BOOK V.

REGULAR POLYGONS.—AREA OF THE CIRCLE.

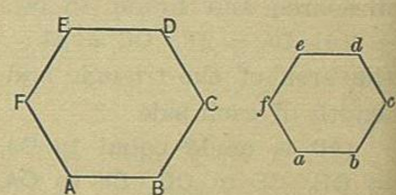
DEFINITION.

1. A Regular Polygon is a polygon which is both equilateral and equiangular.

PROPOSITION I. THEOREM.

Regular polygons of the same number of sides are similar.

Let ABCDEF and *abcdef* be regular polygons of the same number of sides: then they are similar.

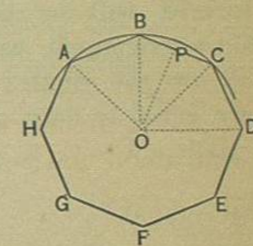


For, the corresponding angles in each are equal, because any angle in either polygon is equal to twice as many right angles as the polygon has sides, less four right angles, divided by the number of angles (B. I., P. XXVI., C. 4); and further, the corresponding sides are proportional, because all the sides of either polygon are equal (D. 1): hence, the polygons are similar (B. IV., D. 1); *which was to be proved.*

PROPOSITION II. THEOREM.

The circumference of a circle may be circumscribed about any regular polygon; a circle may also be inscribed in it.

1°. Let ABCF be a regular polygon: then can the circumference of a circle be circumscribed about it.



For, through three consecutive vertices A, B, C, describe the circumference of a circle (B. III., Problem XIII., S.). Its centre O lies on PO, drawn perpendicular to BC, at its middle point P; draw OA and OD.

Let the quadrilateral OPCD be turned about the line OP, until PC falls on PB; then, because the angle C is equal to B, the side CD will take the direction BA: and because CD is equal to BA, the vertex D, will fall upon the vertex A; and consequently, the line OD will coincide with OA, and is, therefore, equal to it: hence, the circumference which passes through A, B, and C, passes through D. In like manner, it may be shown that it passes through each of the other vertices: hence, it is circumscribed about the polygon; *which was to be proved.*

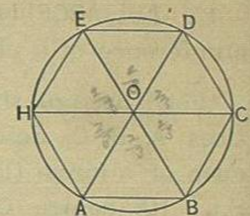
2°. A circle may be inscribed in the polygon.

For, the sides AB, BC, &c., being equal chords of the circumscribed circle, are equidistant from the centre O; hence, a circle described from O as a centre, with OP as a radius, is tangent to each of the sides of the polygon, and consequently, is inscribed in it; *which was to be proved.*

Scholium. If the circumference of a circle is divided into equal arcs, the chords of these arcs are sides of a regular inscribed polygon.

For, the sides are equal, because they are chords of equal arcs, and the angles are equal, because they are measured by halves of equal arcs.

If the vertices A, B, C, &c., of a regular inscribed polygon be joined with the centre O, the triangles thus formed will be equal, because their sides are equal, each to each: hence, all of the angles about the point O are equal to each other.



DEFINITIONS.

1. The Centre of a Regular Polygon is the common centre of the circumscribed and inscribed circles.

2. The Angle at the Centre is the angle formed by drawing lines from the centre to the extremities of any side.

The angle at the centre is equal to four right angles divided by the number of sides of the polygon.

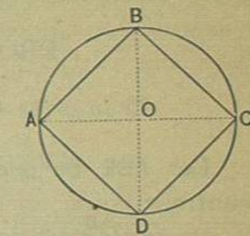
3. The Apothem is the shortest distance from the centre to any side.

The apothem is equal to the radius of the inscribed circle.

PROPOSITION III. PROBLEM.

To inscribe a square in a given circle.

Let ABCD be the given circle. Draw any two diameters AC and BD perpendicular to each other; they divide the circumference into four equal arcs (B. III., P. XVII., S.). Draw the chords AB, BC, CD, and DA: then the figure ABCD is the square required (P. II., S.).



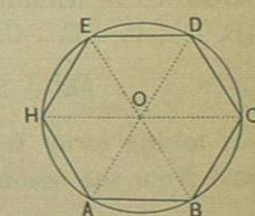
Scholium. The radius is to the side of the inscribed square as 1 is to $\sqrt{2}$.

PROPOSITION IV. THEOREM.

If a regular hexagon is inscribed in a circle, any side is equal to the radius of the circle.

Let ABD be a circle, and ABCDEH a regular inscribed hexagon: then any side, as AB, is equal to the radius of the circle.

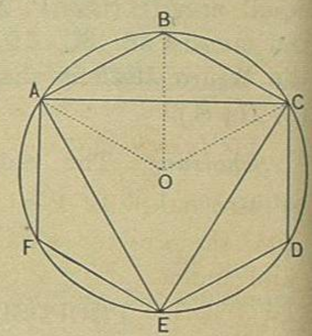
Draw the radii OA and OB. Then the angle AOB is equal to one sixth of four right angles, or to two thirds of one right angle, because it is an angle at the centre (P. II., D. 2). The sum of the two angles OAB and OBA is, consequently, equal to four thirds of a right angle (B. I., P. XXV., C. 1); but, the angles OAB and OBA are equal, because the opposite sides OB and OA are equal: hence, each is equal to two thirds



of a right angle. The three angles of the triangle AOB are therefore equal, and consequently, the triangle is equilateral: hence, AB is equal to OA; *which was to be proved.*

PROPOSITION V. PROBLEM.

To inscribe a regular hexagon in a given circle.



Let ABE be a circle, and O its centre.

Beginning at any point of the circumference, as A, apply the radius OA six times as a chord; then ABCDEF is the hexagon required (P. IV.).

Cor. 1. If the alternate vertices of the regular hexagon are joined by the straight lines AC, CE, and EA, the inscribed triangle ACE is equilateral (P. II., S.).

Cor. 2. If we draw the radii OA and OC, the figure AOCB is a rhombus, because its sides are equal: hence (B. IV., P. XIV., C.), we have,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{OA}^2 + \overline{OC}^2 = \overline{AC}^2 + \overline{OB}^2;$$

or, taking away from the first member the quantity $\overline{OA}^2$, and from the second its equal $\overline{OB}^2$, and reducing, we have,

$$3\overline{OA}^2 = \overline{AC}^2;$$

whence (B. II., P. II.),

$$\overline{AC}^2 : \overline{OA}^2 :: 3 : 1;$$

or (B. II., P. XII., C. 2),

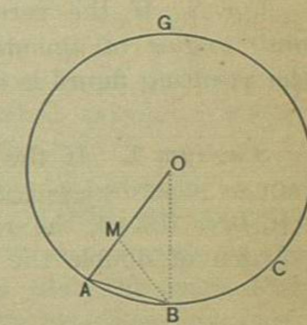
$$AC : OA :: \sqrt{3} : 1;$$

that is, *the side of an inscribed equilateral triangle is to the radius, as the square root of 3 is to 1.*

PROPOSITION VI. THEOREM.

If the radius of a circle is divided in extreme and mean ratio, the greater segment is equal to one side of a regular inscribed decagon.

Let ACG be a circle, OA its radius, and AB, equal to OM, the greater segment of OA when divided in extreme and mean ratio: then AB is equal to the side of a regular inscribed decagon.



Draw OB and BM. We have, by hypothesis,

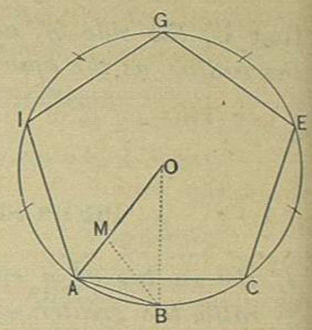
$$AO : OM :: OM : AM;$$

or, since AB is equal to OM, we have,

$$AO : AB :: AB : AM;$$

hence, the triangles OAB and BAM have the sides about their common angle BAM, proportional; they are, therefore, similar (B. IV., P. XX.). But, the triangle OAB is isosceles; hence, BAM is also isosceles, and consequently, the side BM is equal to AB. But, AB is equal to OM, by hypothesis: hence, BM is equal to OM, and consequently, the angles MOB and MBO are equal. The angle

AMB being an exterior angle of the triangle OMB, is equal to the sum of the angles MOB and MBO, or to twice the angle MOB; and because AMB is equal to OAB, and also to OBA, the sum of the angles OAB and OBA is equal to four times the angle AOB: hence, AOB is equal to one fifth of two right angles, or to one tenth of four right angles; and consequently, the arc AB is equal to one tenth of the circumference: hence, the chord AB is equal to the side of a regular inscribed decagon; *which was to be proved.*



Cor. 1. If AB is applied ten times as a chord, the resulting polygon is a regular inscribed decagon.

Cor. 2. If the vertices A, C, E, G, and I, of the alternate angles of the decagon are joined by straight lines, the resulting figure is a regular inscribed pentagon.

Scholium 1. If the arcs subtended by the sides of any regular inscribed polygon are bisected, and chords of the semi-arcs drawn, the resulting figure is a regular inscribed polygon of double the number of sides.

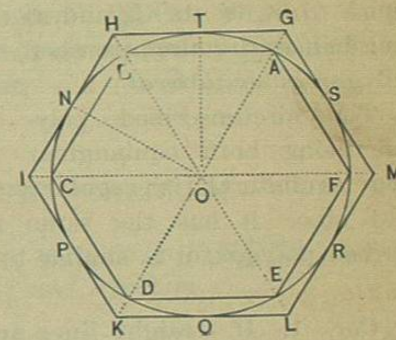
Scholium 2. The area of any regular inscribed polygon is less than that of a regular inscribed polygon of double the number of sides, because a part is less than the whole.

PROPOSITION VII. PROBLEM.

To circumscribe, about a circle, a polygon which shall be similar to a given regular inscribed polygon.

Let TNQ be a circle, O its centre, and ABCDEF a regular inscribed polygon.

At the middle points T, N, P, &c., of the arcs subtended by the sides of the inscribed polygon, draw tangents to the circle, and prolong them till they intersect; then the resulting figure is the polygon required.

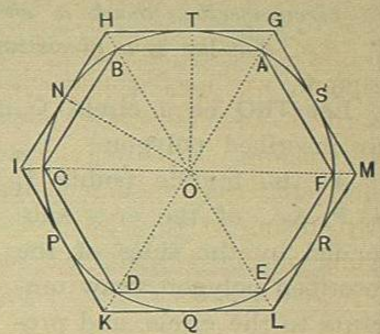


1°. The side HG being parallel to BA, and HI to BC, the angle H is equal to the angle B. In like manner, it may be shown that any other angle of the circumscribed polygon is equal to the corresponding angle of the inscribed polygon: hence, the circumscribed polygon is *equiangular.*

2°. Draw the straight lines OG, OT, OH, ON, and OI. Then, because the lines HT and HN are tangent to the circle, OH bisects the angle NHT, and also the angle NOT (B. III., Prob. XIV., C.); consequently, it passes through the middle point B of the arc NBT. In like manner, it may be shown that the straight line drawn from the centre to the vertex of any other angle of the circumscibed polygon, passes through the corresponding vertex of the inscribed polygon.

The triangles OHG and OHI have the angles OHG and

OHI equal, from what has just been shown; the angles GOH and HOI equal, because they are measured by the equal arcs AB and BC, and the side OH common; they are, therefore, equal in all respects: hence, GH is equal to HI. In like manner, it may be shown that HI is equal to IK, IK to KL, and so on: hence, the circumscribed polygon is *equilateral.*



The circumscribed polygon being both equiangular and equilateral, is *regular;* and since it has the same number of sides as the inscribed polygon, it is similar to it.

Cor. 1. If straight lines are drawn from the centre of a regular circumscribed polygon to its vertices, and the consecutive points in which they intersect the circumference joined by chords, the resulting figure is a regular inscribed polygon similar to the given polygon.

Cor. 2. The sum of the lines HT and HN is equal to the sum of HT and TG, or to HG; that is, to one of the sides of the circumscribed polygon.

Cor. 3. If at the vertices A, B, C, &c., of the inscribed polygon, tangents are drawn to the circle and prolonged till they meet the sides of the circumscribed polygon, the resulting figure is a circumscribed polygon of double the number of sides.

Sch. 1. The area of any regular circumscribed polygon

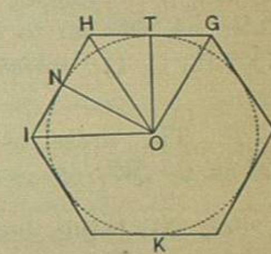
is greater than that of a regular circumscribed polygon of double the number of sides, because the whole is greater than any of its parts.

Sch. 2. By means of a circumscribed and inscribed square, we may construct, in succession, regular circumscribed and inscribed polygons of 8, 16, 32, &c., sides. By means of the regular hexagon we may, in like manner, construct regular polygons of 12, 24, 48, &c., sides. By means of the decagon, we may construct regular polygons of 20, 40, 80, &c., sides.

PROPOSITION VIII. THEOREM.

The area of a regular polygon is equal to half the product of its perimeter and apothem.

Let GHIK be a regular polygon, O its centre, and OT its apothem, or the radius of the inscribed circle: then the area of the polygon is equal to half the product of the perimeter and the apothem.

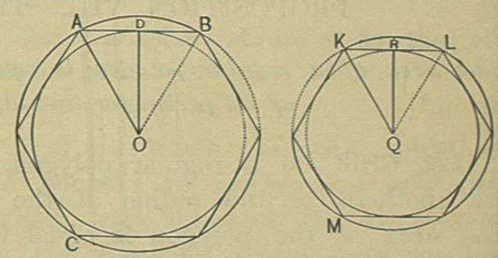


For, draw lines from the centre to the vertices of the polygon. These lines divide the polygon into triangles whose bases are the sides of the polygon, and whose altitudes are equal to the apothem. Now, the area of any triangle, as OHG, is equal to half the product of the side HG and the apothem: hence, the area of the polygon is equal to half the product of the perimeter and the apothem; *which was to be proved.*

PROPOSITION IX. THEOREM.

The perimeters of similar regular polygons are to each other as the radii of their circumscribed or inscribed circles; and their areas are to each other as the squares of those radii.

1°. Let ABC and KLM be similar regular polygons. Let OA and QK be the radii of their circumscribed, OD and QR be the radii of their inscribed circles: then the perimeters of the polygons are to each other as OA is to QK, or as OD is to QR.



For, the lines OA and QK are homologous lines of the polygons to which they belong, as are also the lines OD and QR: hence, the perimeter of ABC is to the perimeter of KLM, as OA is to QK, or as OD is to QR (B. IV., P. XXVII., C. 1); *which was to be proved.*

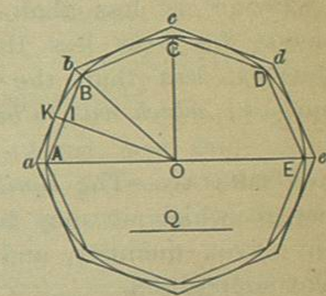
2°. The areas of the polygons are to each other as $\overline{OA}^2$ is to $\overline{QK}^2$, or as $\overline{OD}^2$ is to $\overline{QR}^2$.

For, OA being homologous with QK, and OD with QR, we have, the area of ABC is to the area of KLM as $\overline{OA}^2$ is to $\overline{QK}^2$, or as $\overline{OD}^2$ is to $\overline{QR}^2$ (B. IV., P. XXVII., C. 1); *which was to be proved.*

PROPOSITION X. THEOREM.

Two regular polygons of the same number of sides can be constructed, the one circumscribed about a circle and the other inscribed in it, which shall differ from each other by less than any given surface.

Let ABCE be a circle, O its centre, and Q the side of a square equal to or less than the given surface; then can two similar regular polygons be constructed, the one circumscribed about, and the other inscribed in the given circle, which shall differ from each other by less than the square of Q, and consequently, by less than the given surface.



Inscribe a square in the given circle (P. III.), and by means of it, inscribe, in succession, regular polygons of 8, 16, 32, &c., sides (P. VII., S. 2), until one is found whose side is less than Q; let AB be the side of such a polygon.

Construct a similar circumscribed polygon *abcde*: then these polygons differ from each other by less than the square of Q.

For, from *a* and *b*, draw the lines *a*O and *b*O; they pass through the points A and B. Draw also OK to the point of contact K; it bisects AB at I and is perpendicular to it. Prolong AO to E.

Let P denote the circumscribed, and *p* the inscribed polygon; then, because they are regular and similar, we have (P. IX.),

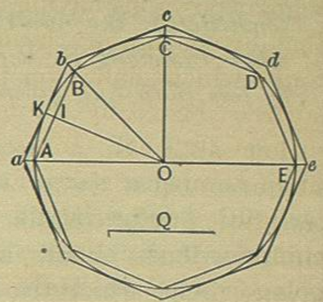
$$P : p :: \overline{OK}^2 \text{ or } \overline{OA}^2 : \overline{OI}^2 ;$$

hence, by division (B. II., P. VI.), we have,

$$P : P - p :: \overline{OA}^2 : \overline{OA}^2 - \overline{OI}^2;$$

or,

$$P : P - p :: \overline{OA}^2 : \overline{AI}^2.$$



Multiplying the terms of the second couplet by 4 (B. II., P. VII.), we have

$$P : P - p :: 4\overline{OA}^2 : 4\overline{AI}^2;$$

whence (B. IV., P. VIII., C.),

$$P : P - p :: \overline{AE}^2 : \overline{AB}^2$$

But P is less than the square of AE (P. VII., S. 1); hence, $P - p$ is less than the square of AB, and consequently, less than the square of Q, or than the given surface; *which was to be proved.*

Definition.—The *limit* of a variable quantity is a quantity to which it may be made to approach nearer than any given quantity, and which it reaches under a particular supposition.

Lemma.—*Two variable quantities which constantly approach to equality, and of which the difference becomes less than any finite magnitude, are ultimately equal.*

For if they are not ultimately equal, let D be their ultimate difference. Now, by hypothesis, the quantities have approached nearer to equality than any given quantity, as D; hence D denotes their difference and a quantity greater than their difference, at the same time, which is impossible; therefore, the two quantities are ultimately equal.*

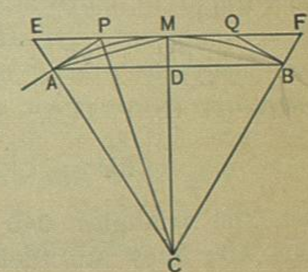
* Newton's Principia, Book I., Lemma I.

Cor. If we take any two similar regular polygons, the one circumscribed about, and the other inscribed in the circle, and bisect the arcs, and then circumscribe and inscribe two regular polygons having double the number of sides, it is plain that by continuing the operation, two new polygons may be found which shall differ from each other by less than any given surface; hence, by the lemma, the two polygons will become ultimately equal. But this equality can not take place for any finite number of sides; hence, the number of sides in each will be infinite, and each will coincide with the circle, which is their common limit. Under this hypothesis, the perimeter of each polygon will coincide with the circumference of the circle.

Scholium. The circle may be regarded as a regular polygon having an infinite number of sides. The circumference may be regarded as the ***perimeter***, and the radius as the ***apothem.***

PROPOSITION XI. PROBLEM.

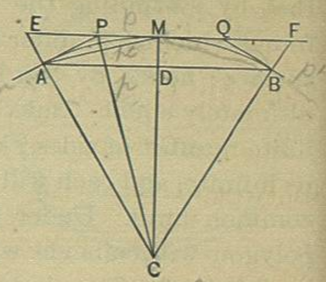
The area of a regular inscribed polygon, and that of a similar circumscribed polygon being given, to find the areas of the regular inscribed and circumscribed polygons having double the number of sides.



Let AB be the side of the given inscribed, and EF that of the given circumscribed polygon. Let C be their common centre, AMB a portion of the circumference of the circle, and M the middle point of the arc AMB.

Draw the chord AM, and at A and B draw the tangents AP and BQ; then AM is the side of the inscribed polygon, and PQ the side of the circumscribed polygon of double the number of sides (P. VII.). Draw CE, CP, CM, and CF.

Denote the area of the given inscribed polygon by p, the area of the given circumscribed polygon by P, and the areas of the inscribed and circumscribed polygons having double the number of sides, respectively by p' and P'.



1°. The triangles CAD, CAM, and CEM, are like parts of the polygons to which they belong: hence, they are proportional to the polygons themselves. But CAM is a mean proportional between CAD and CEM (B. IV., P. XXIV., C.); consequently, p' is a mean proportional between p and P: hence,

$p'^2 = P \times p$

$p' = \sqrt{P \times p}$

$$p' = \sqrt{p \times P}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$P : p' :: p' : p$

2°. Because the triangles CPM and CPE have the common altitude CM, they are to each other as their bases: hence,

$$CPM : CPE :: PM : PE;$$

and because CP bisects the angle ACM, we have (B. IV., P. XVII.),

$$PM : PE :: CM : CE :: CD : CA;$$

hence (B. II., P. IV.),

$$CPM : CPE :: CD : CA \text{ or } CM.$$

But, the triangles CAD and CAM have the common altitude AD; they are, therefore, to each other as their bases: hence,

$$CAD : CAM :: CD : CM;$$

or, because CAD and CAM are to each other as the polygons to which they belong,

$$p : p' :: CD : CM;$$

hence (B. II., P. IV.), we have,

$$CPM : CPE :: p : p';$$

and, by composition,

$$CPM : CPM + CPE \text{ or } CME :: p : p + p';$$

hence (B. II., P. VII.),

$$2CPM \text{ or } CMPA : CME :: 2p : p + p'.$$

But, CMPA and CME are like parts of P' and P; hence,

$$P' : P :: 2p : p + p';$$

or,

$$P' = \frac{2p \times P}{p + p'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$P' \times p + p' = P \times 2p$

Scholium. By means of Equation (1), we can find p' and then, by means of Equation (2), we can find P'.

$P' = \frac{P \times 2p}{p + p'}$

PROPOSITION XII. PROBLEM.

To find the approximate area of a circle whose radius is 1.

The area of an inscribed square is equal to twice the square described on the radius (P. III., S.); the area of a circumscribed square is equal to the square described on the *diameter*. If the radius be taken as the unit of linear measure, and the square described on it as the unit of area, the area of the inscribed square will be 2, and that of the circumscribed square will be 4. Making p equal to 2, and P equal to 4, we have, from Equations (1) and (2) of Proposition XI.,

$$p' = \sqrt{8} = 2.8284271 \quad . \quad . \quad \text{inscribed octagon,}$$

$$P' = \frac{16}{2 + \sqrt{8}} = 3.3137085 \quad . \quad . \quad \text{circumscribed octagon.}$$

Making p equal to 2.8284271, and P equal to 3.3137085, we have, from the same equations,

$p' = 3.0614674$ . . . inscribed polygon of 16 sides.

$P' = 3.1825979$ . . . circumscribed polygon of 16 sides.

By a continued application of these equations, we find the areas indicated below:

Number of Sides.	Inscribed Polygons.	Circumscribed Polygons.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927

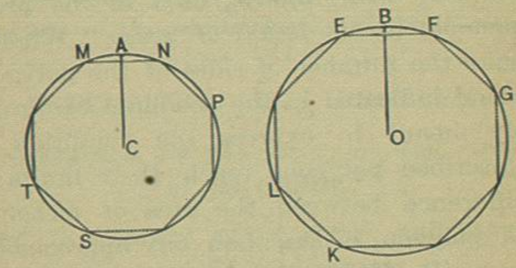
Now, the figures which express the areas of the last two polygons are the same for six decimal places; hence, those areas differ from each other by less than one millionth part of the measuring unit. But the circle differs from either of the polygons by less than they differ from each other. Hence, for all ordinary computation, it is sufficiently accurate to consider the area of a circle, whose radius is 1, equal to 3.141592; the unit of measure being, as shown above, the square described on the radius. This value, 3.141592, is represented by the Greek letter π.

Sch. For ordinary accuracy, π is taken equal to 3.1416.

PROPOSITION XIII. THEOREM.

The circumferences of circles are to each other as their radii, and the areas are to each other as the squares of their radii.

Let C and O be the centres of two circles whose radii are CA and OB: then the circumferences are to each other as their radii, and the areas are to each other as the squares of their radii.

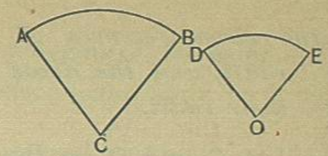


For, let similar regular polygons MNPST and EFGKL be inscribed in the circles: then the perimeters of these polygons are to each other as their apothems, and the areas are to each other as the squares of their apothems, whatever may be the number of their sides (P. IX.).

If the number of sides is made infinite (P. X., Sch.), the polygons coincide with the circles, the perimeters with the circumferences, and the apothems with the radii: hence, the circumferences of the circles are to each other as their radii, and the areas are to each other as the squares of the radii; *which was to be proved.*

Cor. 1. Diameters of circles are proportional to their radii: hence, *the circumferences of circles are proportional to their diameters, and the areas are proportional to the squares of the diameters.*

Cor. 2. Similar arcs, as AB and DE, are like parts of the circumferences to which they belong, and similar sectors, as ACB and DOE, are like parts of the circles to which they belong: hence, *similar arcs are to each other as their radii, and similar sectors are to each other as the squares of their radii.*

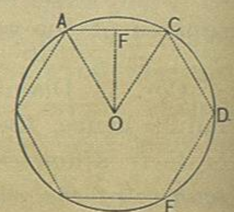


Scholium. The term *infinite*, used in the proposition, is to be understood in its *technical sense.* When it is proposed to make the number of sides of the polygons *infinite*, by the method indicated in the scholium of Proposition X., it is simply meant to express the condition of things, when the inscribed polygons reach their limits; in which case, the difference between the area of either circle and its inscribed polygon, is less than any appreciable quantity. We have seen (P. XII.), that when the number of sides is 16384, the areas differ by less than the millionth part of the measuring unit. By increasing the number of sides, we approximate still nearer.

PROPOSITION XIV. THEOREM.

The area of a circle is equal to half the product of its circumference and radius.

Let O be the centre of a circle, OC its radius, and ACDE its circumference: then the area of the circle is equal to half the product of the circumference and radius.



For, inscribe in it a regular polygon ACDE. Then the area of this polygon is equal to half the product of its perimeter and apothem, whatever may be the number of its sides (P. VIII.).

If the number of sides is made infinite, the polygon coincides with the circle, the perimeter with the circumference, and the apothem with the radius: hence, the area of the circle is equal to half the product of its circumference and radius; *which was to be proved.*

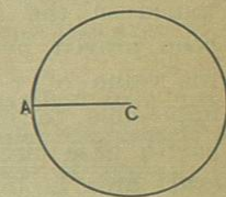
Cor. 1. The area of a sector is equal to half the product of its arc and radius.

Cor. 2. The area of a sector is to the area of the circle, as the arc of the sector to the circumference.

PROPOSITION XV. PROBLEM.

To find an expression for the area of any circle in terms of its radius.

Let C be the centre of a circle, and CA its radius. Denote its area by *area* CA, its radius by R, and the area of a circle whose radius is 1, by π (P. XII., S.).



Then, because the areas of circles are to each other as the squares of their radii (P. XIII.), we have,

$$\textit{area}\ \mathrm{CA} : \pi :: \mathrm{R}^2 : 1;$$

whence,

$$\textit{area}\ \mathrm{CA} = \pi\mathrm{R}^2.$$

That is, *the area of any circle is* 3.1416 *times the square of its radius.*

PROPOSITION XVI. PROBLEM.

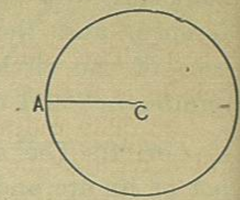
To find an expression for the circumference of a circle, in terms of its radius, or diameter.

Let C be the centre of a circle, and CA its radius.

Denote its circumference by *circ.* CA, its radius by R, and its diameter by D. From the last Proposition, we have,

$$area\ CA = \pi R^2;$$

and, from Proposition XIV., we have,



$$area\ CA = \tfrac{1}{2}\,circ.\ CA \times R;$$

hence, $\tfrac{1}{2}\,circ.\ CA \times R = \pi R^2;$

whence, by reduction,

$$circ.\ CA = 2\pi R, \quad \text{or,} \quad circ.\ CA = \pi D.$$

That is, *the circumference of any circle is equal to* 3.1416 *times its diameter.*

Scholium 1. The abstract number π, equal to 3.1416, denotes the number of times that the diameter of a circle is contained in the circumference, and also the number of times that the square constructed on the radius is contained in the area of the circle (P. XV.). Now, it has been proved by the methods of higher mathematics, that the value of π is incommensurable with 1; hence, it is impossible to express, by means of numbers, the *exact* length of a circumference in terms of the radius, or the *exact* area in terms of the square described on the radius. It is not possible, therefore, to *square the circle*—that is, to construct a square whose area shall be *exactly* equal to that of the circle.

Scholium 2. Besides the approximate value of π, 3.1416, usually employed, the fractions $\frac{22}{7}$ and $\frac{355}{113}$ are also sometimes used to express the ratio of the diameter to the circumference.

EXERCISES.

1. The side of an equilateral triangle inscribed in a circle is 6 feet; find the radius of the circle.

2. The radius of a circle is 10 feet; find the apothem of a regular inscribed hexagon.

3. Find the side of a square inscribed in a circle whose radius is 5 feet.

4. Draw a line whose length shall be $\sqrt{3}$.

5. The radius of a circle is 4 feet; find the area of an inscribed equilateral triangle.

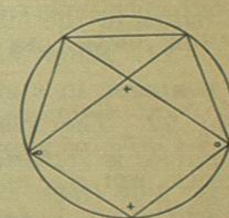
6. Show that the sums of the alternate angles of an octagon inscribed in a circle are equal to each other.

7. The area of a regular hexagon, whose side is 20 feet, is 1039.23 square feet; find the apothem.

8. One side of a regular decagon is 20 feet, and its apothem 15.4 feet; find the perimeter and the area of a similar decagon whose apothem is 8 feet.

9. The area of a regular hexagon inscribed in a circle is 9 square feet, and the area of a similar circumscribed hexagon is 12 square feet; find the areas of regular inscribed and circumscribed polygons of 12 sides.

10. Given two diagonals of a regular pentagon that intersect; show that the greater segments will be equal to each other and to a side of the pentagon, and that the diagonals cut each other in extreme and mean ratio.



11. Show how to inscribe in a given circle a regular polygon of 15 sides.

12. Find the side and the altitude of an equilateral triangle in terms of the radius of the inscribed circle.

13. Given an equilateral triangle inscribed in a circle, and a similar circumscribed triangle; determine the ratio of the two triangles to each other.

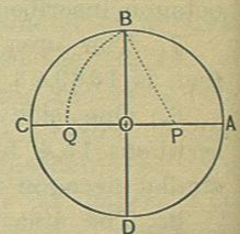
14. The diameter of a circle is 20 feet; find the area of a sector whose arc is 120°.

15. The circumference of a circle is 200 feet; find its area.

16. The area of a circle is 78.54 square yards; find its diameter.

17. The radius of a circle is 10 feet, and the area of a circular sector 100 square feet; find the arc of the sector in degrees.

18. Show that the area of an equilateral triangle circumscribed about a circle is greater than that of a square circumscribed about the same circle.



19. Let AC and BD be diameters perpendicular to each other; from P, the middle point of the radius OA, as a centre, and a radius equal to PB, describe an arc cutting OC in Q; show that the radius OC is divided in extreme and mean ratio at Q.

20. Show that the square of the side of a regular inscribed pentagon is equal to the square of the side of a regular inscribed decagon increased by the square of the radius of the circumscribing circle.

21. Show how, from 19 and 20, to inscribe a regular pentagon in a given circle.

22. The side of a regular pentagon, inscribed in a circle, is 5 feet, and that of a regular inscribed decagon is 2.65 feet; find the side and the area of a regular hexagon inscribed in the same circle.

BOOK VI.

PLANES AND POLYEDRAL ANGLES.

DEFINITIONS.

1. A straight line is PERPENDICULAR TO A PLANE, when it is perpendicular to every straight line of the plane which passes through its FOOT; that is, through the *point* in which it meets the plane.

In this case, the plane is also perpendicular to the line.

2. A straight line is PARALLEL TO A PLANE, when it can not meet the plane, how far soever both may be produced.

In this case, the plane is also parallel to the line.

3. Two PLANES ARE PARALLEL, when they can not meet, how far soever both may be produced.

4. A DIEDRAL ANGLE is the amount of divergence of two planes.

The line in which the planes meet is called the *edge of the angle*, and the planes themselves are called *faces of the angle.*

The measure of a diedral angle is the same as that of a plane angle formed by two straight lines, one in each face, and both perpendicular to the edge at the same point. A diedral angle may be *acute*, *obtuse*, or a *right angle.* In the latter case, the faces are *perpendicular* to each other.