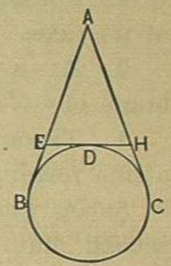
14. Construct an isosceles triangle so that the base shall be a given line and the vertical angle a right angle.

15. Construct a triangle, having given one angle, one of its including sides, and the difference of the two other sides.

16. From a given point, A, without a circle, draw two tangents, AB and AC, and at any point, D, in the included arc, draw a third tangent and produce it to meet the two others; show that the three tangents form a triangle whose perimeter is constant.



17. On a straight line 5 feet long, construct a circular segment that shall contain an angle of 30°.

18. Show that parallel tangents to a circle include semi-circumferences between their points of contact.

19. Show that four circles can be drawn tangent to three intersecting straight lines.

BOOK IV.

MEASUREMENT AND RELATION OF POLYGONS.

DEFINITIONS.

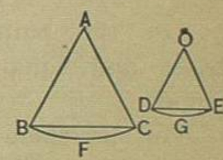
1. Similar Polygons are polygons which are mutually equiangular, and which have the sides about the equal angles, taken in the same order, proportional.

2. In similar polygons, the parts which are similarly placed in each, are called *homologous.*

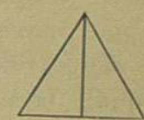
The corresponding angles are *homologous angles,* the corresponding sides are *homologous sides,* the corresponding diagonals are *homologous diagonals,* and so on.

3. Similar Arcs, Sectors, or Segments, in different circles, are those which correspond to equal angles at the centre.

Thus, if the angles A and O are equal, the arcs BFC and DGE are similar, the sectors BAC and DOE are similar, and the segments BFC and DGE are similar.



4. The Altitude of a Triangle is the perpendicular distance from the vertex of any angle to the opposite side, or the opposite side produced.

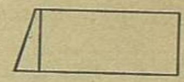


The vertex of the angle from which the distance is measured, is called the *vertex of the triangle,* and the opposite side is called the *base of the triangle.*

5. The ALTITUDE OF A PARALLELOGRAM is the perpendicular distance between two opposite sides.

These sides are called *bases;* one the *upper*, and the other, the *lower base.*

6. The ALTITUDE OF A TRAPEZOID is the perpendicular distance between its parallel sides.



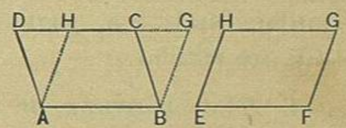
These sides are called *bases;* one the *upper*, and the other, the *lower base.*

7. The AREA OF A SURFACE is its numerical value expressed in terms of some other surface taken as a *unit.* The unit adopted is a square described on the linear unit as a side.

PROPOSITION I. THEOREM.

Parallelograms which have equal bases and equal altitudes, are equal.

Let the parallelograms ABCD and EFGH have equal bases and equal altitudes: then the parallelograms are equal.



For, let them be so placed that their lower bases shall coincide; then, because they have the same altitude, their upper bases will be in the same line DG, parallel to AB.

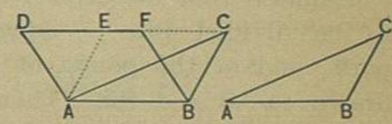
The triangles DAH and CBG, have the sides AD and BC equal, because they are opposite sides of the parallelogram AC (B. I., P. XXVIII.); the sides AH and BG equal, because they are opposite sides of the parallelogram AG; the angles DAH and CBG equal, because their sides are parallel and lie in the same direction (B. I., P. XXIV.): hence, the triangles are equal (B. I., P. V.).

If from the quadrilateral ABGD, we take away the triangle DAH, there will remain the parallelogram AG; if from the same quadrilateral ABGD, we take away the triangle CBG, there will remain the parallelogram AC: hence, the parallelogram AC is equal to the parallelogram EG (A. 3); *which was to be proved.*

PROPOSITION II. THEOREM.

A triangle is equal to one half of a parallelogram having an equal base and an equal altitude.

Let the triangle ABC, and the parallelogram ABFD, have equal bases and equal altitudes: then the triangle is equal to one half of the parallelogram.



For, let them be so placed that the base of the triangle shall coincide with the lower base of the parallelogram; then, because they have equal altitudes, the vertex of the triangle will lie in the upper base of the parallelogram, or in the prolongation of that base.

From A, draw AE parallel to BC, forming the parallelogram ABCE. This parallelogram is equal to the parallelogram ABFD, from Proposition I. But the triangle ABC is equal to half of the parallelogram ABCE (B. I., P. XXVIII., C. 1): hence, it is equal to half of the parallelogram ABFD (A. 7); *which was to be proved.*

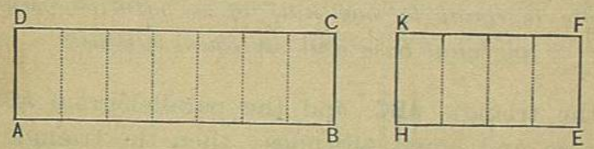
Cor. Triangles having equal bases and equal altitudes are equal, for they are halves of equal parallelograms.

PROPOSITION III. THEOREM.

Rectangles having equal altitudes, are proportional to their bases.

There may be two cases: the bases may be commensurable, or they may be incommensurable.

1°. Let ABCD and HEFK, be two rectangles whose altitudes AD and HK are equal, and whose bases AB and HE are commensurable: then the areas of the rectangles are proportional to their bases.



Suppose that AB is to HE, as 7 is to 4. Conceive AB to be divided into 7 equal parts, and HE into 4 equal parts, and at the points of division, let perpendiculars be drawn to AB and HE. Then will ABCD be divided into 7, and HEFK into 4 rectangles, all of which are equal, because they have equal bases and equal altitudes (P. I.): hence, we have,

ABCD : HEFK :: 7 : 4.

But we have, by hypothesis,

AB : HE :: 7 : 4.

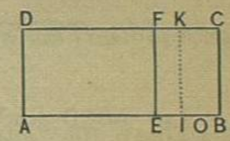
From these proportions, we have (B. II., P. IV.),

ABCD : HEFK :: AB : HE.

Had any other numbers than 7 and 4 been used, the same proportion would have been found; *which was to be proved.*

2°. Let the bases of the rectangles be incommensurable: then the rectangles are proportional to their bases.

For, place the rectangle HEFK upon the rectangle ABCD, so that it shall take the position AEFD. Then, if the rectangles are not proportional to their bases, let us suppose that



ABCD : AEFD :: AB : AO;

in which AO is greater than AE. Divide AB into equal parts, each less than OE; at least one point of division, as I, will fall between E and O; at this point, draw IK perpendicular to AB. Then, because AB and AI are commensurable, we shall have, from what has just been shown,

ABCD : AIKD :: AB : AI.

The above proportions have their antecedents the same in each; hence (B. II., P. IV., C.),

AEFD : AIKD :: AO : AI.

The rectangle AEFD is less than AIKD; and if the above proportion were true, the line AO would be less than AI; whereas, it is greater. The fourth term of the proportion, therefore, cannot be greater than AE. In like manner, it may be shown that it cannot be less than AE; consequently, it must be equal to AE: hence,

ABCD : AEFD :: AB : AE;

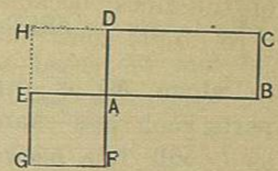
which was to be proved.

Cor. If rectangles have equal bases, they are to each other as their altitudes.

PROPOSITION IV. THEOREM.

Any two rectangles are to each other as the products of their bases and altitudes.

Let ABCD and AEGF be two rectangles: then ABCD is to AEGF, as AB × AD is to AE × AF.



For, place the rectangles so that the angles DAB and EAF shall be opposite or vertical; then, produce the sides CD and GE till they meet in H.

The rectangles ABCD and ADHE have the same altitude AD: hence (P. III.),

ABCD : ADHE : : AB : AE.

The rectangles ADHE and AEGF have the same altitude AE: hence,

ADHE : AEGF : : AD : AF.

Multiplying these proportions, term by term (B. II., P. XII.), and omitting the common factor ADHE (B. II., P. VII.), we have,

ABCD : AEGF : : AB × AD : AE × AF;

which was to be proved.

Cor. If we suppose AE and AF, each to be equal to the linear unit, the rectangle AEGF is the superficial unit, and we have,

ABCD : 1 : : AB × AD : 1;

ABCD = AB × AD:

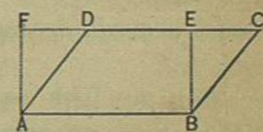
hence, *the area of a rectangle is equal to the product of its base and altitude;* that is, the number of superficial units in the rectangle, is equal to the product of the number of linear units in its base by the number of linear units in its altitude.

The product of two lines is sometimes called the *rectangle* of the lines, because the product is equal to the area of a rectangle constructed with the lines as sides.

PROPOSITION V. THEOREM.

The area of a parallelogram is equal to the product of its base and altitude.

Let ABCD be a parallelogram, AB its base, and BE its altitude: then the area of ABCD is equal to AB × BE.



For, construct the rectangle ABEF, having the same base and altitude: then will the rectangle be equal to the parallelogram (P. I.); but the area of the rectangle is equal to AB × BE: hence, the area of the parallelogram is also equal to AB × BE; *which was to be proved.*

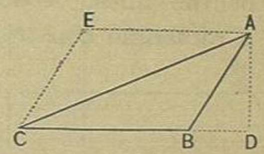
Cor. Parallelograms are to each other as the products of their bases and altitudes. If their altitudes are equal, they are to each other as their bases. If their bases are equal, they are to each other as their altitudes.

PROPOSITION VI. THEOREM.

The area of a triangle is equal to half the product of its base and altitude.

Let ABC be a triangle, BC its base, and AD its altitude: then its area is equal to $\frac{1}{2}BC \times AD$.

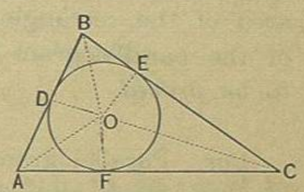
For, from C, draw CE parallel to BA, and from A, draw AE parallel to BC. The area of the parallelogram BCEA is $BC \times AD$ (P. V.); but the triangle ABC is half of the parallelogram BCEA: hence, its area is equal to $\frac{1}{2}BC \times AD$; *which was to be proved.*



Cor. 1. Triangles are to each other, as the products of their bases and altitudes (B. II., P. VII.). If the altitudes are equal, they are to each other as their bases. If the bases are equal, they are to each other as their altitudes.

Cor. 2. The area of a triangle is equal to half the product of its perimeter and the radius of the inscribed circle.

For, let DEF be a circle inscribed in the triangle ABC. Draw OD, OE, and OF, to the points of contact, and OA, OB, and OC, to the vertices.

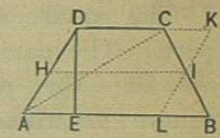


The area of OBC is equal to $\frac{1}{2}OE \times BC$; the area of OAC is equal to $\frac{1}{2}OF \times AC$; and the area of OAB is equal to $\frac{1}{2}OD \times AB$; and since OD, OE, and OF, are equal, the area of the triangle ABC (A. 9), is equal to $\frac{1}{2}OD\,(AB + BC + CA)$.

PROPOSITION VII. THEOREM.

The area of a trapezoid is equal to the product of its altitude and half the sum of its parallel sides.

Let ABCD be a trapezoid, DE its altitude, and AB and DC its parallel sides: then its area is equal to $DE \times \frac{1}{2}(AB + DC)$.



For, draw the diagonal AC, forming the triangles ABC and ACD. The altitude of each of these triangles is equal to DE. The area of ABC is equal to $\frac{1}{2}AB \times DE$ (P. VI.); the area of ACD is equal to $\frac{1}{2}DC \times DE$: hence, the area of the trapezoid, which is the sum of the triangles, is equal to the sum of $\frac{1}{2}AB \times DE$ and $\frac{1}{2}DC \times DE$, or to $DE \times \frac{1}{2}(AB + DC)$; *which was to be proved.*

Scholium. Through I, the middle point of BC, draw IH parallel to AB, and LI parallel to AD, meeting DC produced, at K. Then, since AI and HK are parallelograms, we have $AL = HI = DK$; and therefore, $HI = \frac{1}{2}(AL + DK)$. But since the triangles LIB and CIK are equal in all respects, $LB = CK$; hence, $AL + DK = AB + DC$; and we have $HI = \frac{1}{2}(AB + DC)$: hence,

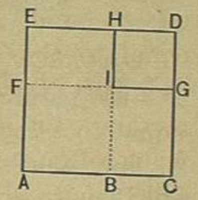
The area of a trapezoid is equal to its altitude multiplied by the line which connects the middle points of its inclined sides.

PROPOSITION VIII. THEOREM.

The square described on the sum of two lines is equal to the sum of the squares described on the lines, increased by twice the rectangle of the lines.

Let AB and BC be two lines, and AC their sum: then $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BC$.

On AC, construct the square AD; from B, draw BH parallel to AE; lay off AF equal to AB, and from F, draw FG parallel to AC: then IG and IH are each equal to BC; and IB and IF, to AB.



The square ACDE is composed of four parts. The part ABIF is a square described on AB; the part IGDH is equal to a square described on BC; the part BCGI is equal to the rectangle of AB and BC; and the part FIHE is also equal to the rectangle of AB and BC: hence, we have (A. 9),

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BC;$$

which was to be proved.

Cor. If the lines AB and BC are equal, the four parts of the square on AC are also equal: hence, *the square described on a line is equal to four times the square described on half the line.*

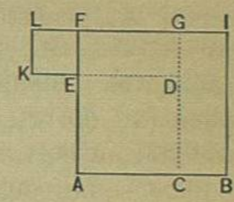
PROPOSITION IX. THEOREM.

The square described on the difference of two lines is equal to the sum of the squares described on the lines, diminished by twice the rectangle of the lines.

Let AB and BC be two lines, and AC their difference; then

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times BC.$$

On AB construct the square ABIF; from C draw CG parallel to BI; lay off CD equal to AC, and from D draw DK parallel and equal to BA; complete the square EFLK; then EK is equal to BC, and EFLK is equal to the square of BC.



The whole figure ABILKE is equal to the sum of the squares described on AB and BC. The part CBIG is equal to the rectangle of AB and BC; the part DGLK is also equal to the rectangle of AB and BC. If from the whole figure ABILKE, the two parts CBIG and DGLK be taken, there will remain the part ACDE, which is equal to the square of AC: hence,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times BC;$$

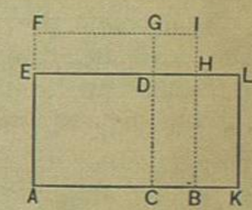
which was to be proved.

PROPOSITION X. THEOREM.

9 *The rectangle contained by the sum and difference of two lines, is equal to the difference of their squares.*

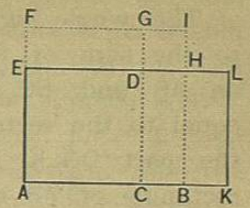
Let AB and BC be two lines, of which AB is the greater: then

$$(AB + BC)(AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$



On AB, construct the square ABIF; prolong AB, and make BK equal to BC; then AK is equal to AB + BC; from K, draw KL parallel to BI, and make it equal to AC; draw LE parallel to KA, and CG parallel to BI: then DG is equal to BC, and the figure DHIG is equal to the square on BC, and EDGF is equal to BKLH.

If we add to the figure ABHE, the rectangle BKLH, we have the rectangle AKLE, which is equal to the rectangle of AB + BC and AB − BC. If to the same figure ABHE, we add the rectangle DGFE, equal to BKLH, we have the figure ABHDGF, which is equal to the difference of the squares of AB and BC. But the sums of equals are equal (A. 2), hence,



$$(AB + BC)\,(AB - BC) = \overline{AB}^2 - \overline{BC}^2;$$

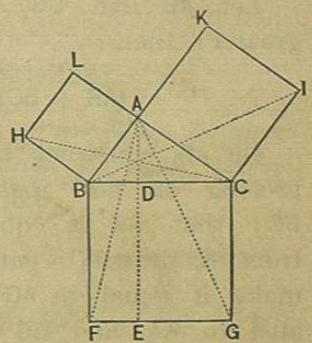
which was to be proved.

PROPOSITION XI. THEOREM.

The square described on the hypothenuse of a right-angled triangle, is equal to the sum of the squares described on the two other sides.

Let ABC be a triangle, right-angled at A: then

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2.$$



Construct the square BG on the side BC, the square AH on the side AB, and the square AI on the side AC; from A draw AD perpendicular to BC, and prolong it to E: then DE is parallel to BF; draw AF and HC.

In the triangles HBC and ABF, we have HB equal to AB, because they are sides of the same square; BC equal

to BF, for the same reason, and the included angles HBC and ABF equal, because each is equal to the angle ABC plus a right angle: hence, the triangles are equal in all respects (B. I., P. V.).

The triangle ABF, and the rectangle BE, have the same base BF, and because DE is the prolongation of DA, their altitudes are equal: hence, the triangle ABF is equal to half the rectangle BE (P. II.). The triangle HBC, and the square BL, have the same base BH, and because AC is the prolongation of LA (B. I., P. IV.), their altitudes are equal: hence, the triangle HBC is equal to half the square of AH. But, the triangles ABF and HBC are equal: hence, the rectangle BE is equal to the square AH. In the same manner, it may be shown that the rectangle DG is equal to the square AI: hence, the sum of the rectangles BE and DG, or the square BG, is equal to the sum of the squares AH and AI; or, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$; *which was to be proved.*

Cor. 1. The square of either side about the right angle is equal to the square of the hypothenuse diminished by the square of the other side: thus,

$$\overline{AB}^2 = \overline{BC}^2 - \overline{AC}^2; \quad \text{or,} \quad \overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2.$$

Cor. 2. If from the vertex of the right angle, a perpendicular be drawn to the hypothenuse, dividing it into two *segments*, BD and DC, *the square of the hypothenuse is to the square of either of the other sides, as the hypothenuse is to the segment adjacent to that side.*

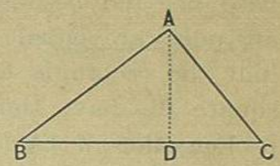
For, the square BG, is to the rectangle BE, as BC to BD (P. III.); but the rectangle BE is equal to the square AH: hence,

$$\overline{BC}^2 : \overline{AB}^2 :: BC : BD.$$

In like manner, we have,

$$\overline{BC}^2 : \overline{AC}^2 :: BC : DC.$$

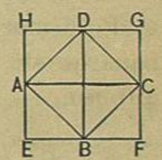
Cor. 3. *The squares of the sides about the right angle are to each other as the adjacent segments of the hypothenuse.*



For, by combining the proportions of the preceding corollary (B. II., P. IV., C.), we have,

$$\overline{AB}^2 : \overline{AC}^2 :: BD : DC.$$

Cor. 4. *The square described on the diagonal of a square is double the given square.*



For, the square of the diagonal is equal to the sum of the squares of the two sides; but the square of each side is equal to the given square: hence,

$$\overline{AC}^2 = 2\overline{AB}^2; \quad \text{or,} \quad \overline{AC}^2 = 2\overline{BC}^2.$$

Cor. 5. From the last corollary, we have,

$$\overline{AC}^2 : \overline{AB}^2 :: 2 : 1;$$

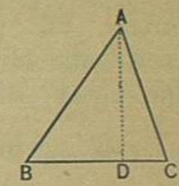
hence, by extracting the square root of each term, we have,

$$AC : AB :: \sqrt{2} : 1;$$

that is, *the diagonal of a square is to the side, as the square root of two is to one;* consequently, *the diagonal and the side of a square are incommensurable.*

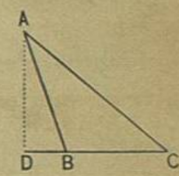
PROPOSITION XII. THEOREM.

In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the base and the other side, diminished by twice the rectangle of the base and the distance from the vertex of the acute angle to the foot of the perpendicular drawn from the vertex of the opposite angle to the base, or to the base produced.



Let ABC be a triangle, C one of its acute angles, BC its base, and AD the perpendicular drawn from A to BC, or BC produced; then

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD.$$



For, whether the perpendicular meets the base, or the base produced, we have BD equal to the difference of BC and CD: hence (P. IX.),

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2BC \times CD.$$

Adding $\overline{AD}^2$ to both members, we have,

$$\overline{BD}^2 + \overline{AD}^2 = \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2 - 2BC \times CD.$$

But, $\quad \overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2,$

and $\quad \overline{CD}^2 + \overline{AD}^2 = \overline{AC}^2:$

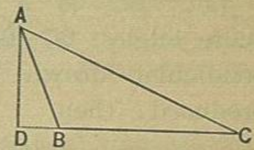
hence, $\quad \overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD;$

which was to be proved.

PROPOSITION XIII. THEOREM.

In any obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the base and the other side, increased by twice the rectangle of the base and the distance from the vertex of the obtuse angle to the foot of the perpendicular drawn from the vertex of the opposite angle to the base produced.

Let ABC be an obtuse-angled triangle, B its obtuse angle, BC its base, and AD the perpendicular drawn from A to BC produced; then



$$\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 + 2BC \times BD.$$

For, CD is the sum of BC and BD: hence (P. VIII.),

$$\overline{CD}^2 = \overline{BC}^2 + \overline{BD}^2 + 2BC \times BD.$$

Adding $\overline{AD}^2$ to both members, and reducing, we have,

$$\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 + 2BC \times BD;$$

which was to be proved.

Scholium. The right-angled triangle is the only one in which the sum of the squares described on two sides is equal to the square described on the third side.

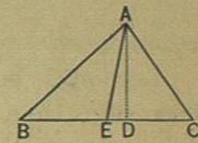
PROPOSITION XIV. THEOREM.

10 *In any triangle, the sum of the squares described on two sides is equal to twice the square of half the third side, increased by twice the square of the line drawn from the middle point of that side to the vertex of the opposite angle.*

Let ABC be any triangle, and EA a line drawn from the middle of the base BC to the vertex A: then

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{BE}^2 + 2\overline{EA}^2.$$

Draw AD perpendicular to BC; then, from Proposition XII., we have,



$$\overline{AC}^2 = \overline{EC}^2 + \overline{EA}^2 - 2EC \times ED.$$

From Proposition XIII., we have,

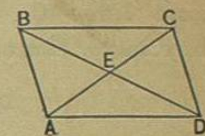
$$\overline{AB}^2 = \overline{BE}^2 + \overline{EA}^2 + 2BE \times ED.$$

Adding these equations, member to member (A. 2), recollecting that BE is equal to EC, we have,

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{BE}^2 + 2\overline{EA}^2;$$

which was to be proved.

Cor. Let ABCD be a parallelogram, and BD, AC, its diagonals. Then, since the diagonals mutually bisect each other (B. I., P. XXXI.), we have,



$$\overline{AB}^2 + \overline{BC}^2 = 2\overline{AE}^2 + 2\overline{BE}^2;$$

and, $\overline{CD}^2 + \overline{DA}^2 = 2\overline{CE}^2 + 2\overline{DE}^2;$

whence, by addition, recollecting that AE is equal to CE, and BE to DE, we have,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\overline{CE}^2 + 4\overline{DE}^2;$$

but, $4\overline{CE}^2$ is equal to $\overline{AC}^2$, and $4\overline{DE}^2$ to $\overline{BD}^2$ (P. VIII., C.): hence,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2.$$

That is, *the sum of the squares of the sides of a parallelogram, is equal to the sum of the squares of its diagonals.*

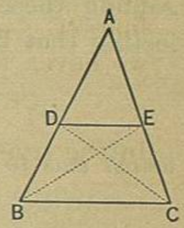
PROPOSITION XV. THEOREM.

In any triangle, a line drawn parallel to the base divides the other sides proportionally.

Let ABC be a triangle, and DE a line parallel to the base BC: then

$$AD : DB :: AE : EC.$$

Draw EB and DC. Then, because the triangles AED and DEB have their bases in the same line AB, and their vertices at the same point E, they have a common altitude: hence (P. VI., C.),



$$AED : DEB :: AD : DB.$$

The triangles AED and EDC, have their bases in the same line AC, and their vertices at the same point D; they have, therefore, a common altitude; hence,

$$AED : EDC :: AE : EC.$$

But the triangles DEB and EDC have a common base DE, and their vertices in the line BC, parallel to DE: they are, therefore, equal: hence, the two preceding proportions have a couplet in each equal; and consequently, the remaining terms are proportional (B. II., P. IV.), hence,

$$AD : DB :: AE : EC;$$

which was to be proved.

Cor. 1. We have, by composition (B. II., P. VI.),

$$AD + DB : AD :: AE + EC : AE;$$

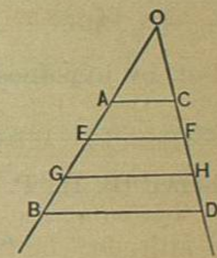
or,

$$AB : AD :: AC : AE;$$

and, in like manner,

$$AB : DB :: AE : EC.$$

Cor. 2. If any number of parallels be drawn cutting two lines, they divide the lines proportionally.



For, let O be the point where AB and CD meet. In the triangle OEF, the line AC being parallel to the base EF, we have,

$$OE : AE :: OF : CF.$$

In the triangle OGH, we have,

$$OE : EG :: OF : FH;$$

hence (B. II., P. IV., C.),

$$AE : EG :: CF : FH.$$

In like manner,

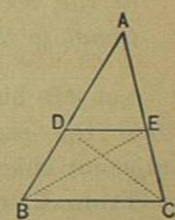
$$EG : GB :: FH : HD;$$

and so on.

PROPOSITION XVI. THEOREM.

If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

Let ABC be a triangle, and let DE divide AB and AC, so that



$$AD : DB :: AE : EC;$$

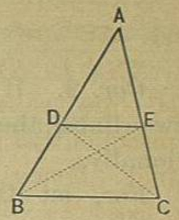
then DE is parallel to BC.

Draw DC and EB. Then the triangles

ADE and DEB have a common altitude; and consequently, we have,

ADE : DEB :: AD : DB.

The triangles ADE and EDC have also a common altitude; and consequently, we have,



ADE : EDC :: AE : EC;

but, by hypothesis,

AD : DB :: AE : EC;

hence (B. II., P. IV.),

ADE : DEB :: ADE : EDC.

The antecedents of this proportion being equal, the consequents are equal; that is, the triangles DEB and EDC are equal. But these triangles have a common base DE: hence, their altitudes are equal (P. VI., C.); that is, the points B and C, of the line BC, are equally distant from DE, or DE prolonged: hence, BC and DE are parallel (B. I., P. XXX., C.); *which was to be proved.*

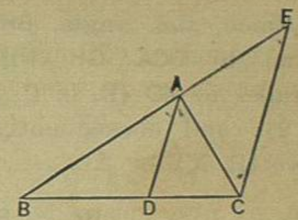
PROPOSITION XVII. THEOREM.

11 *In any triangle, the straight line which bisects the angle at the vertex, divides the base into two segments proportional to the adjacent sides.*

Let AD bisect the vertical angle A of the triangle BAC: then the segments BD and DC are proportional to the adjacent sides BA and CA.

From C, draw CE parallel to DA, and produce it until

it meets BA prolonged, at E. Then, because CE and DA are parallel, the angles BAD and AEC are equal (B. I., P. XX., C. 3); the angles DAC and ACE are also equal (B. I., P. XX., C. 2). But, BAD and DAC are equal, by hypothesis; consequently, AEC and ACE are equal: hence, the triangle ACE is isosceles, AE being equal to AC.



In the triangle BEC, the line AD is parallel to the base EC: hence (P. XV.),

BA : AE :: BD : DC;

or, substituting AC for its equal AE,

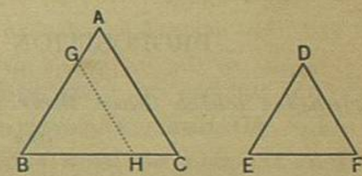
BA : AC :: BD : DC;

which was to be proved.

PROPOSITION XVIII. THEOREM.

Triangles which are mutually equiangular, are similar.

Let the triangles ABC and DEF have the angle A equal to the angle D, the angle B to the angle E, and the angle C to the angle F: then they are similar.



For, place the triangle DEF upon the triangle ABC, so that the angle E shall coincide with the angle B; then will the point F fall at some point H, of BC; the point D at some point G, of BA;