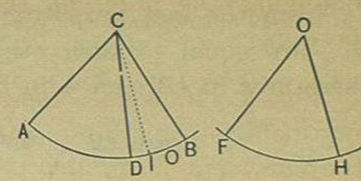
if the proposition is not true, let us suppose that the angle ACB is to the angle FOH, or its equal ACD, as the arc AB is to an arc AO, greater than FH, or its equal AD; whence,



angle ACB : angle ACD : : arc AB : arc AO.

Conceive the arc AB to be divided into equal parts, each less than DO: there will be at least one point of division between D and O; let I be that point; and draw CI. Then the arcs AB, AI, will be commensurable, and we shall have (P. XVI.),

angle ACB : angle ACI : : arc AB : arc AI.

Comparing the two proportions, we see that the antecedents are the same in both: hence, the consequents are proportional (B. II., P. IV., C.); hence,

angle ACD : angle ACI : : arc AO : arc AI.

But, AO is greater than AI: hence, if this proportion is true, the angle ACD must be greater than the angle ACI. On the contrary, it is less: hence, the fourth term of the assumed proportion can not be greater than AD.

In a similar manner, it may be shown that the fourth term can not be less than AD: hence, it must be equal to AD; therefore, we have,

angle ACB : angle ACD : : arc AB : arc AD;

which was to be proved.

Cor. 1. The intercepted arcs are proportional to the corresponding angles at the centre, as may be shown by changing the order of the couplets in the preceding proportion.

Cor. 2. In equal circles, angles at the centre are proportional to their intercepted arcs, and the reverse, whether they are commensurable or incommensurable.

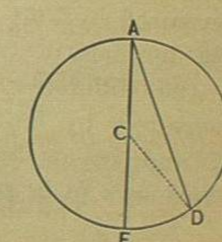
Cor. 3. In equal circles, sectors are proportional to their angles, and also to their arcs.

Scholium. Since the intercepted arcs are proportional to the corresponding angles at the centre, the arcs may be taken as the measures of the angles. That is, if a circumference be described from the vertex of any angle, as a centre, and with a fixed radius, the arc intercepted between the sides of the angle may be taken as the measure of the angle. In Geometry, the right angle, which is measured by a quarter of a circumference, or a *quadrant*, is taken as a unit. If, therefore, any angle is measured by one half or two thirds of a quadrant, it is equal to one half or two thirds of a right angle.

PROPOSITION XVIII. THEOREM.

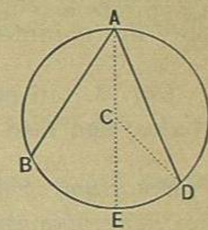
An inscribed angle is measured by half of the arc included between its sides.

There may be three cases: the centre of the circle may lie on one of the sides of the angle; it may lie within the angle; or, it may lie without the angle.



1°. Let EAD be an inscribed angle, one of whose sides AE passes through the centre: then it is measured by half of the arc DE.

For, draw the radius CD. The external angle DCE, of the triangle DCA, is equal to the sum of the opposite interior angles CAD and CDA (B. I., P. XXV., C. 6). But, the triangle DCA being isosceles, the angles D and A are equal; therefore, the angle DCE is double the angle DAE. Because DCE is at the centre, it is measured by the arc DE (P. XVII., S.): hence, the angle DAE is measured by half of the arc DE; *which was to be proved.*

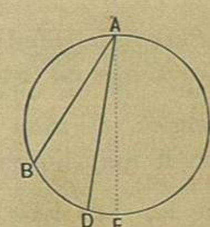


2°. Let DAB be an inscribed angle, and let the centre lie within it: then the angle is measured by half of the arc BED.

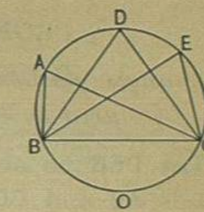
For, draw the diameter AE. Then, from what has just been proved, the angle DAE is measured by half of DE, and the angle EAB by half of EB: hence, BAD, which is the sum of EAB and DAE, is measured by half of the sum of DE and EB, or by half of BED; *which was to be proved.*

3°. Let BAD be an inscribed angle, and let the centre lie without it: then it is measured by half of the arc BD.

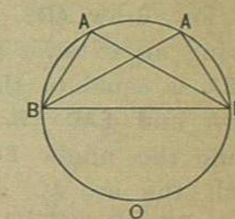
For, draw the diameter AE. Then, from what precedes, the angle DAE is measured by half of DE, and the angle BAE by half of BE: hence, BAD, which is the difference of BAE and DAE, is measured by half of the difference of BE and DE, or by half of the arc BD; *which was to be proved.*



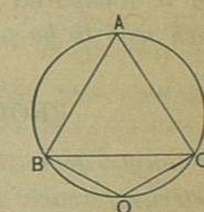
Cor. 1. All the angles BAC, BDC, BEC, inscribed in the same segment, are equal; because they are each measured by half of the same arc BOC.



Cor. 2. Any angle BAD, inscribed in a semicircle, is a right angle; because it is measured by half the semi-circumference BOD, or by a quadrant (P. XVII., S.).

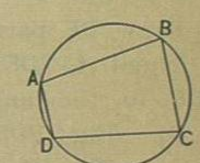


Cor. 3. Any angle BAC, inscribed in a segment greater than a semicircle, is acute; for it is measured by half the arc BOC, less than a semi-circumference.



Any angle BOC, inscribed in a segment less than a semicircle, is obtuse; for it is measured by half the arc BAC, greater than a semi-circumference.

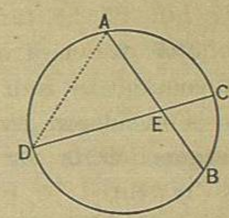
Cor. 4. The opposite angles A and C, of an inscribed quadrilateral ABCD, are together equal to two right angles; for the angle DAB is measured by half the arc DCB, the angle DCB by half the arc DAB: hence, the two angles, taken together, are measured by half the circumference: hence, their sum is equal to two right angles.



PROPOSITION XIX. THEOREM.

Any angle formed by two chords, which intersect, is measured by half the sum of the included arcs.

Let DEB be an angle formed by the intersection of the chords AB and CD: then it is measured by half the sum of the arcs AC and DB.

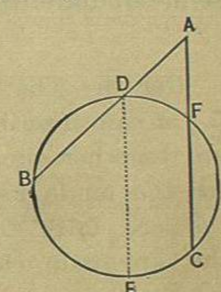


For, draw AD: then, the angle DEB, being an exterior angle of the triangle DEA, is equal to the sum of the angles EDA and EAD (B. I., P. XXV., C. 6). But, the angle EDA is measured by half the arc AC, and EAD by half the arc DB (P. XVIII.): hence, the angle DEB is measured by half the sum of the arcs AC and DB; *which was to be proved.*

PROPOSITION XX. THEOREM.

8 *The angle formed by two secants, intersecting without the circumference, is measured by half the difference of the included arcs.*

Let AB, AC, be two secants: then the angle BAC is measured by half the difference of the arcs BC and DF.

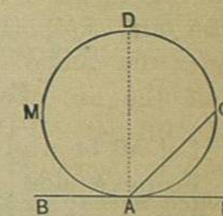


Draw DE parallel to AC: the arc EC is equal to DF (P. X.), and the angle BDE to the angle BAC (B. I., P. XX., C. 3). But BDE is measured by half the arc BE (P. XVIII.): hence, BAC is also measured by half the arc BE; that is, by half the difference of BC and EC, or by half the difference of BC and DF; *which was to be proved.*

PROPOSITION XXI. THEOREM.

An angle formed by a tangent and a chord meeting it at the point of contact, is measured by half the included arc.

Let BE be tangent to the circle AMC, and let AC be a chord drawn from the point of contact A: then BAC is measured by half of the arc AMC.



For, draw the diameter AD. The angle BAD is a right angle (P. IX.), and is measured by half the semi-circumference AMD (P. XVII., S.); the angle DAC is measured by half of the arc DC (P. XVIII.): hence, the angle BAC, which is equal to the sum of the angles BAD and DAC, is measured by half the sum of the arcs AMD and DC, or by half of the arc AMC; *which was to be proved.*

The angle CAE, which is the difference of DAE and DAC, is measured by half the difference of the arcs DCA and DC, or by half the arc CA.

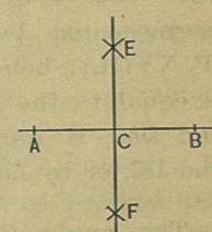
PRACTICAL APPLICATIONS.

PROBLEM I.

To bisect a given straight line.

Let AB be a given straight line.

From A and B, as centres, with a radius greater than one half of AB, describe arcs intersecting at E and F: join E and F, by the straight line EF. Then EF bisects the given line AB. For, E and F are each equally distant from A and B; and consequently, the line EF bisects AB (B. I., P. XVI., C.).

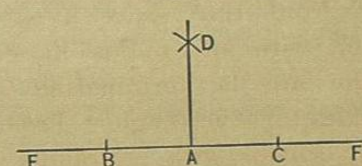


PROBLEM II.

To erect a perpendicular to a given straight line, at a given point of that line.

Let EF be a given line, and let A be a given point of that line.

From A, lay off the equal distances AB and AC; from B and C, as centres, with a radius greater than one half



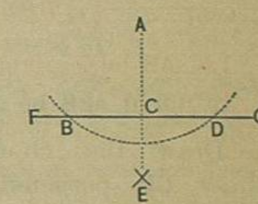
of BC, describe arcs intersecting at D; draw the line AD: then AD is the perpendicular required. For, D and A are each equally distant from B and C; consequently, DA is perpendicular to BC at the given point A (B. I., P. XVI., C.).

PROBLEM III.

To draw a perpendicular to a given straight line, from a given point without that line.

Let FG be the given line, and A the given point.

From A, as a centre, with a radius sufficiently great, describe an arc cutting FG in two points, B and D; with B and D as centres, and a radius greater than one half of BD, describe arcs intersecting at E; draw AE: then AE is the perpendicular required. For, A and E are each equally distant from B and D: hence, AE is perpendicular to BD (B. I., P. XVI., C.).

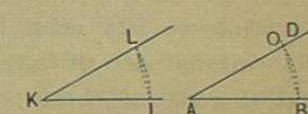


PROBLEM IV.

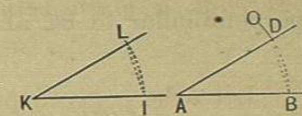
At a point on a given straight line, to construct an angle equal to a given angle.

Let A be the given point, AB the given line, and IKL the given angle.

From the vertex K as a center, with any radius KI, describe the arc IL, terminating in the sides of the angle. From A as a centre, with a radius AB, equal to KI, describe the



indefinite arc BO; then, with a radius equal to the chord LI, from B as a centre, describe an arc cutting the arc BO in D; draw AD: then BAD is equal to the angle K.



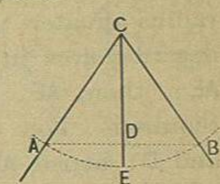
For the arcs BD, IL, have equal radii and equal chords: hence, they are equal (P. IV.); therefore, the angles BAD, IKL, measured by them, are also equal (P. XV.).

PROBLEM V.

To bisect a given arc or a given angle.

1°. Let AEB be a given arc, and C its centre.

Draw the chord AB; through C, draw CD perpendicular to AB (Prob. III.): then CD bisects the arc AEB (P. VI.).



2°. Let ACB be a given angle.

With C as a centre, and any radius CB, describe the arc BA; bisect it by the line CD, as just explained: then CD bisects the angle ACB.

For, the arcs AE and EB are equal, from what was just shown; consequently, the angles ACE and ECB are also equal (P. XV.).

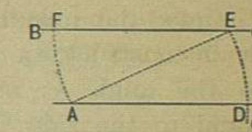
Scholium. If each half of an arc or angle is bisected, the original arc or angle is divided into four equal parts; and if each of these is bisected, the original arc or angle is divided into eight equal parts; and so on.

PROBLEM VI.

Through a given point, to draw a straight line parallel to a given straight line.

Let A be a given point, and BC a given line.

From the point A as a centre, with a radius AE, greater than the shortest distance from A to BC, describe an indefinite arc EO; from E as a centre, with the same radius, describe the arc AF; lay off ED equal to AF, and draw AD: then AD is the parallel required.



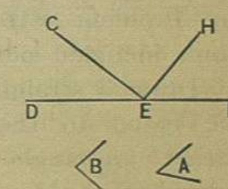
For, drawing AE, the angles AEF, EAD, are equal (P. XV.); therefore, the lines AD, EF are parallel (B. I., P. XIX., C. 1).

PROBLEM VII.

Given, two angles of a triangle, to construct the third angle.

Let A and B be given angles of a triangle.

Draw a line DF, and at some point of it, as E, construct the angle FEH equal to A, and HEC equal to B. Then, CED is equal to the required angle.



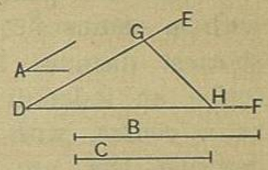
For, the sum of the three angles at E is equal to two right angles (B. I., P. I., C. 2), as is also the sum of the three angles of a triangle (B. I., P. XXV.). Consequently, the third angle CED must be equal to the third angle of the triangle.

PROBLEM VIII.

Given, two sides and the included angle of a triangle, to construct the triangle.

Let B and C denote the given sides, and A the given angle.

Draw the indefinite line DF, and at D construct an angle FDE, equal to the angle A; on DF, lay off DH equal to the side C, and on DE, lay off DG equal to the side B; draw GH: then DGH is the required triangle (B. I., P. V.).

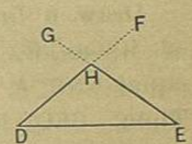


PROBLEM IX.

Given, one side and two angles of a triangle, to construct the triangle.

The two angles may be either both adjacent to the given side, or one may be adjacent and the other opposite to it. In the latter case, construct the third angle by Problem VII. We shall then have two angles and their included side.

Draw a straight line, and on it lay off DE equal to the given side; at D construct an angle equal to one of the adjacent angles, and at E construct an angle equal to the other adjacent angle; produce the sides DF and EG till they intersect at H: then DEH is the triangle required (B. I., P. VI.).

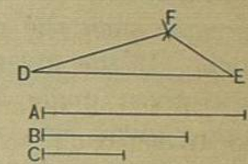


PROBLEM X.

Given, the three sides of a triangle, to construct the triangle.

Let A, B, and C, be the given sides.

Draw DE, and make it equal to the side A; from D as a centre, with a radius equal to the side B, describe an arc; from E as a centre, with a radius equal to the side C, describe an arc intersecting the former at F; draw DF and EF: then DEF is the triangle required (B. I., P. X.).



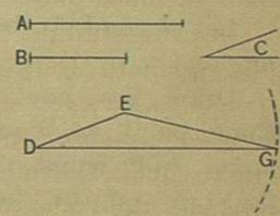
Scholium. In order that the construction may be possible, any one of the given sides must be *less* than the sum of the two others, and *greater* than their difference (B. I., P. VII., S.).

PROBLEM XI.

Given, two sides of a triangle, and the angle opposite one of them, to construct the triangle.

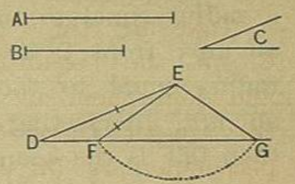
Let A and B be the given sides, and C the given angle.

Draw an indefinite line DG, and at some point of it, as D, construct an angle GDE equal to the given angle; on one side of this angle lay off the distance DE equal to the side B adjacent to the given angle; from E as a centre, with a radius equal to the side opposite the given angle, describe an arc cutting the side DG at G: draw EG. Then DEG is the required triangle.



For, the sides DE and EG are equal to the given sides, and the angle D, opposite one of them, is equal to the given angle.

Scholium. If the side opposite the given angle is greater than the other given side, there is but one solution. If the given angle is acute, and the side opposite the given angle is less than the other given side, and greater than the shortest distance from E to DG, there are two solutions, DEG and DEF. If the side opposite the given angle is equal to the shortest distance from E to DG, the arc will be tangent to DG, the angle opposite DE is a right angle, and there is but one solution. If the side opposite the given angle is shorter than the distance from E to DG, there is no solution.

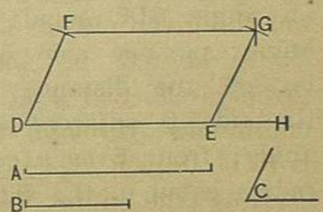


PROBLEM XII.

Given, two adjacent sides of a parallelogram and their included angle, to construct the parallelogram.

Let A and B be the given sides, and C the given angle.

Draw the line DH, and at some point as D, construct the angle HDF equal to the angle C. Lay off DE equal to the side A, and DF equal to the side B; draw FG parallel to DE, and EG parallel to DF; then DFGE is the parallelogram required.

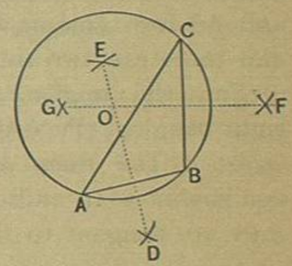


For, the opposite sides are parallel by construction; and consequently, the figure is a parallelogram (D. 28); it is also formed with the given sides and given angle.

PROBLEM XIII.

To find the centre of a given circumference or arc.

Take any three points A, B, and C, on the circumference or arc, and join them by the chords AB, BC; bisect these chords by the perpendiculars DE and FG: then their point of intersection, O, is the centre required (P. VII.).



Scholium. The same construction enables us to pass a circumference through any three points not in a straight line. If the points are vertices of a triangle, the circle is circumscribed about it.

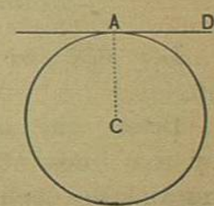
PROBLEM XIV.

Through a given point, to draw a tangent to a given circle.

There may be two cases: the given point may lie on the circumference of the given circle, or it may lie without the given circle.

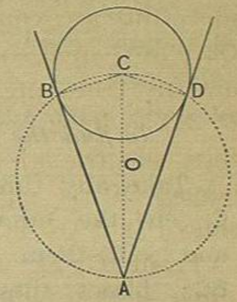
1°. Let C be the centre of the given circle, and A a point on the circumference, through which the tangent is to be drawn.

Draw the radius CA, and at A draw AD perpendicular to AC: then AD is the tangent required (P. IX.).



2°. Let C be the centre of the given circle, and A a point without the circle, through which the tangent is to be drawn.

Draw the line AC; bisect it at O, and from O as a centre, with a radius OC, describe the circumference ABCD; join the point A with the points of intersection D and B: then both AD and AB are tangent to the given circle and there are two solutions.



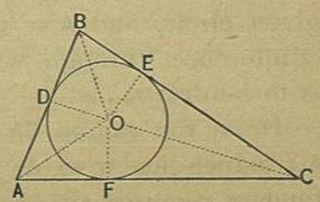
For, the angles ABC and ADC are right angles (P. XVIII., C. 2): hence, each of the lines AB and AD is perpendicular to a radius at its extremity; and consequently, they are tangent to the given circle (P. IX.).

Corollary. The right-angled triangles ABC and ADC, have a common hypothenuse AC, and the side BC equal to DC; and consequently, they are equal in all respects (B. I., P. XVII.): hence, AB is equal to AD, and the angle CAB is equal to the angle CAD. The tangents are therefore equal, and the line AC bisects the angle between them.

PROBLEM XV.

To inscribe a circle in a given triangle.

Let ABC be the given triangle.



Bisect the angles A and B, by the lines AO and BO, meeting in the point O (Prob. V.); from the point O let fall the

perpendiculars OD, OE, OF, on the sides of the triangle: these perpendiculars are all equal.

For, in the triangles BOD and BOE, the angles OBE and OBD are equal, by construction; the angles ODB and OEB are equal, because each is a right angle; and consequently, the angles BOD and BOE are also equal (B. I., P. XXV., C. 2), and the side OB is common; and therefore, the triangles are equal in all respects (B. I., P. VI.): hence, OD is equal to OE. In like manner, it may be shown that OD is equal to OF.

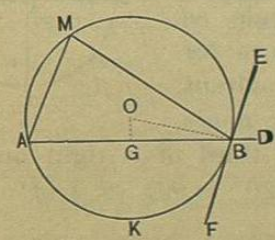
From O as a centre, with a radius OD, describe a circle, and it will be the circle required. For, each side is perpendicular to a radius at its extremity, and is therefore tangent to the circle.

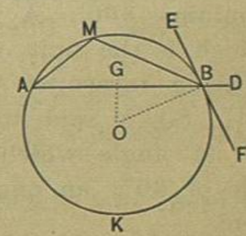
Corollary. The lines that bisect the three angles of a triangle all meet in one point.

PROBLEM XVI.

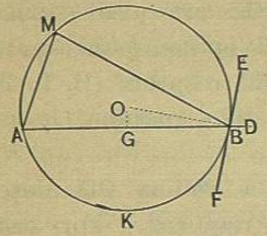
On a given straight line, to construct a segment that shall contain a given angle.

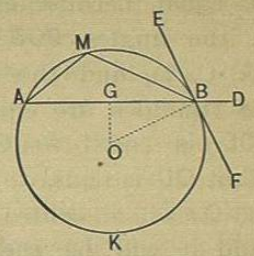
Let AB be the given line.





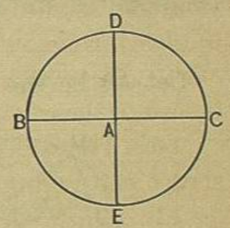
Produce AB towards D; at B construct the angle DBE equal to the given angle; draw BO perpendicular to BE, and at the middle point G, of AB, draw GO perpendicular to AB; from their point of intersection O, as a centre, with a radius OB, describe the arc AMB: then the segment AMB is the segment required.





For, the angle ABF, equal to EBD, is measured by half of the arc AKB (P. XXI.); and the inscribed angle AMB is measured by half of the same arc: hence, the angle AMB is equal to the angle EBD, and consequently, to the given angle.

NOTE.—A *quadrant* or quarter of a circumference, as CD, is, for convenience, divided into 90 equal parts, each of which is called a *degree*. A degree is denoted by the symbol °; thus, 25° is read 25 degrees, etc. Since a quadrant contains 90°, the whole circumference contains 360°. A right angle, as CAD, which is the unit of measure for angles, being measured by a quadrant (P. XVII., S.), is said to be an angle of 90°; an angle which is one third of a right angle is an angle of 30°; an angle of 120° is $\frac{120}{90}$ or $\frac{4}{3}$ of a right angle, etc.



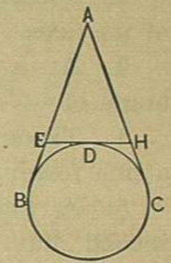
EXERCISES.

1. Draw a circumference of given radius through two given points.
2. Construct an equilateral triangle, having given one of its sides.
3. At a point on a given straight line, construct an angle of 30°.
4. Through a given point without a given line, draw a line forming with the given line an angle of 30°.
5. A line 8 feet long is met at one extremity by a second line, making with it an angle of 30°; find the centre of the circle of which the first line is a chord and the second a tangent.
6. How many degrees in an angle inscribed in an arc of 135°?
7. How many degrees in the angle formed by two secants meeting without the circle and including arcs of 60° and 110°?
8. At one extremity of a chord, which divides the circumference into two arcs of 290° and 70° respectively, a tangent is drawn; how many degrees in each of the angles formed by the tangent and the chord?
9. Show that the sum of the alternate angles of an inscribed hexagon is equal to four right angles.
10. The sides of a triangle are 3, 5, and 7 feet; construct the triangle.
11. Show that the three perpendiculars erected at the middle points of the three sides of a triangle meet in a common point.
12. Construct an isosceles triangle with a given base and a given vertical angle.
13. At a point on a given straight line, construct an angle of 45°

14. Construct an isosceles triangle so that the base shall be a given line and the vertical angle a right angle.

15. Construct a triangle, having given one angle, one of its including sides, and the difference of the two other sides.

16. From a given point, A, without a circle, draw two tangents, AB and AC, and at any point, D, in the included arc, draw a third tangent and produce it to meet the two others; show that the three tangents form a triangle whose perimeter is constant.



17. On a straight line 5 feet long, construct a circular segment that shall contain an angle of 30°.

18. Show that parallel tangents to a circle include semi-circumferences between their points of contact.

19. Show that four circles can be drawn tangent to three intersecting straight lines.

BOOK IV.

MEASUREMENT AND RELATION OF POLYGONS.

DEFINITIONS.

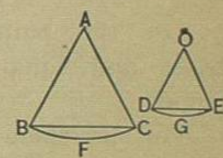
1. Similar Polygons are polygons which are mutually equiangular, and which have the sides about the equal angles, taken in the same order, proportional.

2. In similar polygons, the parts which are similarly placed in each, are called *homologous.*

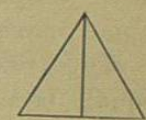
The corresponding angles are *homologous angles*, the corresponding sides are *homologous sides*, the corresponding diagonals are *homologous diagonals*, and so on.

3. Similar Arcs, Sectors, or Segments, in different circles, are those which correspond to equal angles at the centre.

Thus, if the angles A and O are equal, the arcs BFC and DGE are similar, the sectors BAC and DOE are similar, and the segments BFC and DGE are similar.



4. The Altitude of a Triangle is the perpendicular distance from the vertex of any angle to the opposite side, or the opposite side produced.



The vertex of the angle from which the distance is measured, is called the *vertex of the triangle*, and the opposite side is called the *base of the triangle.*