the symbol $\sqrt{\ }$, the cube root by $\sqrt[3]{\ }$, the fourth root by $\sqrt[4]{\ }$, etc.

163. The following results are important:

$$1^2 = 1,$$
$$10^2 = 100,$$
$$100^2 = 10000,$$
$$1000^2 = 1000000.$$

The squares of all numbers between 1 and 10 lie between 1 and 100, the squares of all numbers between 10 and 100 lie between 100 and 10000, etc. Hence, the square of a number of one digit is a number of one or two digits, the square of a number of two digits is a number of three or four digits, etc. It will be noticed that the addition of a digit to a number adds two digits to the square.

164.

$$1^3 = 1,$$
$$10^3 = 1000,$$
$$100^3 = 1000000, \text{ etc.}$$

It will be noticed in this case that the addition of a digit to a number adds three digits to its cube.

165.

$$0.1^2 = 0.01,$$
$$0.01^2 = 0.0001,$$
$$0.001^2 = 0.000001, \text{ etc.}$$

Hence, the square of a decimal number contains twice as many digits as the number itself.

166.

$$0.1^3 = 0.001,$$
$$0.01^3 = 0.000001,$$
$$0.001^3 = 0.000000001, \text{ etc.}$$

Hence, the cube of a decimal number contains three times as many digits as the number itself.

167. Laws of exponents:

Since $2^2 = 2 \times 2$ and $2^3 = 2 \times 2 \times 2$,

then $2^2 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$.

This may be written in the form

$$2^2 \times 2^3 = 2^{2+3} = 2^5.$$

Or, in general, $a^m \times a^n = a^{m+n}$. **I.**

Also,

$$(2^3)^4 = (2 \times 2 \times 2)(2 \times 2 \times 2)(2 \times 2 \times 2)(2 \times 2 \times 2) = 2^{3+3+3+3} = 2^{12}.$$

This may be written in the form

$$(2^3)^4 = 2^{3 \times 4} = 2^{12}.$$

Or, in general, $(a^m)^n = a^{m \times n}$. **II.**

Also, $2^5 \div 2^3 = \dfrac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 \times 2 = 2^2$.

This may be written in the form

$$2^5 \div 2^3 = 2^{5-3} = 2^2.$$

Or, in general, $a^m \div a^n = a^{m-n}$. **III.**

Also, since $\frac{2^2}{2^2} = 2^{2-2} = 2^0$ (Principle III), and $\frac{2^2}{2^2} = 1$, $\therefore 2^0 = 1$.

Or, since $2^2 \times 2^0 = 2^{2+0} = 2^2$ (Principle I), and $2^2 \times 1 = 2^2$, $\therefore 2^0 = 1$.

Or, in general, $a^0 = 1$. **IV.**

Also, since $\frac{2}{2^2} = 2^{1-2} = 2^{-1}$ (Principle III), and $\frac{2}{2^2} = \frac{2}{4} = \frac{1}{2}$, $\therefore 2^{-1} = \frac{1}{2}$.

Or, since $2^2 \times 2^{-1} = 2$ (Principle I), and $2^2 \times \frac{1}{2} = 2$, $\therefore 2^{-1} = \frac{1}{2}$.

Or, in general, $a^{-n} = \frac{1}{a^n}$. **V.**

That is, any number with a negative exponent is equal to the reciprocal of the same number with a numerically equal positive exponent.

Also, $(2^{\frac{1}{2}})^2 = 2$, since $(2^{\frac{1}{2}})^2 = 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$.

$\therefore 2^{\frac{1}{2}} = \sqrt{2}$ (extracting the square root of both members of the equation $(2^{\frac{1}{2}})^2 = 2$).

And $(2^{\frac{2}{3}})^3 = 2^2$, or $2^{\frac{2}{3}} = \sqrt[3]{2^2}$.

Or, in general,

$$(a^{\frac{m}{n}})^n = a^m, \text{ or } a^{\frac{m}{n}} = \sqrt[n]{a^m}. \qquad \textbf{VI.}$$

Ex. Show that $\frac{3^3}{3^2} = 3^1 = 3$; $\frac{3^2}{3^2} = 3^0 = 1$; $\frac{3}{3^2} = 3^{-1} = \frac{1}{3}$; $10^{-1} = 0.1$; $10^{-2} = 0.01$; $10^{-5} = 0.00001$.

Show that $8^{\frac{2}{3}} = 4$; $27^{\frac{2}{3}} = 9$; $2^{\frac{12}{4}} = 8$; $81^{\frac{3}{4}} = 27$; $32^{\frac{6}{5}} = 64$.

EXERCISE 26

1. How many figures are there in the square of a number of 3 figures? of 4 figures?

2. How many figures are there in 31^2? in 32^2?

3. How many figures are there in the cube of a number of 2 figures? of 3 figures?

4. How many figures are there in 30^3? in 32^3?

5. How many figures are there in the fourth power of a number of 3 figures? in the fifth power of a number of 2 figures?

6. How many figures are there in the cube of a number of 5 figures?

7. How many figures are there in the cube of a number of 4 figures?

8. How many figures are there in $\sqrt{5929}$? in $\sqrt{1038361}$?

9. How many figures are there in $\sqrt{0.04}$? in $\sqrt{0.36}$?

10. How many figures are there in $\sqrt{37.21}$? in $\sqrt{4.8841}$?

11. How many figures are there in $\sqrt[3]{1030301}$?

12. How many figures are there in $\sqrt[3]{10793861}$?

13. Show by multiplication that $(a+b)^2 = a^2 + 2ab + b^2$, and by use of this formula square 32 and 65.

14. Show by multiplication that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, and by use of the formula cube 41 and 98.

15. Square 234, first considering it as $230 + 4$ and then as $200 + 34$.

16. Show that no number ending in 2, 3, 7 or 8 can be a perfect square.

17. Prove that the cube of a number may end in any digits.

168. Square Root. The square root can often be determined by inspection if the number can readily be separated into prime factors. Thus:

$$32400 = 2^4 \times 3^4 \times 5^2.$$

$$\therefore \sqrt{32400} = 2^2 \times 3^2 \times 5 = 180.$$

Ex. By separating into prime factors find the square root of (*a*) 64, (*b*) 17424, (*c*) 7056, (*d*) 99225, (*e*) 680625, (*f*) 2800625, (*g*) 11025, (*h*) 81, (*i*) 1764, (*j*) 9801.

169. Since $43^2 = (40+3)^2 = 40^2 + 2 \times 40 \times 3 + 3^2 = 1849$, by reversing the process we can find $\sqrt{1849}$.

	$40^2 + 2 \times 40 \times 3 + 3^2$	$40 + 3$
	40^2	
$2 \times 40 + 3$	$2 \times 40 \times 3 + 3^2$	
	$2 \times 40 \times 3 + 3^2$	

170. In the above we notice that 40 is the square root of the first part. After subtracting 40^2 the remainder is $2 \times 40 \times 3 + 3^2$. The **trial divisor,** 2×40, is contained in the remainder 3 times. By adding 3 to 2×40 the **complete divisor,** $2 \times 40 + 3$, is formed. The complete divisor is contained exactly 3 times in the remainder. By this division the 3, the second figure of the root, is found.

171. The above is equivalent to the following:

	4 3
	1849
$40^2 =$	1600
$2 \times 40 + 3$	249
$= 83$	249

Separating 1849 into periods of two figures each (why?) we find that 1600 is the greatest square in 1800. 4 is therefore the first figure in the root. Subtracting 1600 and using 2×40 as the trial divisor, the next figure in the root is found to be 3. Completing the divisor by adding the 3, it is found to be exactly contained in the remainder. If the number contains more than two periods, the process is repeated.

172. The whole process of extracting the square root of a number is contained in the formula $(a+b)^2 = a^2 + 2ab + b^2$. The process of extracting the cube root is contained in $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. In general, the process of extracting any root can be derived from the corresponding power of $(a+b)$.

Ex. Extract the square root of 4529.29.

The *a* of the formula will always represent the part of the root already found and the *b* the next figure.

	6 7 3
$(a+b)^2 = a^2 + 2ab + b^2$	4529.29
a^2	$= 3600$
$2a = 120$	929.29
$b = 7$	
$(2a+b)b = 127 \times 7 =$	889.
$2a = 134$	40.29
$b = 0.3$	
$(2a+b)b = 134.3 \times 0.3 =$	40.29

In practice the work may be arranged as follows:

	6 7 3
	4529.29
	36
120	929.
127	889.
134	40.29
134.3	40.29

EXERCISE 27

1. In extracting the square root, why should the number be separated into periods of two figures each?

2. Where do you begin to separate into periods?

3. Separate each of the following into periods: 312, 4.162, 0.0125, 30000.4.

4. Will the division by $2a$ always give the next figure of the root?

5. Why is $2a$ called the trial divisor?

6. Why is $2a + b$ called the complete divisor?

7. In the above example explain how $2a$ can equal both 120 and 134.

8. Explain how $2ab + b^2$ can equal both 889 and 40.29.

9. Show how square root may be checked by casting out the 9's.

10. Extract the square root of each of the following, using the formula: (*a*) 2916, (*b*) 5.3361, (*c*) 65.61, (*d*) 0.003721, (*e*) 1632.16, (*f*) 289444, (*g*) 0.597529, (*h*) 103.4289, (*i*) 978121.

11. Extract the square root of 14400.

12. By first reducing to a decimal, extract the square root of $\frac{2}{3}$, $1\frac{1}{4}$, $\frac{3}{8}$, $\frac{11}{17}$, $6\frac{4}{5}$.

13. By first making the denominator a perfect square, extract the square root of $\frac{2}{5}$, $\frac{3}{7}$, $\frac{7}{13}$, $\frac{5}{11}$.

$$\frac{2}{5} = \frac{10}{25};\ \text{hence}\ \sqrt{\frac{2}{5}} = \sqrt{\frac{10}{25}} = \frac{3.162+}{5} = 0.632+.$$

14. Which of the two methods given in Ex. 12 and 13 is to be preferred?

15. Extract to 0.01 the square root of 16, 1.6, 0.016.

173. Cube Root.

Since $37^3 = (30 + 7)^3 = 30^3 + 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 = 50653$, by reversing the process we can find $\sqrt[3]{50653}$.

$$\begin{array}{r|l} & 30^3 + 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 (30 + 7 \\ & 30^3 \\ \hline 3 \times 30^2 + 3 \times 30 \times 7 + 7^2 & 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 \\ & 3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3 \\ \hline \end{array}$$

174. In the preceding we notice that 30 is the cube root of the first part.

After subtracting 30^3 the remainder is $3 \times 30^2 \times 7 + 3 \times 30 \times 7^2 + 7^3$. The **trial divisor,** 3×30^2, is contained in the remainder 7 times.

By adding $3 \times 30 \times 7 + 7^2$ the **complete divisor** is formed. This complete divisor is contained 7 times in the remainder. By this division the 7, the second figure in the root, is found.

175. The above is equivalent to the following:

$$\begin{array}{lr} & 3\ \ 7 \\ & 50653 \\ 30^3 = & 27000 \\ \hline 3 \times 30^2 + 3 \times 30 \times 7 + 7^2 = 3379 & 23653 \\ 3379 \times 7 = & 23653 \end{array}$$

Separating 50653 into periods of three figures each (Why?) we find that 27000 is the greatest cube in 50000. The first figure of the root is therefore 3. Subtracting 27000 and using 3×30^2 as the trial divisor, the next figure in the root (since 8, when the trial divisor is completed, proves to be too large) is found to be 7. Completing the divisor by adding $3 \times 30 \times 7 + 7^2$, it is found to be contained exactly 7 times in the remainder. If the number contains more than two periods, the process is to be repeated.

Ex. Extract the cube root of 362467.097.

As in extracting the square root of a number, the a of the formula will always represent the part of the root already found and the b the next figure.

$$\begin{array}{lr} & 7\ \ 1.\ \ 3 \\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 & 362467.097 \\ a^3 = & 343000 \\ \hline 3a^2 = 14700 & 19467.097 \\ b = \quad 1 & \\ (3a^2 + 3ab + b^2) = 14911 & 14911 \\ \hline 3a^2 = 15123 & 4556.097 \\ b = \quad 0.3 & \\ (3a^2 + 3ab + b^2) = 15186.99 & 4556.097 \\ \hline \end{array}$$

In practice the work may be arranged as follows:

	7 1. 3
	362467.097
	343
14700	19467.
14911	14911.
15123	4556.097
15186.99	4556.097

EXERCISE 28

1. Separate each of the following numbers into periods: 2500, 2.5, 3046.2971, 0.0125, 486521.3.

2. In extracting the cube root of 208527857, does the division by $3a^2$ give the second figure of the root correctly? Why is $3a^2$ called the trial divisor? Why is $3a^2 + 3ab + b^2$ called the complete divisor?

3. In the above example explain how $3a^2$ can equal both 14700 and 15123. Explain how $3a^2 + 3ab + b^2$ can equal both 14911 and 15186.99.

4. Show how cube root may be checked by casting out the 9's.

5. Extract the cube root of each of the following, using the formula: (*a*) 472729139, (*b*) 278.445077. (*c*) 1054.977832, (*d*) 19683, (*e*) 205379, (*f*) 25153.757.

6. Extract the cube root of $\frac{1331}{3375}$, $\frac{216}{343}$, $\frac{10648}{13824}$.

7. By reducing to a decimal, extract the cube root of $\frac{2}{3}$, $\frac{4}{9}$, $\frac{51}{56}$, $16\frac{2}{3}$, $33\frac{1}{3}$.

8. By first making the denominator a perfect cube, extract the cube root of $\frac{7}{8}$, $\frac{11}{14}$, $\frac{3}{7}$, $\frac{5}{9}$.

9. Find correct to 0.01 the cube root of 12.5, 125, 1.25.

MENSURATION

176. Certain measurements have been in very common use in the development of arithmetical knowledge from the earliest times.

177. The Babylonians and Egyptians used a great variety of geometrical figures in decorating their walls and in tile floors. The sense perception of these geometrical figures led to their actual measurement and finally to abstract geometrical reasoning.

178. The Greeks credited the Egyptians with the invention of geometry and gave as its origin the measurement of plots of land. Herodotus says that the Egyptian king, Sesostris (about 1400 B.C.), divided Egypt into equal rectangular plots of ground, and that the annual overflow of the Nile either washed away portions of the plot or obliterated the boundaries, making new measurements necessary. These measurements gave rise to the study of geometry (from *ge*, earth, and *metron*, to measure).

179. Ahmes, in his arithmetical work, calculates the contents of barns and the area of squares, rectangles, isosceles triangles, isosceles trapezoids and circles. There is no clew to his method of calculating volumes. In finding the area of the isosceles triangle he multiplies a side by half of the base, giving the area of a triangle whose sides are 10 and base 4 as 20 instead of 19.6, the result obtained by multiplying the altitude by half of the base. For the area of the isosceles trapezoid he multiplies a side by half the sum of the parallel bases, instead of finding the altitude and multiplying that by half the sum of the two parallel sides. He finds the area of the circle by subtracting from the diameter $\frac{1}{9}$ of its length and squaring the remainder. This leads to the fairly correct value of 3.1604 for π.

180. The Egyptians are also credited with knowing that in special cases the square on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. They were careful to locate their temples and other public buildings on north and south,