13. An Hypothesis is a supposition made, either in the statement of a proposition, or in the course of a demonstration.

14. Magnitudes are equal to each other, when each contains the same unit an equal number of times.

15. Magnitudes are equal *in all respects*, when they may be so placed as to coincide throughout their whole extent; they are equal *in all their parts* when each part of one is equal to the corresponding part of the other, when taken either in the same or in the reverse order.

ELEMENTS OF GEOMETRY.

BOOK I.

ELEMENTARY PRINCIPLES.

DEFINITIONS.

1. Geometry is that branch of Mathematics which treats of the properties, relations, and measurements of the Geometrical Magnitudes.

2. A Point is that which has position, but not magnitude.

3. A Line is that which has length, but neither breadth nor thickness.

Lines are divided into two classes, *straight* and *curved.*

4. A Straight Line is one which does not change its direction at any point.

5. A Curved Line is one which changes its direction at every point.

When the sense is obvious, to avoid repetition, the word *line*, alone, is commonly used for *straight line;* and the word *curve*, alone, for *curved line.*

6. A line made up of straight lines, not lying in the same direction, is called a *broken line.*

7. A Surface is that which has length and breadth without thickness.

Surfaces are divided into two classes, *plane* and *curved surfaces.*

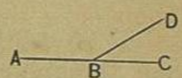
8. A Plane is a surface, such, that if any two of its points be joined by a straight line, that line will lie wholly in the surface.

9. A Curved Surface is a surface which is neither a plane nor composed of planes.

10. A Plane Angle is the amount of divergence of two straight lines lying in the same plane.

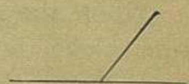
Thus, the amount of divergence of the lines AB and AC, is an angle. The lines AB and AC are called *sides*, and their common point A, is called the *vertex*. An angle is designated by naming its sides, or sometimes by simply naming its vertex; thus, the above is called the angle BAC, or simply, the angle A.

11. When one straight line meets another, the two angles which they form are called *adjacent angles*. Thus, the angles ABD and DBC are adjacent.



12. A Right Angle is formed by one straight line meeting another so as to make the adjacent angles *equal*. The first line is then said to be *perpendicular* to the second.

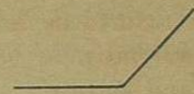
13. An Oblique Angle is formed by one straight line meeting another so as to make the adjacent angles *unequal*.



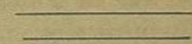
Oblique angles are subdivided into two classes, *acute angles*, and *obtuse angles*.

14. An Acute Angle is less than a right angle.

15. An Obtuse Angle is greater than a right angle.



16. Two straight lines are *parallel*, when they lie in the same plane and can not meet, how far soever, either way, both may be produced. They then have the *same direction*.



17. A Plane Figure is a portion of a plane bounded by lines, either straight or curved.

18. A Polygon is a plane figure bounded by straight lines.

The bounding lines are called *sides* of the polygon. The broken line, made up of all the sides of the polygon, is called the *perimeter* of the polygon. The angles formed by the sides are called *angles* of the polygon.

19. Polygons are classified according to the number of their sides or angles.

A Polygon of three sides is called a *triangle;* one of four sides, a *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon;* one of seven sides, a *heptagon;* one of eight sides, an *octagon;* one of ten sides, a *decagon;* one of twelve sides, a *dodecagon*, &c.

20. An Equilateral Polygon is one whose sides are all equal.

An Equiangular Polygon is one whose angles are all equal.

A Regular Polygon is one which is both equilateral and equiangular.

21. Two polygons are *mutually equilateral*, when their sides, taken in the same order, are equal, each to each: that is, following their perimeters in the same direction, the first

side of the one is equal to the first side of the other, the second side of the one to the second side of the other, and so on.

22. Two polygons are *mutually equiangular*, when their angles, taken in the same order, are equal, each to each.

23. A DIAGONAL of a polygon is a straight line joining the vertices of two angles, not consecutive.

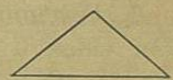
24. A BASE of a polygon is any one of its sides on which the polygon is supposed to stand.

25. Triangles may be classified with reference to either their sides, or their angles.

When classified with reference to their sides, there are two classes: *scalene* and *isosceles*.

1st. A SCALENE TRIANGLE is one which has no two of its sides equal.

2d. An ISOSCELES TRIANGLE is one which has two of its sides equal.



When all of the sides are equal, the triangle is EQUILATERAL.

When classified with reference to their angles, there are two classes: *right-angled* and *oblique-angled.*

1st. A RIGHT-ANGLED TRIANGLE is one that has one right angle.

The side opposite the right angle is called the *hypothenuse.*

2d. An OBLIQUE-ANGLED TRIANGLE is one whose angles are all oblique.

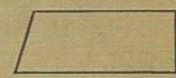
If one angle of an oblique-angled triangle is obtuse, the triangle is said to be OBTUSE-ANGLED. If all of the angles are acute, the triangle is said to be ACUTE-ANGLED.

26. Quadrilaterals are classified with reference to the relative directions of their sides. There are then two classes; the *first class* embraces those which have no two sides parallel; the *second class* embraces those which have at least two sides parallel.

Quadrilaterals of the first class, are called *trapeziums.*

Quadrilaterals of the second class, are divided into two species: *trapezoids* and *parallelograms.*

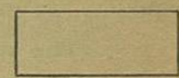
27. A TRAPEZOID is a quadrilateral which has only two of its sides parallel.



28. A PARALLELOGRAM is a quadrilateral which has its opposite sides parallel, two and two.

There are two varieties of parallelograms: *rectangles* and *rhomboids.*

1st. A RECTANGLE is a parallelogram whose angles are all right angles.



A SQUARE is an equilateral rectangle.

2d. A RHOMBOID is a parallelogram whose angles are all oblique.

A RHOMBUS is an equilateral rhomboid.

29. SPACE is indefinite extension.

30. A VOLUME is a limited portion of space, combining the three dimensions of length, breadth, and thickness.

AXIOMS.

1. Things which are equal to the same thing, are equal to each other.

2. If equals are added to equals, the sums are equal.

3. If equals are subtracted from equals, the remainders are equal.

4. If equals are added to unequals, the sums are unequal.

5. If equals are subtracted from unequals, the remainders are unequal.

6. If equals are multiplied by equals, the products are equal.

7. If equals are divided by equals, the quotients are equal.

8. The whole is greater than any of its parts.

9. The whole is equal to the sum of all its parts.

10. All right angles are equal.

11. Only one straight line can be drawn joining two given points.

12. The shortest distance from one point to another is measured on the straight line which joins them.

13. Through the same point, only one straight line can be drawn parallel to a given straight line.

POSTULATES.

1. A straight line can be drawn joining any two points.

2. A straight line may be prolonged to any length.

3. If two straight lines are unequal, the length of the less may be laid off on the greater.

4. A straight line may be bisected; that is, divided into two equal parts.

5. An angle may be bisected.

6. A perpendicular may be drawn to a given straight line, either from a point without, or from a point on the line.

7. A straight line may be drawn, making with a given straight line an angle equal to a given angle.

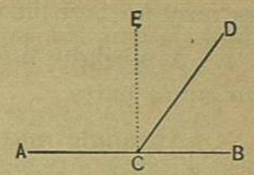
8. A straight line may be drawn through a given point, parallel to a given line.

NOTE.

In making references, the following abbreviations are employed, viz.: A. for Axiom; B. for Book; C. for Corollary; D. for Definition; I. for Introduction; P. for Proposition; Prob. for Problem; Post. for Postulate; and S. for Scholium. In referring to the same Book, the number of the Book *is not* given; in referring to any other Book, the number of the Book *is* given.

PROPOSITION I. THEOREM.

If a straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.



Let DC meet AB at C: then is the sum of the angles DCA and DCB equal to two right angles.

At C, let CE be drawn perpendicular to AB (Post. 6); then, by definition (D. 12), the angles ECA and ECB are both right angles, and consequently, their sum is equal to *two right angles.*

The angle DCA is equal to the sum of the angles ECA and ECD (A. 9); hence,

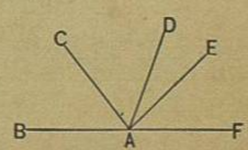
$$DCA + DCB = ECA + ECD + DCB;$$

But, ECD + DCB is equal to ECB (A. 9); hence,

$$DCA + DCB = ECA + ECB.$$

The sum of the angles ECA and ECB, is equal to two right angles; consequently, its equal, that is, the sum of the angles DCA and DCB, must also be equal to two right angles; *which was to be proved.*

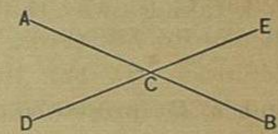
Cor. 1. If one of the angles DCA, DCB, is a right angle, the other must also be a right angle.



Cor. 2. The sum of the angles BAC, CAD, DAE, EAF, formed about a given point on the same side of a straight line BF, is equal to two right angles. For, their sum is equal to the sum of the angles EAB and EAF; which, from the proposition just demonstrated, is equal to two right angles.

DEFINITIONS.

If two straight lines intersect each other, they form four angles about the point of intersection, which have received different names, with respect to each other.

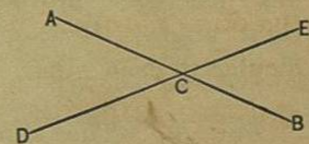


1°. ADJACENT ANGLES are those which lie on the same side of one line, and on opposite sides of the other; thus, ACE and ECB, or ACE and ACD, are adjacent angles.

2°. OPPOSITE, or VERTICAL ANGLES, are those which lie on opposite sides of both lines; thus, ACE and DCB, or ACD and ECB, are opposite angles. From the proposition just demonstrated, the sum of any two adjacent angles is equal to two right angles.

PROPOSITION II. THEOREM.

If two straight lines intersect each other, the opposite or vertical angles are equal.

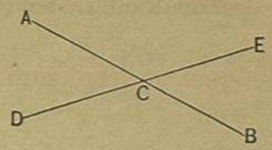


Let AB and DE intersect at C: then are the opposite or vertical angles equal.

The sum of the adjacent angles ACE and ACD, is equal to two right angles (P. I.): the sum of the adjacent angles ACE and ECB, is also equal to two right angles. But things which are equal to the same thing, are equal to each other (A. 1); hence,

$$ACE + ACD = ACE + ECB;$$

Taking from both the common angle ACE (A. 3), there remains,



$$ACD = ECB.$$

In like manner, we find,

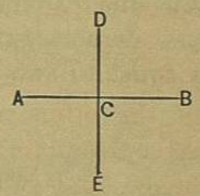
$$ACD + ACE = ACD + DCB;$$

and, taking away the common angle ACD, we have,

$$ACE = DCB.$$

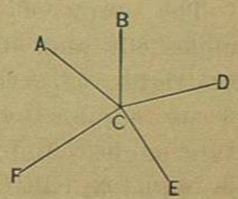
Hence, *the proposition is proved.*

Cor. 1. If one of the angles about C is a right angle, all of the others are right angles also. For, (P. I., C. 1), each of its adjacent angles is a right angle; and from the proposition just demonstrated, its opposite angle is also a right angle.



Cor. 2. If one line DE, is perpendicular to another AB, then is the second line AB perpendicular to the first DE. For, the angles DCA and DCB are right angles, by definition (D. 12); and from what has just been proved, the angles ACE and BCE are also right angles. Hence, the two lines are mutually perpendicular to each other.

Cor. 3. The sum of all the angles ACB, BCD, DCE, ECF, FCA, that can be formed about a point, is equal to four right angles.

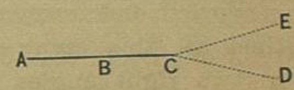


For, if two lines are drawn through the point, mutually perpendicular to each other, the sum of the angles which they form is equal to four right angles, and it is also equal to the sum of the given angles (A. 9). Hence, the sum of the given angles is equal to four right angles.

PROPOSITION III. THEOREM.

If two straight lines have two points in common, they coincide throughout their whole extent, and form one and the same line.

Let A and B be two points common to two lines: then the lines coincide throughout.



Between A and B they must coincide (A. 11). Suppose, now, that they begin to separate at some point C, beyond AB, the one becoming ACE, and the other ACD. If the lines do separate at C, one or the other must change direction at this point; but this is contradictory to the definition of a straight line (D. 4): hence, the supposition that they separate at any point is absurd. They must, therefore, coincide throughout; *which was to be proved.*

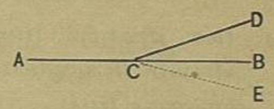
Cor. Two straight lines can intersect in only one point.

Note.—The method of demonstration employed above, is called the *reductio ad absurdum.* It consists in assuming an hypothesis which is the contradictory of the proposition to be proved, and then continuing the reasoning until the assumed hypothesis is shown to be false. Its contradictory is thus proved to be true. This method of demonstration is often used in Geometry.

PROPOSITION IV. THEOREM.

If a straight line meets two other straight lines at a common point, making the sum of the contiguous angles equal to two right angles, the two lines met form one and the same straight line.

Let DC meet AC and BC at C, making the sum of the angles DCA and DCB equal to two right angles: then is CB the prolongation of AC.



For, if not, suppose CE to be the prolongation of AC; then is the sum of the angles DCA and DCE equal to two right angles (P. I.): consequently, we have (A. 1),

$$DCA + DCB = DCA + DCE;$$

Taking from both the common angle DCA, there remains

$$DCB = DCE,$$

which is impossible, since a part can not be equal to the whole (A. 8). Hence, CB must be the prolongation of AC; *which was to be proved.*

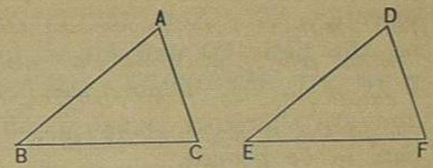
PROPOSITION V. THEOREM.

If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles are equal in all respects.

In the triangles ABC and DEF, let AB be equal to DE,

AC to DF, and the angle A to the angle D: then are the triangles equal in all respects.

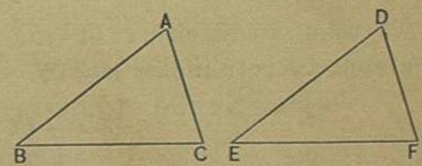
For, let ABC be applied to DEF, in such a manner that the angle A shall coincide with the angle D, the side AB taking the direction DE, and the side AC the direction DF. Then, because AB is equal to DE, the vertex B will coincide with the vertex E; and because AC is equal to DF, the vertex C will coincide with the vertex F; consequently, the side BC will coincide with the side EF (A. 11). The two triangles, therefore, coincide throughout, and are consequently equal in all respects (I., D. 15); *which was to be proved.*



PROPOSITION VI. THEOREM.

If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles are equal in all respects.

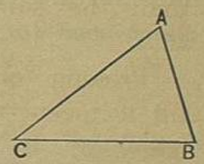
In the triangles ABC and DEF, let the angle B be equal to the angle E, the angle C to the angle F, and the side BC to the side EF: then are the triangles equal in all respects.



For, let ABC be applied to DEF in such a manner that the angle B shall coincide with the angle E, the side BC taking the direction EF, and the side BA the direction ED. Then, because BC is equal to EF, the vertex C will coincide with the vertex F; and because the angle C is equal to the angle F, the side CA will take the direction FD. Now, the vertex A being at the same time on the lines ED and FD, it must be at their intersection D (P. III., C.): hence, the triangles coincide throughout, and are therefore equal in all respects (I., D. 15); *which was to be proved.*

PROPOSITION VII. THEOREM.

The sum of any two sides of a triangle is greater than the third side.



Let ABC be a triangle: then will the sum of any two sides, as AB, BC, be greater than the third side AC.

For, the distance from A to C, measured on any broken line AB, BC, is greater than the distance measured on the straight line AC (A. 12): hence, the sum of AB and BC is greater than AC; *which was to be proved.*

Cor. If from both members of the inequality,

$$AC < AB + BC,$$

we take away either of the sides AB, BC, as BC, for example, there remains (A. 5),

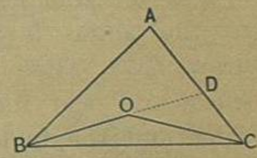
$$AC - BC < AB;$$

that is, *the difference between any two sides of a triangle is less than the third side.*

Scholium. In order that any three given lines may represent the sides of a triangle, the sum of any two must be greater than the third, and the difference of any two must be less than the third.

PROPOSITION VIII. THEOREM.

If from any point within a triangle two straight lines are drawn to the extremities of any side, their sum is less than that of the two remaining sides of the triangle.



Let O be any point within the triangle BAC, and let the lines OB, OC, be drawn to the extremities of any side, as BC: then the sum of BO and OC is less than the sum of the sides BA and AC.

Prolong one of the lines, as BO, till it meets the side AC in D; then, from Prop. VII., we have,

$$OC < OD + DC;$$

adding BO to both members of this inequality, recollecting that the sum of BO and OD is equal to BD, we have (A. 4),

$$BO + OC < BD + DC.$$

From the triangle BAD, we have (P. VII.),

$$BD < BA + AD;$$

adding DC to both members of this inequality, recollecting that the sum of AD and DC is equal to AC, we have,

$$BD + DC < BA + AC.$$

But it was shown that BO + OC is less than BD + DC; still more, then, is BO + OC less than BA + AC; *which was to be proved.*

PROPOSITION IX. THEOREM.

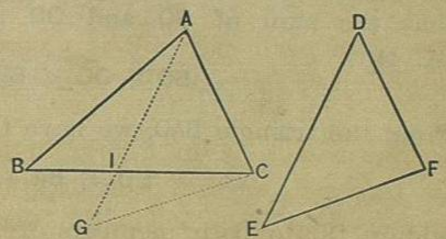
2 *If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides are unequal; and the greater side belongs to the triangle which has the greater included angle.*

In the triangles BAC and DEF, let AB be equal to DE, AC to DF, and the angle A greater than the angle D: then is BC greater than EF.

Let the line AG be drawn, making the angle CAG equal to the angle D (Post. 7); make AG equal to DE, and draw GC. Then the triangles AGC and DEF have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each; consequently, GC is equal to EF (P. V.).

Now, the point G may be without the triangle ABC, it may be on the side BC, or it may be within the triangle ABC. Each case will be considered separately.

1°. When G is without the triangle ABC.



In the triangles GIC and AIB, we have, (P. VII.),

$$GI + IC > GC, \quad \text{and} \quad BI + IA > AB;$$

whence, by addition, recollecting that the sum of BI and IC is equal to BC, and the sum of GI and IA, to GA, we have,

$$AG + BC > AB + GC.$$

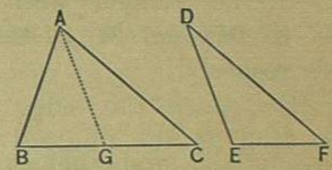
Or, since $AG = AB$, and $GC = EF$, we have,

$$AB + BC > AB + EF.$$

Taking away the common part AB, there remains (A. 5),

$$BC > EF.$$

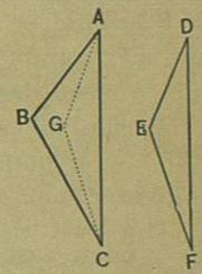
2°. When G is on BC.



In this case, it is obvious that GC is less than BC; or since $GC = EF$, we have,

$$BC > EF.$$

3°. When G is within the triangle ABC.



From Proposition VIII., we have,

$$BA + BC > GA + GC;$$

or, since $GA = BA$, and $GC = EF$, we have,

$$BA + BC > BA + EF.$$

Taking away the common part AB, there remains,

$$BC > EF.$$

Hence, in each case, BC is greater than EF; *which was to be proved.*

Conversely: If in two triangles ABC and DEF, the side AB is equal to the side DE, the side AC to DF, and BC greater than EF, then is the angle BAC greater than the angle EDF.

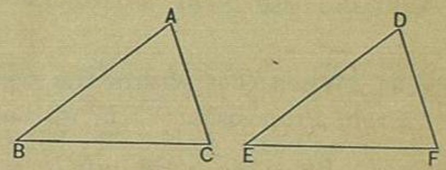
For, if not, BAC must either be equal to, or less than, EDF. In the former case, BC would be equal to EF (P. V.), and in the latter case, BC would be less than EF; either of which would contradict the hypothesis: hence, BAC must be greater than EDF.

PROPOSITION X. THEOREM.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are equal in all respects.

In the triangles ABC and DEF, let AB be equal to DE, AC to DF, and BC to EF: then are the triangles equal in all respects.

For, since the sides AB, AC, are equal to DE, DF, each to each, if the angle A were greater than D, it would follow, by the last Proposition, that the side BC would be greater than EF; and if the angle A were less than D, the side BC would be less than EF. But BC is equal to EF, by hypothesis; therefore, the angle A can neither be greater nor less than D: hence, it must be equal to it. The two triangles have, therefore, two sides and the included angle of the one equal to two sides and the included angle of the other, each to each; and, consequently, they are equal in all respects (P. V.); *which was to be proved.*



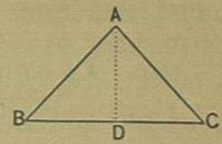
Scholium. In triangles, equal in all respects, the equal sides lie opposite the equal angles; and conversely.

PROPOSITION XI. THEOREM.

In an isosceles triangle the angles opposite the equal sides are equal.

Let BAC be an isosceles triangle, having the side AB equal to the side AC: then the angle C is equal to the angle B.

Join the vertex A and the middle point D of the base BC. Then, AB is equal to AC, by hypothesis, AD common, and BD equal to DC, by construction: hence, the triangles BAD, and DAC, have the three sides of the one equal to those of the other, each to each; therefore, by the last Proposition, the angle B is equal to the angle C; *which was to be proved.*



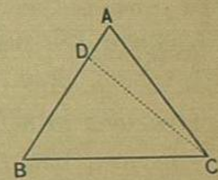
Cor. 1. An equilateral triangle is equiangular.

Cor. 2. The angle BAD is equal to DAC, and BDA to CDA: hence, the last two are right angles. Consequently, *a straight line drawn from the vertex of an isosceles triangle to the middle of the base, bisects the angle at the vertex, and is perpendicular to the base.*

PROPOSITION XII. THEOREM.

If two angles of a triangle are equal, the sides opposite to them are also equal, and consequently, the triangle is isosceles.

In the triangle ABC, let the angle ABC be equal to the angle ACB: then is AC equal to AB, and consequently, the triangle is isosceles.



For, if AB and AC are not equal, suppose one of them, as AB, to be the greater. On this, take BD equal to AC (Post. 3), and draw DC. Then, in the triangles ABC, DBC, we have the side BD equal to AC, by construction, the side BC common, and the included angle ACB equal to the included angle DBC, by hypothesis: hence, the two triangles are equal