BOOK II.

RATIOS AND PROPORTIONS.

DEFINITIONS.

1. The Ratio of one quantity to another of the same kind. is the quotient obtained by dividing the second by the first. The first quantity is called the Antecedent, and the second, the Consequent.

2. A Proportion is an expression of equality between two equal ratios. Thus,

$$\frac{B}{A} = \frac{D}{C},$$

expresses the fact that the ratio of A to B is equal to the ratio of C to D. In Geometry, the proportion is written thus,

$$A : B :: C : D,$$

and read, A is to B, as C is to D.

3. A Continued Proportion is one in which several ratios are successively equal to each other; as,

$$A : B :: C : D :: E : F :: G : H, \text{ \&c.}$$

4. There are four terms in every proportion. The first and second form the *first couplet*, and the third and fourth,

the *second couplet*. The first and fourth terms are called *extremes;* the second and third, *means*, and the fourth term, a *fourth proportional* to the three others. When the second term is equal to the third, it is said to be a *mean proportional* between the extremes. In this case, there are but three different quantities in the proportion, and the last is said to be a *third proportional to the two others.* Thus, if we have,

$$A : B :: B : C,$$

B is a *mean* proportional between A and C, and C is a *third* proportional to A and B.

5. Quantities are in proportion by *alternation*, when antecedent is compared with antecedent, and consequent with consequent.

6. Quantities are in proportion by *inversion*, when antecedents are made consequents, and consequents, antecedents.

7. Quantities are in proportion by *composition*, when the sum of antecedent and consequent is compared with either antecedent or consequent.

8. Quantities are in proportion by *division*, when the difference of the antecedent and consequent is compared with either antecedent or consequent.

9. Four quantities are *reciprocally* proportional, when the first is to the second as the fourth is to the third. *Two varying* quantities are reciprocally proportional, when their product is a fixed quantity, as $xy = m$.

10. Equimultiples of two or more quantities, are the products obtained by multiplying each by the same quantity. Thus, mA and mB, are equimultiples of A and B.

PROPOSITION I. THEOREM.

If four quantities are in proportion, the product of the means is equal to the product of the extremes.

Assume the proportion,

$$A : B :: C : D; \quad \text{whence} \quad \frac{B}{A} = \frac{D}{C};$$

clearing of fractions, we have,

$$BC = AD;$$

which was to be proved.

Cor. If B is equal to C, there are but three proportional quantities; in this case, *the square of the mean is equal to the product of the extremes.*

PROPOSITION II. THEOREM.

If the product of two factors is equal to the product of two other factors, either pair of factors may be made the extremes and the other pair the means of a proportion.

Assume

$$B \times C = A \times D;$$

dividing each member by $A \times C$, we have,

$$\frac{B}{A} = \frac{D}{C}, \quad \text{or} \quad A : B :: C : D;$$

in like manner, we have,

$$\frac{A}{B} = \frac{C}{D}, \quad \text{or} \quad B : A :: D : C;$$

which was to be proved.

PROPOSITION III. THEOREM.

If four quantities are in proportion, they are in proportion by alternation.

Assume the proportion,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C}.$$

Multiplying each member by $\frac{C}{B}$, we have,

$$\frac{C}{A} = \frac{D}{B}; \quad \text{or} \quad A : C :: B : D;$$

which was to be proved.

PROPOSITION IV. THEOREM.

If one couplet in each of two proportions is the same, the other couplets form a proportion.

Assume the proportions,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C};$$

and $\quad A : B :: F : G; \quad$ whence, $\quad \frac{B}{A} = \frac{G}{F}.$

From Axiom 1, we have,

$$\frac{D}{C} = \frac{G}{F}; \quad \text{whence,} \quad C : D :: F : G;$$

which was to be proved.

Cor. If the antecedents, in two proportions, are the same, the consequents are proportional. For, the antecedents of the second couplets may be made the consequents of the first, by alternation (P. III.).

PROPOSITION V. THEOREM.

If four quantities are in proportion, they are in proportion by inversion.

Assume the proportion,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C}.$$

If we take the reciprocals of each member (A. 7), we have,

$$\frac{A}{B} = \frac{C}{D}; \quad \text{whence,} \quad B : A :: D : C;$$

which was to be proved.

PROPOSITION VI. THEOREM.

If four quantities are in proportion, they are in proportion by composition or division.

Assume the proportion,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C}.$$

If we add 1 to each member, and subtract 1 from each member, we have,

$$\frac{B}{A} + 1 = \frac{D}{C} + 1; \quad \text{and} \quad \frac{B}{A} - 1 = \frac{D}{C} - 1;$$

whence, by reducing to a common denominator, we have,

$$\frac{B+A}{A} = \frac{D+C}{C}, \quad \text{and} \quad \frac{B-A}{A} = \frac{D-C}{C}; \quad \text{whence,}$$

$$A : B+A :: C : D+C, \quad \text{and} \quad A : B-A :: C : D-C;$$

which was to be proved.

PROPOSITION VII. THEOREM.

Equimultiples of two quantities are proportional to the quantities themselves.

Let A and B be any two quantities; then $\frac{B}{A}$ will denote their ratio.

If we multiply each term of this fraction by m, its value will not be changed; and we shall have,

$$\frac{mB}{mA} = \frac{B}{A}; \quad \text{whence,} \quad mA : mB :: A : B;$$

which was to be proved.

PROPOSITION VIII. THEOREM.

If four quantities are in proportion, any equimultiples of the first couplet are proportional to any equimultiples of the second couplet.

Assume the proportion,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C}.$$

If we multiply each term of the first member by m, and each term of the second member by n, we have,

$$\frac{mB}{mA} = \frac{nD}{nC}; \quad \text{whence,} \quad mA : mB :: nC : nD;$$

which was to be proved.

PROPOSITION IX. THEOREM.

If two quantities are increased or diminished by like parts of each, the results are proportional to the quantities themselves.

We have, Prop. VII.,

$$A : B :: mA : mB.$$

If we make $m = 1 \pm \frac{p}{q}$, in which $\frac{p}{q}$ is any fraction, we have,

$$A : B :: A \pm \frac{p}{q}A : B \pm \frac{p}{q}B;$$

which was to be proved.

PROPOSITION X. THEOREM.

If both terms of the first couplet of a proportion are increased or diminished by like parts of each; and if both terms of the second couplet are increased or diminished by any other like parts of each, the results are in proportion.

Since we have, Prop. VIII.,

$$mA : mB :: nC : nD;$$

if we make $m = 1 \pm \frac{p}{q}$, and $n = 1 \pm \frac{p'}{q'}$, we have,

$$A \pm \frac{p}{q}A : B \pm \frac{p}{q}B :: C \pm \frac{p'}{q'}C : D \pm \frac{p'}{q'}D;$$

which was to be proved.

PROPOSITION XI. THEOREM.

In any continued proportion, the sum of the antecedents is to the sum of the consequents, as any antecedent to its corresponding consequent.

From the definition of a continued proportion (D. 3),

$$A : B :: C : D :: E : F :: G : H, \text{ \&c.};$$

hence,

$$\frac{B}{A} = \frac{B}{A}; \quad \text{whence,} \quad BA = AB;$$

$$\frac{B}{A} = \frac{D}{C}; \quad \text{whence,} \quad BC = AD;$$

$$\frac{B}{A} = \frac{F}{E}; \quad \text{whence,} \quad BE = AF;$$

$$\frac{B}{A} = \frac{H}{G}; \quad \text{whence,} \quad BG = AH;$$

&c., &c.

Adding and factoring, we have,

$$B(A + C + E + G + \text{\&c.}) = A(B + D + F + H + \text{\&c.}):$$

hence, from Proposition II.,

$$A + C + E + G + \text{\&c.} : B + D + F + H + \text{\&c.} :: A : B;$$

which was to be proved.

PROPOSITION XII. THEOREM.

The products of the corresponding terms of two proportions are proportional.

Assume the two proportions,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C};$$

and $$E : F :: G : H; \quad \text{whence,} \quad \frac{F}{E} = \frac{H}{G}.$$

Multiplying the equations, member by member, we have,

$$\frac{BF}{AE} = \frac{DH}{CG}; \quad \text{whence,} \quad AE : BF :: CG : DH;$$

which was to be proved.

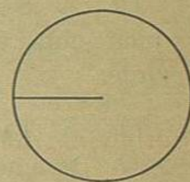
Cor. 1. If the corresponding terms of two proportions are equal, each term of the resulting proportion is the square of the corresponding term in either of the given proportions: hence, *If four quantities are proportional, their squares are proportional.*

Cor. 2. If the principle of the proposition be extended to three or more proportions, and the corresponding terms of each be supposed equal, it will follow that, *like powers of proportional quantities are proportionals.*

BOOK III.

THE CIRCLE AND THE MEASUREMENT OF ANGLES.

DEFINITIONS.



1. A CIRCLE is a plane figure, bounded by a curved line, every point of which is equally distant from a point within, called the *centre.*

The bounding line is called the *circumference.*

2. A RADIUS is a straight line drawn from the centre to any point of the circumference.

3. A DIAMETER is a straight line drawn through the centre and terminating in the circumference.

All radii of the same circle are equal. All diameters are also equal, and each is double the radius.

4. An ARC is any part of a circumference.

5. A CHORD is a straight line joining the extremities of an arc.

Any chord belongs to two arcs: the smaller one is meant, unless the contrary is expressed.

6. A SEGMENT is a part of a circle included between an arc and its chord.

7. A SECTOR is a part of a circle included between an arc and the two radii drawn to its extremities.