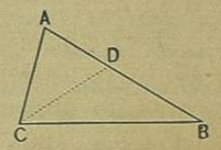
in all respects (P. V.). But this is impossible, because a part can not be equal to the whole (A. 8): hence, the hypothesis that AB and AC are unequal, is false. They must, therefore, be equal; *which was to be proved.*

Cor. An equiangular triangle is equilateral.

PROPOSITION XIII. THEOREM.

In any triangle, the greater side is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.



In the triangle ABC, let the angle ACB be greater than the angle ABC: then the side AB is greater than the side AC.

For, draw CD, making the angle BCD equal to the angle B (Post. 7): then, in the triangle DCB, we have the angles DCB and DBC equal: hence, the opposite sides DB and DC are equal (P. XII.). In the triangle ACD, we have (P. VII.),

$$AD + DC > AC;$$

or, since $DC = DB$, and $AD + DB = AB$, we have,

$$AB > AC;$$

which was to be proved.

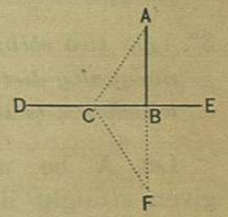
Conversely: Let AB be greater than AC: then the angle ACB is greater than the angle ABC.

For, if ACB were less than ABC, the side AB would be less than the side AC, from what has just been proved; if ACB were equal to ABC, the side AB would be equal to AC, by Prop. XII.; but both conclusions contradict

the hypothesis: hence, ACB can neither be less than, nor equal to, ABC; it must, therefore, be greater; *which was to be proved.*

PROPOSITION XIV. THEOREM.

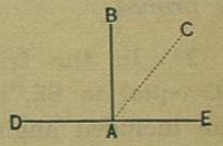
From a given point only one perpendicular can be drawn to a given straight line.



Let A be a given point, and AB a perpendicular to DE: then can no other perpendicular to DE be drawn from A.

For, suppose a second perpendicular AC to be drawn. Prolong AB till BF is equal to AB, and draw CF. Then, the triangles ABC and FBC have AB equal to BF, by construction, CB common, and the included angles ABC and FBC equal, because both are right angles: hence, the angles ACB and FCB are equal (P. V.). But ACB is, by a hypothesis, a right angle: hence, FCB must also be a right angle, and consequently, the line ACF must be a straight line (P. IV.). But this is impossible (A. 11). The hypothesis that two perpendiculars can be drawn is, therefore, absurd; consequently, only one such perpendicular can be drawn; *which was to be proved.*

If the given point is on the given line, the proposition is equally true. For, if from A two perpendiculars AB and AC could be drawn to DE, we should have BAE and CAE each equal to a right angle; and consequently, equal to each other; which is absurd (A. 8).



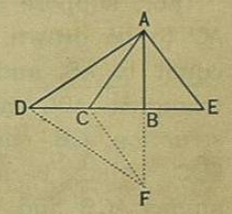
PROPOSITION XV. THEOREM.

If from a point without a straight line a perpendicular is let fall on the line, and oblique lines are drawn to different points of it:

1°. *The perpendicular is shorter than any oblique line.*

2°. *Any two oblique lines that meet the given line at points equally distant from the foot of the perpendicular, are equal.*

3°. *Of two oblique lines that meet the given line at points unequally distant from the foot of the perpendicular, the one which meets it at the greater distance is the longer.*



Let A be a given point, DE a given straight line, AB a perpendicular to DE, and AD, AC, AE oblique lines, BC being equal to BE, and BD greater than BC. Then AB is less than any of the oblique lines, AC is equal to AE, and AD greater than AC.

Prolong AB until BF is equal to AB, and draw FC, FD.

1°. In the triangles ABC, FBC, we have the side AB equal to BF, by construction, the side BC common, and the included angles ABC and FBC equal, because both are right angles: hence, FC is equal to AC (P. V.). But, AF is shorter than ACF (A. 12): hence, AB, the half of AF, is shorter than AC, the half of ACF; *which was to be proved.*

2°. In the triangles ABC and ABE, we have the side BC equal to BE, by hypothesis, the side AB common, and the included angles ABC and ABE equal, because both are right angles: hence, AC is equal to AE; *which was to be proved.*

3°. It may be shown, as in the first case, that AD is equal to DF. Then, because the point C lies within the triangle ADF, the sum of the lines AD and DF is greater than the sum of the lines AC and CF (P. VIII.): hence, AD, the half of ADF, is greater than AC, the half of ACF; *which was to be proved.*

Cor. 1. The perpendicular is the shortest distance from a point to a line.

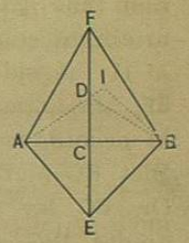
Cor. 2. From a given point to a given straight line, only two equal straight lines can be drawn; for, if there could be more, there would be at least two equal oblique lines on the same side of the perpendicular; which is impossible.

PROPOSITION XVI. THEOREM.

If a perpendicular is drawn to a given straight line at its middle point:

1°. *Any point of the perpendicular is equally distant from the extremities of the line:*

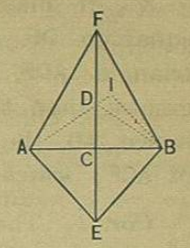
2°. *Any point, without the perpendicular, is unequally distant from the extremities.*



Let AB be a given straight line, C its middle point, and EF the perpendicular. Then any point of EF is equally distant from A and B; and any point without EF, is unequally distant from A and B.

1°. From any point of EF, as D, draw the lines DA and DB. Then DA and DB are equal (P. XV.): hence, D is equally distant from A and B; *which was to be proved.*

2°. From any point without EF, as I, draw IA and IB. One of these lines, as IA, will cut EF in some point D; draw DB. Then, from what has just been shown, DA and DB are equal; but IB is less than the sum of ID and DB (P. VII.); and because the sum of ID and DB is equal to the sum of ID and DA, or IA, we have IB less than IA: hence, I is unequally distant from A and B; *which was to be proved.*

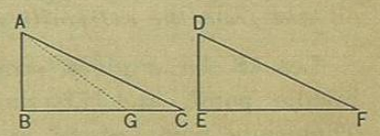


Cor. If a straight line, EF, has two of its points, E and F, each equally distant from A and B, it is perpendicular to the line AB at its middle point.

PROPOSITION XVII. THEOREM.

If two right-angled triangles have the hypothenuse and a side of the one equal to the hypothenuse and a side of the other, each to each, the triangles are equal in all respects.

Let the right-angled triangles ABC and DEF have the hypothenuse AC equal to DF, and the side AB equal to DE: then the triangles are equal in all respects.



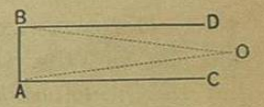
If the side BC is equal to EF, the triangles are equal, in accordance with Proposition X. Let us suppose then, that BC and EF are unequal, and that BC is the longer. On BC lay off BG equal to EF, and draw AG. The triangles ABG and DEF have AB equal to DE, by hypothesis, BG equal to EF, by construction, and the angles B and E

equal, because both are right angles; consequently, AG is equal to DF (P. V.). But, AC is equal to DF, by hypothesis: hence, AG and AC are equal, which is impossible (P. XV.). The hypothesis that BC and EF are unequal, is, therefore, absurd: hence, the triangles have all their sides equal, each to each, and are, consequently, equal in all respects; *which was to be proved.*

PROPOSITION XVIII. THEOREM.

If two straight lines are perpendicular to a third straight line, they are parallel.

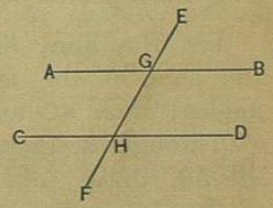
Let the two lines AC, BD, be perpendicular to AB: then they are parallel.



For, if they could meet in a point O, there would be two perpendiculars OA, OB, drawn from the same point to the same straight line; which is impossible (P. XIV.): hence, the lines are parallel; *which was to be proved.*

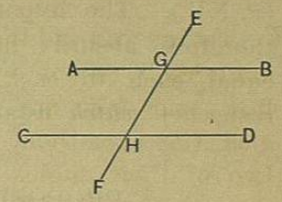
DEFINITIONS.

If a straight line EF intersect two other straight lines AB and CD, it is called a *secant*, with respect to them. The eight angles formed about the points of intersection have different names, with respect to each other.



1°. Interior angles on the same side, are those that lie on the same side of the secant and *within* the other two lines. Thus, BGH and GHD are interior angles on the same side.

2°. Exterior angles on the same side are those that lie on the same side of the secant and *without* the other two lines. Thus, EGB and DHF are exterior angles on the same side.



3°. Alternate angles are those that lie on opposite sides of the secant and *within* the other two lines, but not adjacent. Thus, AGH and GHD are alternate angles.

4°. Alternate exterior angles are those that lie on opposite sides of the secant and *without* the other two lines. Thus, AGE and FHD are alternate exterior angles.

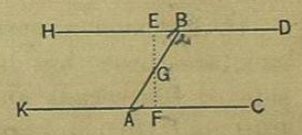
5°. Opposite exterior and interior angles are those that lie on the same side of the secant, the one *within* and the other *without* the other two lines, but not adjacent. Thus, EGB and GHD are opposite exterior and interior angles.

PROPOSITION XIX. THEOREM.

3 *If two straight lines meet a third straight line, making the sum of the interior angles on the same side equal to two right angles, the two lines are parallel.*

Let the lines KC and HD meet the line BA, making the sum of the angles BAC and ABD equal to two right angles; then KC and HD are parallel.

Through G, the middle point of AB, draw GF perpendicular to KC, and prolong it to E.



The sum of the angles GBE and GBD is equal to two right

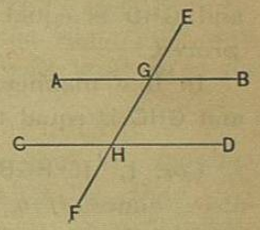
angles (P. I.); the sum of the angles FAG and GBD is equal to two right angles, by hypothesis: hence (A. 1),

$$GBE + GBD = FAG + GBD.$$

Taking away the common part GBD, we have the angle GBE equal to the angle FAG. Again, the angles BGE and AGF are equal, because they are vertical angles (P. II.): hence, the triangles GEB and GFA have two of their angles and the included side equal, each to each; they are, therefore, equal in all respects (P. VI.): hence, the angle GEB is equal to the angle GFA. But, GFA is a right angle, by construction; GEB must, therefore, be a right angle: hence, the lines KC and HD are perpendicular to EF, and are, therefore, parallel (P. XVIII.); *which was to be proved.*

Cor. 1. If two straight lines are cut by a third straight line, making the alternate angles equal to each other, the two straight lines are parallel.

Let the angle HGA be equal to GHD. Adding to both the angle HGB, we have,



$$HGA + HGB = GHD + HGB.$$

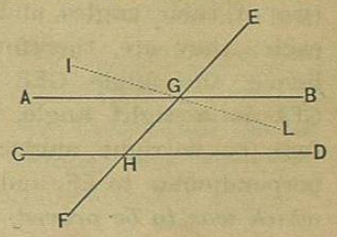
But the first sum is equal to two right angles (P. I.): hence, the second sum is also equal to two right angles; therefore, from what has just been shown, AB and CD are parallel.

Cor. 2. If two straight lines are cut by a third, making the opposite exterior and interior angles equal, the two straight lines are parallel. Let the angles EGB and GHD be equal: Now EGB and AGH are equal, because they are vertical angles (P. II.); and consequently, AGH and GHD are equal: hence, from *Cor.* 1, AB and CD are parallel.

PROPOSITION XX. THEOREM.

4 *If a straight line intersects two parallel straight lines, the sum of the interior angles on the same side is equal to two right angles.*

Let the parallels AB, CD, be cut by the secant line FE: then the sum of HGB and GHD is equal to two right angles.

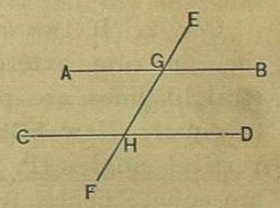


For, if the sum of HGB and GHD is not equal to two right angles, let IGL be drawn, making the sum of HGL and GHD equal to two right angles; then IL and CD are parallel (P. XIX.); and consequently, we have two lines, GB, GL, drawn through the same point G and parallel to CD, which is impossible (A. 13): hence, the sum of HGB and GHD is equal to two right angles; *which was to be proved.*

In like manner, it may be proved that the sum of HGA and GHC is equal to two right angles.

Cor. 1. If HGB is a right angle, GHD is a right angle also: hence, *if a line is perpendicular to one of two parallels, it is perpendicular to the other also.*

Cor. 2. *If a straight line intersects two parallels, the alternate angles are equal.*



For, if AB and CD are parallel, the sum of BGH and GHD is equal to two right angles; the sum of BGH and HGA is also equal to two right angles (P. I.): hence, these sums are equal. Taking away the

common part BGH, there remains the angle GHD equal to HGA. In like manner, it may be shown that BGH and GHC are equal.

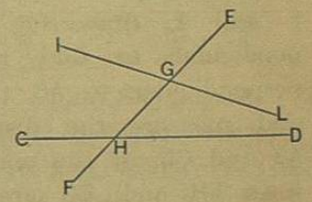
Cor. 3. *If a straight line intersects two parallels, the opposite exterior and interior angles are equal.* The angles DHG and HGA are equal, from what has just been shown. The angles HGA and BGE are equal, because they are vertical: hence, DHG and BGE are equal. In like manner, it may be shown that CHG and AGE are equal.

Scholium. Of the eight angles formed by a line cutting two parallel lines obliquely, the four acute angles are equal, and so, also, are the four obtuse angles.

PROPOSITION XXI. THEOREM.

If two straight lines intersect a third straight line, making the sum of the interior angles on the same side less than two right angles, the two lines will meet if sufficiently produced.

Let the two lines CD, IL, meet the line EF, making the sum of the interior angles HGL, GHD, less than two right angles: then will IL and CD meet if sufficiently produced.

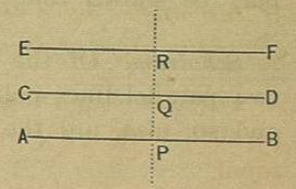


For, if they do not meet, they must be parallel (D. 16). But, if they were parallel, the sum of the interior angles HGL, GHD, would be equal to two right angles (P. XX.), which contradicts the hypothesis: hence, IL, CD, will meet if sufficiently produced; *which was to be proved.*

Cor. It is evident that IL and CD will meet on that side of EF, on which the sum of the two angles is less than two right angles.

PROPOSITION XXII. THEOREM.

If two straight lines are parallel to a third line, they are parallel to each other.



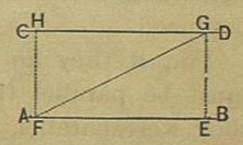
Let AB and CD be respectively parallel to EF: then are they parallel to each other.

For, draw PR perpendicular to EF; then is it perpendicular to AB, and also to CD (P. XX., C. 1): hence, AB and CD are perpendicular to the same straight line, and consequently, they are parallel to each other (P. XVIII.); *which was to be proved.*

PROPOSITION XXIII. THEOREM.

Two parallels are every-where equally distant.

Let AB and CD be parallel: then are they every-where equally distant.



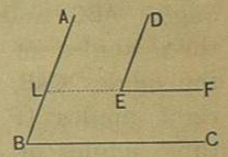
From any two points of AB, as F and E, draw FH and EG perpendicular to CD; they are also perpendicular to AB (P. XX., C. 1), and measure the distance between AB and CD, at the points F and E. Draw also FG. The lines FH and EG are parallel (P. XVIII.): hence, the alternate angles HFG and FGE are equal (P. XX., C. 2). The lines AB and CD are parallel, by hypothesis: hence,

the alternate angles EFG and FGH are equal. The triangles FGE and FGH have, therefore, the angle HGF equal to GFE, GFH equal to FGE, and the side FG common; they are, therefore, equal in all respects (P. VI.): hence, FH is equal to EG; and consequently, AB and CD are every-where equally distant; *which was to be proved.*

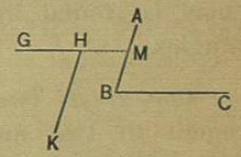
PROPOSITION XXIV. THEOREM.

If two angles have their sides parallel, and lying either in the same or in opposite directions, they are equal.

1°. Let the angles ABC and DEF have their sides parallel, and lying in the same direction: then are they equal.



Prolong FE to L. Then, because DE and AL are parallel, the exterior angle DEF is equal to its opposite interior angle ALE (P. XX., C. 3); and, because BC and LF are parallel, the exterior angle ALE is equal to its opposite interior angle ABC: hence, DEF is equal to ABC; *which was to be proved.*



2°. Let the angles ABC and GHK have their sides parallel, and lying in opposite directions: then are they equal.

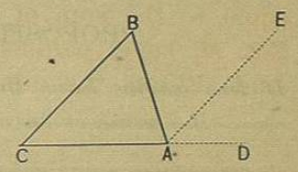
Prolong GH to M. Then, because KH and BM are parallel, the exterior angle GHK is equal to its opposite interior angle HMB; and because HM and BC are parallel, the angle HMB is equal to its alternate angle MBC (P. XX., C. 2): hence, GHK is equal to ABC; *which was to be proved.*

Cor. The opposite angles of a parallelogram are equal.

PROPOSITION XXV. THEOREM.

In any triangle, the sum of the three angles is equal to two right angles.

Let CBA be any triangle: then the sum of the angles C, A, and B, is equal to two right angles.



For, prolong CA to D, and draw AE parallel to BC.

Then, since AE and CB are parallel, and CD cuts them, the exterior angle DAE is equal to its opposite interior angle C (P. XX., C. 3). In like manner, since AE and CB are parallel, and AB cuts them, the alternate angles ABC and BAE are equal: hence, the sum of the three angles of the triangle BAC is equal to the sum of the angles CAB, BAE, EAD; but this sum is equal to two right angles (P. I., C. 2); consequently, the sum of the three angles of the triangle, is equal to two right angles (A. 1); *which was to be proved.*

Cor. 1. Two angles of a triangle being given, the third may be found by subtracting their sum from two right angles.

Cor. 2. If two angles of one triangle are respectively equal to two angles of another, the two triangles are mutually equiangular.

Cor. 3. In any triangle, there can be but one right angle; for if there were two, the third angle would be zero. Nor can a triangle have more than one obtuse angle.

Cor. 4. In any right-angled triangle, the sum of the acute angles is equal to a right angle.

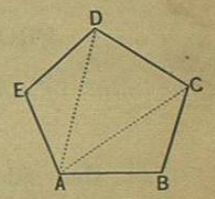
Cor. 5. Since every equilateral triangle is also equiangular (P. XI., C. 1), each of its angles is equal to the third part of two right angles; so that, if the right angle is expressed by 1, each angle of an equilateral triangle is expressed by $\frac{2}{3}$.

Cor. 6. In any triangle ABC, the exterior angle BAD is equal to the sum of the interior opposite angles B and C. For, AE being parallel to BC, the part BAE is equal to the angle B, and the other part DAE, is equal to the angle C.

PROPOSITION XXVI. THEOREM.

The sum of the interior angles of a polygon is equal to two right angles taken as many times, less two, as the polygon has sides.

Let ABCDE be any polygon; then the sum of its interior angles A, B, C, D, and E, is equal to two right angles taken as many times, less two, as the polygon has sides.



From the vertex of any angle A, draw diagonals AC, AD. The polygon will be divided into as many triangles, less two, as it has sides, having the point A for a common vertex, and for bases, the sides of the polygon, except the two which form the angle A. It is evident, also, that the sum of the angles of these triangles does not differ from the sum of the angles of the polygon: hence, the sum of the angles of the polygon is equal to two right angles, taken as many times as there are triangles; that is, as many times, less two, as the polygon has sides; ***which was to be proved.***

Cor. 1. The sum of the interior angles of a quadrilateral is equal to two right angles taken twice; that is, to four right angles. If the angles of a quadrilateral are equal, each is a right angle.

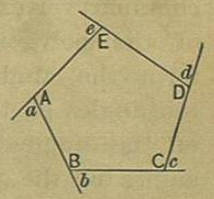
Cor. 2. The sum of the interior angles of a pentagon is equal to two right angles taken three times; that is, to six right angles: hence, when a pentagon is equiangular, each angle is equal to the fifth part of six right angles, or to $\frac{6}{5}$ of one right angle.

Cor. 3. The sum of the interior angles of a hexagon is equal to eight right angles: hence, in the equiangular hexagon, each angle is the sixth part of eight right angles, or $\frac{4}{3}$ of one right angle.

Cor. 4. In any equiangular polygon, any interior angle is equal to twice as many right angles as the figure has sides, less four right angles, divided by the number of angles.

PROPOSITION XXVII. THEOREM.

The sum of the exterior angles of a polygon is equal to four right angles.

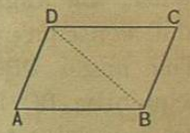


Let the sides of the polygon ABCDE be prolonged, in the same order, forming the exterior angles *a*, *b*, *c*, *d*, *e*; then the sum of these exterior angles is equal to four right angles.

For, each interior angle, together with the corresponding exterior angle, is equal to two right angles (P. I.); hence, the sum of all the interior and exterior angles is equal to two right angles taken as many times as the polygon has sides. But the sum of the interior angles is equal to two right angles taken as many times, less two, as the polygon has sides: hence, the sum of the exterior angles is equal to two right angles taken twice; that is, equal to four right angles; *which was to be proved.*

PROPOSITION XXVIII. THEOREM.

In any parallelogram, the opposite sides are equal, each to each.



Let ABCD be a parallelogram: then AB is equal to DC, and AD to BC.

For, draw the diagonal BD. Then, because AB and DC are parallel, the angle DBA is equal to its alternate angle BDC (P. XX., C. 2); and, because AD and BC are parallel, the angle BDA is equal to its alternate angle DBC. The triangles ABD and CDB, have, therefore, the angle DBA equal to CDB, the angle BDA equal to DBC, and the included side DB common; consequently, they are equal in all respects: hence, AB is equal to DC, and AD to BC; *which was to be proved.*

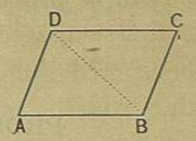
Cor. 1. A diagonal of a parallelogram divides it into two triangles equal in all respects.

Cor. 2. Two parallels included between two other parallels, are equal.

Cor. 3. If two parallelograms have two sides and the included angle of the one, equal to two sides and the included angle of the other, each to each, they are equal.

PROPOSITION XXIX. THEOREM.

If the opposite sides of a quadrilateral are equal, each to each, the figure is a parallelogram.

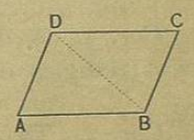


In the quadrilateral ABCD, let AB be equal to DC, and AD to BC: then is it a parallelogram.

Draw the diagonal DB. Then, the triangles ADB and CBD, have the sides of the one equal to the sides of the other, each to each; and therefore, the triangles are equal in all respects: hence, the angle ABD is equal to the angle CDB (P. X., S.); and consequently, AB is parallel to DC (P. XIX., C. 1). The angle DBC is also equal to the angle BDA, and consequently, BC is parallel to AD: hence, the opposite sides are parallel, two and two; that is, the figure is a parallelogram (D. 28); *which was to be proved.*

PROPOSITION XXX. THEOREM.

If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.



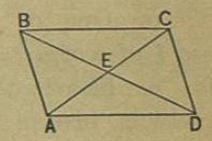
In the quadrilateral ABCD, let AB be equal and parallel to DC: then the figure is a parallelogram.

Draw the diagonal DB. Then, because AB and DC are parallel, the angle ABD is equal to its alternate angle CDB. Now, the triangles ABD and CDB have the side DC equal to AB, by hypothesis, the side DB common, and the included angle ABD equal to BDC, from what has just been shown; hence, the triangles are equal in all respects (P. V.); and consequently, the alternate angles ADB and DBC are equal. The sides BC and AD are, therefore, parallel, and the figure is a parallelogram; *which was to be proved.*

Cor. If two points are taken at equal distances from a given straight line, and on the same side of it, the straight line joining them is parallel to the given line.

PROPOSITION XXXI. THEOREM.

The diagonals of a parallelogram divide each other into equal parts, or mutually bisect each other.



Let ABCD be a parallelogram, and AC, BD, its diagonals: then AE is equal to EC, and BE to ED.

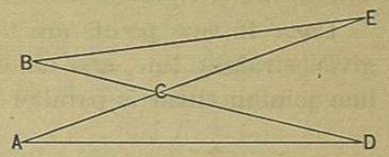
For, the triangles BEC and AED, have the angles EBC and ADE equal (P. XX., C. 2), the angles ECB and DAE equal, and the included sides BC and AD equal: hence, the triangles are equal in all respects (P. VI.); consequently, AE is equal to EC, and BE to ED; *which was to be proved.*

Scholium. In a rhombus, the sides AB, BC, being equal, the triangles AEB, EBC, have the sides of the one equal to the corresponding sides of the other; they are, therefore, equal in all respects: hence, the angles AEB, BEC, are equal, and therefore, the two diagonals bisect each other at right angles.

EXERCISES.

1. Show that the lines which bisect (*halve*) two vertical angles, form one and the same straight line.

2. Given two lines, BE and AD; join B with D and A with E, and show that BD + AE is greater than BE + AD. (P. VII.)

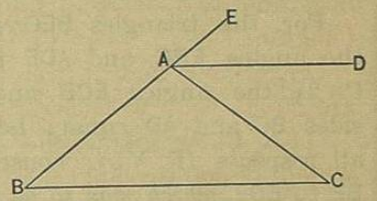


3. One of the two interior angles on the same side, formed by a straight line meeting two parallels, is one-half of a right angle; what is the other angle equal to?

4. The sum of two angles of a triangle is $\frac{4}{3}$ of a right angle; what is the other angle equal to?

5. One of the acute angles of a right-angled triangle is $\frac{2}{5}$ of a right angle; what is the other?

6. Show that the line which bisects the exterior vertical angle of an isosceles triangle is parallel to the base of the triangle. (P. XXV., C. 6; P. XIX., C. 1.)



7. The sum of the interior angles of a polygon is 12 right angles; what is the polygon?

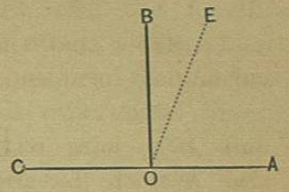
8. What is the sum of the interior angles of a heptagon equal to?

9. The sum of five angles of a given equiangular polygon is 8 right angles; what is the polygon?

10. What part of a right angle is an angle of an equiangular decagon?

11. How many sides has a polygon in which the sum of the interior angles is equal to the sum of the exterior angles?

12. Construct a square, having given one of its diagonals.



NOTE 1.—The *complement* of an angle is the difference between that angle and a right angle; thus, EOB is the complement of AOE.

NOTE 2.—The *supplement* of an angle is the difference between that angle and two right angles; thus, EOC is the supplement of AOE.

13. An angle is $\frac{6}{7}$ of a right angle; what is its complement? and what its supplement?

14. Show that any two adjacent angles of a parallelogram are supplements of each other.

15. Show that if two parallelograms have one angle in each equal, their remaining angles are equal each to each.

16. Show that if two sides of a quadrilateral are parallel and two opposite angles equal, the figure is a parallelogram.

17. Show that if the opposite angles of a quadrilateral are equal, each to each, the figure is a parallelogram.

18. Show that the lines which bisect the angles of any quadrilateral form, by their intersection, another quadrilateral, the opposite angles of which are supplements of each other. [Twice the angle B is equal to the sum of the angles CDE and DEF.]