PROPOSITION XII. THEOREM.

The products of the corresponding terms of two proportions are proportional.

Assume the two proportions,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C};$$

and

$$E : F :: G : H; \quad \text{whence,} \quad \frac{F}{E} = \frac{H}{G}.$$

Multiplying the equations, member by member, we have,

$$\frac{BF}{AE} = \frac{DH}{CG}; \quad \text{whence,} \quad AE : BF :: CG : DH;$$

which was to be proved.

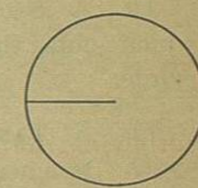
Cor. 1. If the corresponding terms of two proportions are equal, each term of the resulting proportion is the square of the corresponding term in either of the given proportions: hence, *If four quantities are proportional, their squares are proportional.*

Cor. 2. If the principle of the proposition be extended to three or more proportions, and the corresponding terms of each be supposed equal, it will follow that, *like powers of proportional quantities are proportionals.*

BOOK III.

THE CIRCLE AND THE MEASUREMENT OF ANGLES.

DEFINITIONS.



1. A CIRCLE is a plane figure, bounded by a curved line, every point of which is equally distant from a point within, called the *centre.*

The bounding line is called the *circumference.*

2. A RADIUS is a straight line drawn from the centre to any point of the circumference.

3. A DIAMETER is a straight line drawn through the centre and terminating in the circumference.

All radii of the same circle are equal. All diameters are also equal, and each is double the radius.

4. An ARC is any part of a circumference.

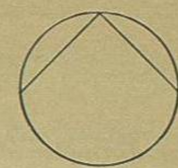
5. A CHORD is a straight line joining the extremities of an arc.

Any chord belongs to two arcs: the smaller one is meant, unless the contrary is expressed.

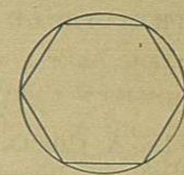
6. A SEGMENT is a part of a circle included between an arc and its chord.

7. A SECTOR is a part of a circle included between an arc and the two radii drawn to its extremities.

8. An INSCRIBED ANGLE is an angle whose vertex is in the circumference, and whose sides are chords.

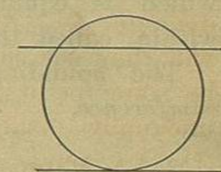


9. An INSCRIBED POLYGON is a polygon whose vertices are all in the circumference. The sides are chords.

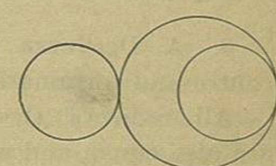


10. A SECANT is a straight line which cuts the circumference in two points.

11. A TANGENT is a straight line which touches the circumference in one point only. This point is called, the *point of contact*, or the *point of tangency.*

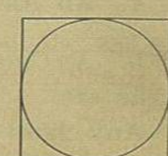


12. Two circles are *tangent to each other*, when they touch each other in one point only. This point is called, the *point of contact*, or the *point of tangency.*



13. A Polygon is *circumscribed about a circle*, when each of its sides is tangent to the circumference.

14. A Circle is *inscribed in a polygon*, when its circumference touches each of the sides of the polygon.



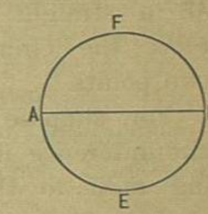
POSTULATE.

A circumference can be described from any point as a *centre*, and with any *radius.*

PROPOSITION I. THEOREM.

Any diameter divides the circle, and also its circumference, into two equal parts.

Let AEBF be a circle, and AB any diameter: then will it divide the circle and its circumference into two equal parts.



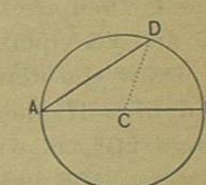
For, let AFB be applied to AEB, the diameter AB remaining common; then will they coincide; otherwise there would be some points in either one or the other of the curves unequally distant from the centre; which is impossible (D. 1): hence, AB divides the circle, and also its circumference, into two equal parts; *which was to be proved.*

PROPOSITION II. THEOREM.

A diameter is greater than any other chord.

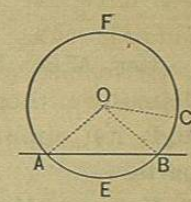
Let AD be a chord, and AB a diameter through one extremity, as A: then will AB be greater than AD.

Draw the radius CD. In the triangle ACD, we have AD less than the sum of AC and CD (B. I., P. VII.). But this sum is equal to AB (D. 3): hence, AB is greater than AD; *which was to be proved.*



PROPOSITION III. THEOREM.

A straight line can not meet a circumference in more than two points.

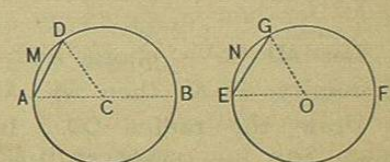


Let AEBF be a circumference, and AB a straight line: then AB can not meet the circumference in more than two points.

For, suppose that AB could meet the circumference in three points. By drawing radii to these points, we should have three equal straight lines drawn from the same point to the same straight line; which is impossible (B. I., P. XV., C. 2): hence, AB can not meet the circumference in more than two points; *which was to be proved.*

PROPOSITION IV. THEOREM.

In equal circles, equal arcs are subtended by equal chords; and conversely, equal chords subtend equal arcs.



1°. In the equal circles ADB and EGF, let the arcs AMD and ENG be equal: then are the chords AD and EG equal.

Draw the diameters AB and EF. If the semicircle ADB be applied to the semicircle EGF, it will coincide with it, and the semi-circumference ADB will coincide with the semi-circumference EGF. But the part AMD is equal to the part ENG, by hypothesis: hence, the point D will fall on G; therefore,

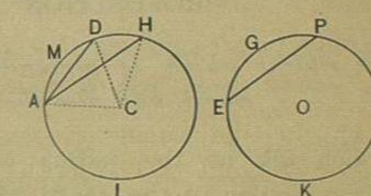
the chord AD will coincide with EG (A. 11), and is, therefore, equal to it; *which was to be proved.*

2°. Let the chords AD and EG be equal: then will the arcs AMD and ENG be equal.

Draw the radii CD and OG. The triangles ACD and EOG have all the sides of the one equal to the corresponding sides of the other; they are, therefore, equal in all respects: hence, the angle ACD is equal to EOG. If, now, the sector ACD be placed upon the sector EOG, so that the angle ACD shall coincide with the angle EOG, the sectors will coincide throughout; and, consequently, the arcs AMD and ENG will coincide: hence, they are equal; *which was to be proved.*

PROPOSITION V. THEOREM.

In equal circles, a greater arc is subtended by a greater chord; and conversely, a greater chord subtends a greater arc.

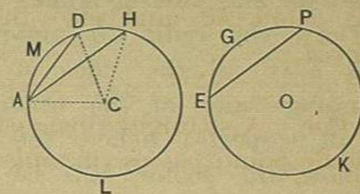


1°. In the equal circles ADL and EGK, let the arc EGP be greater than the arc AMD: then is the chord EP greater than the chord AD.

For, place the circle EGK upon AHL, so that the centre O shall fall upon the centre C, and the point E upon A; then, because the arc EGP is greater than AMD, the point P will fall at some point H, beyond D, and the chord EP will take the position AH.

Draw the radii CA, CD, and CH. Now, the sides AC, CH, of the triangle ACH, are equal to the sides AC, CD, of the triangle ACD, and the angle ACH is greater than ACD: hence, the side AH, or its equal EP, is greater than the side AD (B. I., P. IX.); *which was to be proved.*

2°. Let the chord EP, or its equal AH, be greater than AD: then is the arc EGP, or its equal ADH, greater than AMD.

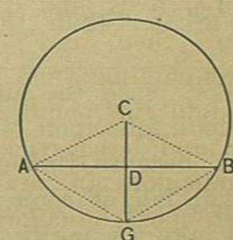


For, if ADH were equal to AMD, the chord AH would be equal to the chord AD (P. IV.); which contradicts the hypothesis. And, if the arc ADH were less than AMD, the chord AH would be less than AD; which also contradicts the hypothesis. Then, since the arc ADH, subtended by the greater chord, can neither be equal to, nor less than AMD, it must be greater than AMD; *which was to be proved.*

PROPOSITION VI. THEOREM.

6 *The radius which is perpendicular to a chord, bisects that chord, and also the arc subtended by it.*

Let CG be the radius which is perpendicular to the chord AB: then this radius bisects the chord AB, and also the arc AGB.



For, draw the radii CA and CB. Then, the right-angled triangles CDA and CDB have the hypothenuse CA equal to CB, and the side CD common; the triangles are, therefore, equal in all respects: hence, AD is equal to DB. Again, because CG is perpendicular to AB, at its middle point, the chords GA and GB are equal (B. I., P. XVI.); and consequently, the arcs GA and GB are also equal (P. IV.): hence, CG bisects the chord AB, and also the arc AGB; *which was to be proved.*

Cor. A straight line, perpendicular to a chord, at its middle point, passes through the centre of the circle.

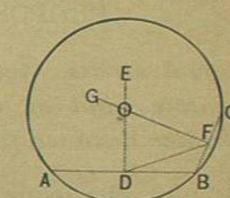
Scholium. The centre C, the middle point D of the chord AB, and the middle point G of the subtended arc, are points of the radius perpendicular to the chord. But two points determine the position of a straight line (A. 11): hence, any straight line which passes through two of these points, passes through the third, and is perpendicular to the chord.

PROPOSITION VII. THEOREM.

Through any three points, not in the same straight line, one circumference may be made to pass, and but one.

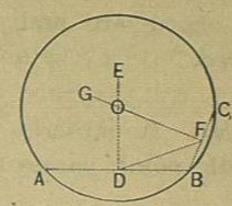
Let A, B, and C, be any three points, not in a straight line: then may one circumference be made to pass through them, and but one.

Join the points by the lines AB, BC, and bisect these lines by perpendiculars DE and FG: then will these perpendiculars meet in some point O. For, if they do not meet, they are parallel. Draw DF. The sum of the angles EDF and GFD is less than the sum of the angles EDB and GFB, *i. e.*,



less than two right angles: therefore, DE and FG are not parallel, and will meet at some point, as O (B. I., P. XXI.)

Now, O is on a perpendicular to AB at its middle point; it is, therefore, equally distant from A and B (B. I., P. XVI.). For a like reason, O is equally distant from B and C. If, therefore, a circumference be described from O as a centre, with a radius equal to the distance from O to A, it will pass through A, B, and C.



Again, O is the only point which is equally distant from A, B, and C: for, DE contains all of the points which are equally distant from A and B; and FG all of the points which are equally distant from B and C; and consequently, their point of intersection O, is the only point that is equally distant from A, B, and C: hence, one circumference may be made to pass through these points, and but one; *which was to be proved.*

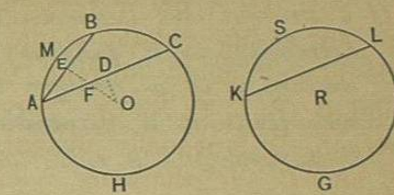
Cor. Two circumferences can not intersect in more than two points; for, if they could intersect in three points, there would be two circumferences passing through the same three points; which is impossible.

PROPOSITION VIII. THEOREM.

7 *In equal circles, equal chords are equally distant from the centres; and of two unequal chords, the less is at the greater distance from the centre.*

1°. In the equal circles ACH and KLG, let the chords AC and KL be equal; then are they equally distant from the centres.

For, let the circle KLG be placed upon ACH, so that the centre R shall fall upon the centre O, and the point K upon the point A: then will the chord KL coincide with AC (P. IV.); and consequently, they are equally distant from the centre; *which was to be proved.*



2°. Let AB be less than KL: then is it at a greater distance from the centre.

For, place the circle KLG upon ACH, so that R shall fall upon O, and K upon A. Then, because the chord KL is greater than AB, the arc KSL is greater than AMB; and consequently, the point L will fall at a point C, beyond B, and the chord KL will take the direction AC.

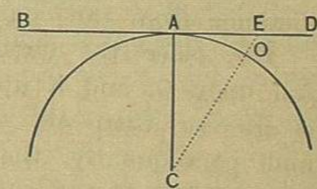
Draw OD and OE, respectively perpendicular to AC and AB; then OE is greater than OF (A. 8), and OF than OD (B. I., P. XV.): hence, OE is greater than OD. But, OE and OD are the distances of the two chords from the centre (B. I., P. XV., C. 1): hence, the less chord is at the greater distance from the centre; *which was to be proved.*

Scholium. All the propositions relating to chords and arcs of equal circles, are also true for chords and arcs of one and the same circle. For, any circle may be regarded as made up of two equal circles, so placed that they coincide in all their parts.

PROPOSITION IX. THEOREM.

If a straight line is perpendicular to a radius at its outer extremity, it is tangent to the circle at that point; conversely, if a straight line is tangent to a circle at any point, it is perpendicular to the radius drawn to that point.

1°. Let BD be perpendicular to the radius CA, at A: then is it tangent to the circle at A.



For, take any other point of BD, as E, and draw CE: then CE is greater than CA (B. I., P. XV.); and consequently, the point E lies without the circle: hence, BD touches the circumference at the point A; it is, therefore, tangent to it at that point (D. 11); *which was to be proved.*

2°. Let BD be tangent to the circle at A: then is it perpendicular to CA.

For, let E be any point of the tangent, except the point of contact, and draw CE. Then, because BD is a tangent, E lies without the circle; and consequently, CE is greater than CA: hence, CA is shorter than any other line that can be drawn from C to BD; it is, therefore, perpendicular to BD (B. I., P. XV., C. 1); *which was to be proved.*

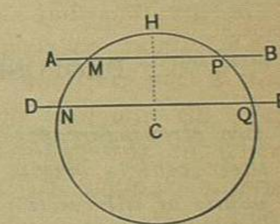
Cor. At a given point of a circumference, only one tangent can be drawn. For, if two tangents could be drawn, they would both be perpendicular to the same radius at the same point; which is impossible (B. I., P. XIV.).

PROPOSITION X. THEOREM.

Two parallels intercept equal arcs of a circumference.

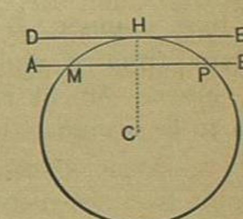
There may be three cases: both parallels may be secants; one may be a secant and the other a tangent; or, both may be tangents.

1°. Let the secants AB and DE be parallel: then the intercepted arcs MN and PQ are equal.



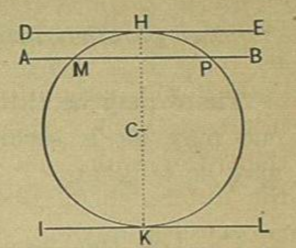
For, draw the radius CH perpendicular to the chord MP; it is also perpendicular to NQ (B. I., P. XX., C. 1), and H is at the middle point of the arc MHP, and also of the arc NHQ: hence, MN, which is the difference of HN and HM, is equal to PQ, which is the difference of HQ and HP (A. 3); *which was to be proved.*

2°. Let the secant AB and tangent DE be parallel; then the intercepted arcs MH and PH are equal.



For, draw the radius CH to the point of contact H; it will be perpendicular to DE (P. IX.), and also to its parallel MP. But, because CH is perpendicular to MP, H is the middle point of the arc MHP (P. VI.): hence, MH and PH are equal; *which was to be proved.*

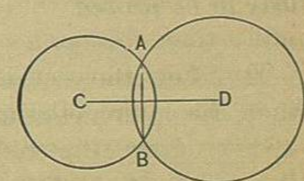
3°. Let the tangents DE and IL be parallel, and let H and K be their points of contact: then the intercepted arcs HMK and HPK are equal.



For, draw the secant AB parallel to DE; then, from what has just been shown, we have HM equal to HP, and MK equal to PK: hence, HMK, which is the sum of HM and MK, is equal to HPK, which is the sum of HP and PK; *which was to be proved.*

PROPOSITION XI. THEOREM.

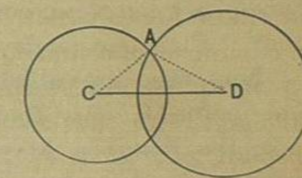
If two circumferences intersect each other, the line joining their centres bisects at right angles the line joining the points of intersection.



Let the circumferences, whose centres are C and D, intersect at the points A and B: then CD bisects AB at right angles. For the point C, being the centre of the circle, is equally distant from A and B; in like manner, D is equally distant from A and B: hence, CD bisects AB at right angles (B. I., P. XVI., C.); *which was to be proved.*

PROPOSITION XII. THEOREM.

If two circumferences intersect each other, the distance between their centres is less than the sum, and greater than the difference, of their radii.



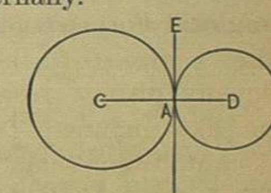
Let the circumferences, whose centres are C and D, intersect at A: then CD is less than the sum, and greater than the difference of the radii of the two circles.

For, draw AC and AD, forming the triangle ACD. Then CD is less than the sum of AC and AD, and greater than their difference (B. I., P. VII.); *which was to be proved.*

PROPOSITION XIII. THEOREM.

If the distance between the centres of two circles is equal to the sum of their radii, the circles are tangent externally.

Let C and D be the centres of two circles, and let the distance between the centres be equal to the sum of the radii: then the circles are tangent externally.

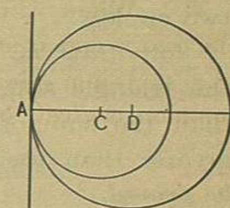


For, they have at least one point, A, on the line CD, common; for, if not, the distance between their centres would be greater than the sum of their radii, which contradicts the hypothesis, and is, therefore, impossible. Again, they have no other point in common; for, if they had two points in common, the distance between their centres would be less than the sum of their radii, which contradicts the hypothesis: hence, they have one and only one point in common, and are tangent externally; *which was to be proved.*

PROPOSITION XIV. THEOREM.

If the distance between the centres of two circles is equal to the difference of their radii, one circle is tangent to the other internally.

Let C and D be the centres of two circles, and let the distance between these centres be equal to the difference of the radii: then one circle is tangent to the other internally.



For, the circles will have at least one point, A, on DC, common; for, if not, the distance between the centres would be less than the difference of their radii, which contradicts the hypothesis. Again, they will have no other point in common; for, if they had two points in common, the distance between their centres would be greater than the difference of their radii, which contradicts the hypothesis: hence, they have one and only one point in common, and one is tangent to the other internally; *which was to be proved.*

Cor. 1. If two circles are tangent, either externally or internally, the point of contact is on the straight line drawn through their centres.

Cor. 2. All circles whose centres are on the same straight line, and which pass through a common point of that line, are tangent to each other at that point. And if a straight line be drawn tangent to one of the circles at their common point, it is tangent to them all at that point.

Scholium. From the preceding propositions, we infer that two circles may have any one of six positions with respect to each other, depending upon the distance between their centres:

1°. When the distance between their centres is greater

than the sum of their radii, *they are external, one to the other:*

2°. When this distance is equal to the sum of the radii, *they are tangent*, externally:

3°. When this distance is less than the sum, and greater than the difference of the radii, *they intersect each other:*

4°. When this distance is equal to the difference of their radii, *one is tangent to the other*, internally:

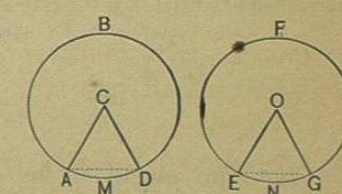
5°. When this distance is less than the difference of the radii, *one is wholly within the other:*

6°. When this distance is equal to zero, *they have a common centre;* or, *they are concentric.*

PROPOSITION XV. THEOREM.

In equal circles, radii making equal angles at the centre, intercept equal arcs of the circumference; conversely, radii which intercept equal arcs, make equal angles at the centre.

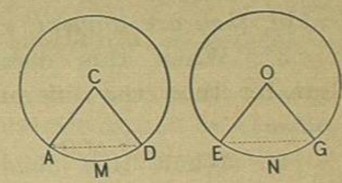
1°. In the equal circles ADB and EGF, let the angles ACD and EOG be equal: then the arcs AMD and ENG are equal.



For, draw the chords AD and EG; then the triangles ACD and EOG have two sides and their included angle, in the one, equal to two sides and their included angle, in the other, each to each. They are, therefore, equal in all respects; consequently, AD is equal to EG. But, since the chords AD and EG are equal, the arcs AMD and ENG are also equal (P. IV.); *which was to be proved.*

2°. Let the arcs AMD and ENG be equal: then the angles ACD and EOG are equal.

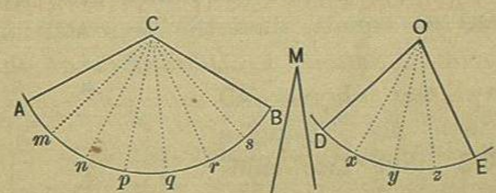
For, since the arcs AMD and ENG are equal, the chords AD and EG are equal (P. IV.); consequently, the triangles ACD and EOG have their sides equal, each to each; they are, therefore, equal in all respects: hence, the angle ACD is equal to the angle EOG; *which was to be proved.*



PROPOSITION XVI. THEOREM.

In equal circles, commensurable angles at the centre are proportional to their intercepted arcs.

In the equal circles, whose centres are C and O, let the angles ACB and DOE be commensurable; that is, be exactly measured by a common unit: then are they proportional to the intercepted arcs AB and DE.



Let the angle M be a common unit; and suppose, for example, that this unit is contained 7 times in the angle ACB, and 4 times in the angle DOE. Then, suppose ACB be divided into 7 angles, by the radii Cm, Cn, Cp, &c.; and DOE into 4 angles, by the radii Ox, Oy, and Oz, each equal to the unit M.

From the last proposition, the arcs Am, mn, &c., Dx, xy, &c., are equal to each other; and because there are 7 of these arcs in AB, and 4 in DE, we shall have,

arc AB : arc DE :: 7 : 4.

But, by hypothesis, we have,

angle ACB : angle DOE :: 7 : 4;

hence, from (B. II., P. IV.), we have,

angle ACB : angle DOE :: arc AB : arc DE.

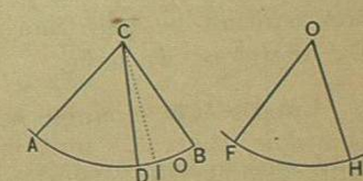
If any other numbers than 7 and 4 had been used, the same proportion would have been found; *which was to be proved.*

Cor. If the intercepted arcs are commensurable, they are proportional to the corresponding angles at the centre, as may be shown by changing the order of the couplets in the above proportion.

PROPOSITION XVII. THEOREM.

In equal circles, incommensurable angles at the centre are proportional to their intercepted arcs.

In the equal circles, whose centres are C and O, let ACB and FOH be incommensurable: then are they proportional to the arcs AB and FH.



For, let the less angle FOH, be placed upon the greater angle ACB, so that it shall take the position ACD. Then,