13. Given an equilateral triangle inscribed in a circle, and a similar circumscribed triangle; determine the ratio of the two triangles to each other.

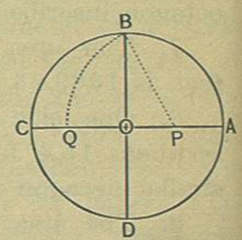
14. The diameter of a circle is 20 feet; find the area of a sector whose arc is 120°.

15. The circumference of a circle is 200 feet; find its area.

16. The area of a circle is 78.54 square yards; find its diameter.

17. The radius of a circle is 10 feet, and the area of a circular sector 100 square feet; find the arc of the sector in degrees.

18. Show that the area of an equilateral triangle circumscribed about a circle is greater than that of a square circumscribed about the same circle.



19. Let AC and BD be diameters perpendicular to each other; from P, the middle point of the radius OA, as a centre, and a radius equal to PB, describe an arc cutting OC in Q; show that the radius OC is divided in extreme and mean ratio at Q.

20. Show that the square of the side of a regular inscribed pentagon is equal to the square of the side of a regular inscribed decagon increased by the square of the radius of the circumscribing circle.

21. Show how, from 19 and 20, to inscribe a regular pentagon in a given circle.

22. The side of a regular pentagon, inscribed in a circle, is 5 feet, and that of a regular inscribed decagon is 2.65 feet; find the side and the area of a regular hexagon inscribed in the same circle.

BOOK VI.

PLANES AND POLYEDRAL ANGLES.

DEFINITIONS.

1. A straight line is PERPENDICULAR TO A PLANE, when it is perpendicular to every straight line of the plane which passes through its FOOT; that is, through the *point* in which it meets the plane.

In this case, the plane is also perpendicular to the line.

2. A straight line is PARALLEL TO A PLANE, when it can not meet the plane, how far soever both may be produced.

In this case, the plane is also parallel to the line.

3. Two PLANES ARE PARALLEL, when they can not meet, how far soever both may be produced.

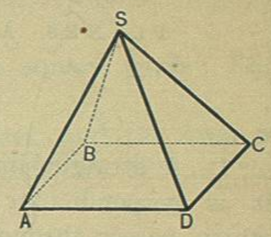
4. A DIEDRAL ANGLE is the amount of divergence of two planes.

The line in which the planes meet is called the *edge of the angle*, and the planes themselves are called *faces of the angle*.

The measure of a diedral angle is the same as that of a plane angle formed by two straight lines, one in each face, and both perpendicular to the edge at the same point. A diedral angle may be *acute*, *obtuse*, or a *right angle*. In the latter case, the faces are *perpendicular* to each other.

5. A POLYEDRAL ANGLE is the amount of divergence of several planes meeting at a common point.

This point is called the *vertex of the angle;* the lines in which the planes meet are called *edges of the angle*, and the portions of the planes lying between the edges are called *faces of the angle*. Thus, S is the vertex of the polyedral angle, whose edges are SA, SB, SC, SD, and whose faces are ASB, BSC, CSD, DSA.



A polyedral angle which has but three faces, is called a *triedral angle*.

POSTULATE.

A straight line may be drawn perpendicular to a plane from any point of the plane, or from any point without the plane.

PROPOSITION I. THEOREM.

If a straight line has two of its points in a plane, it lies wholly in that plane.

For, by definition, a plane is a surface such, that if any two of its points are joined by a straight line, that line lies wholly in the surface (B. I., D. 8).

Cor. Through any point of a plane, an infinite number of straight lines may be drawn which lie in the plane. For, if a straight line is drawn from the given point to any other point of the plane, that line lies wholly in the plane.

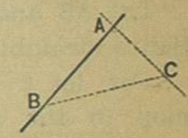
Scholium. If any two points of a plane are joined by a straight line, the plane may be turned about that line as

an axis, so as to take an infinite number of positions. Hence, we infer that an infinite number of planes may be passed through a given straight line.

PROPOSITION II. THEOREM.

Through three points, not in the same straight line, one plane can be passed, and only one.

Let A, B, and C be the three points: then can one plane be passed through them, and only one.



Join two of the points, as A and B, by the line AB. Through AB let a plane be passed, and let this plane be turned around AB until it contains the point C; in this position it will pass through the three points A, B, and C. If now, the plane be turned about AB, in either direction, it will no longer contain the point C: hence, one plane can always be passed through three points, and only one; *which was to be proved.*

Cor. 1. Three points, not in a straight line, determine the position of a plane, because only one plane can be passed through them.

Cor. 2. A straight line and a point without that line determine the position of a plane, because only one plane can be passed through them.

Cor. 3. Two straight lines which intersect determine the position of a plane. For, let AB and AC intersect at A: then either line, as AB, and one point of the other, as C, determine the position of a plane.

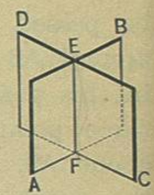
Cor. 4. Two parallel straight lines determine the position of a plane. For, let AB and CD be parallel. By definition (B. I., D. 16) two parallel lines always lie in the same plane. But either line, as AB, and any point of the other, as F, determine the position of a plane: hence, two parallels determine the position of a plane.

PROPOSITION III. THEOREM.

The intersection of two planes is a straight line.

Let AB and CD be two planes: then is their intersection a straight line.

For, let E and F be any two points common to the planes; draw the straight line EF. This line having two points in the plane AB, lies wholly in that plane; and having two points in the plane CD, lies wholly in that plane: hence, every point of EF is common to both planes. Furthermore, the planes can have no common point lying without EF, otherwise there would be two planes passing through a straight line and a point lying without it, which is impossible (P. II., C. 2); hence, the intersection of the two planes is a straight line; *which was to be proved.*



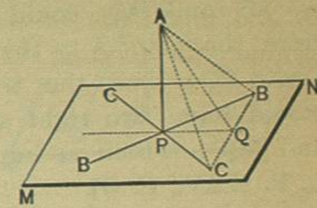
PROPOSITION IV. THEOREM.

If a straight line is perpendicular to two straight lines at their point of intersection, it is perpendicular to the plane of those lines.

Let MN be the plane of the two lines BB, CC, and let AP be perpendicular to these lines at P: then is AP per-

pendicular to every straight line of the plane which passes through P, and consequently, to the plane itself.

For, through P, draw in the plane MN, any line PQ; through any point of this line, as Q, draw the line BC, so that BQ shall be equal to QC (B. IV., Prob. V.); draw AB, AQ, and AC.



The base BC, of the triangle BPC, being bisected at Q, we have (B. IV., P. XIV.),

$$\overline{PC}^2 + \overline{PB}^2 = 2\overline{PQ}^2 + 2\overline{QC}^2.$$

In like manner, we have, from the triangle ABC,

$$\overline{AC}^2 + \overline{AB}^2 = 2\overline{AQ}^2 + 2\overline{QC}^2.$$

Subtracting the first of these equations from the second, member from member, we have,

$$\overline{AC}^2 - \overline{PC}^2 + \overline{AB}^2 - \overline{PB}^2 = 2\overline{AQ}^2 - 2\overline{PQ}^2.$$

But, from Proposition XI., C. 1, Book IV., we have,

$$\overline{AC}^2 - \overline{PC}^2 = \overline{AP}^2, \quad \text{and} \quad \overline{AB}^2 - \overline{PB}^2 = \overline{AP}^2;$$

hence, by substitution,

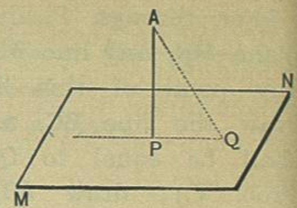
$$2\overline{AP}^2 = 2\overline{AQ}^2 - 2\overline{PQ}^2;$$

whence,

$$\overline{AP}^2 = \overline{AQ}^2 - \overline{PQ}^2; \quad \text{or,} \quad \overline{AP}^2 + \overline{PQ}^2 = \overline{AQ}^2.$$

The triangle APQ is, therefore, right-angled at P (B. IV., P. XIII., S.), and consequently, AP is perpendicular to PQ: hence, AP is perpendicular to every line of the plane MN passing through P, and consequently, to the plane itself; *which was to be proved.*

Cor. 1. Only one perpendicular can be drawn to a plane from a point without the plane. For, suppose two perpendiculars, as AP and AQ, could be drawn from the point A to the plane MN. Draw PQ; then the triangle APQ would have two right angles, APQ and AQP; which is impossible (B. I., P. XXV., C. 3).



Cor. 2. Only one perpendicular can be drawn to a plane from a point of that plane. For, suppose that two perpendiculars could be drawn to the plane MN, from the point P. Pass a plane through the perpendiculars, and let PQ be its intersection with MN; then we should have two perpendiculars drawn to the same straight line from a point of that line; which is impossible (B. I., P. XIV.).

PROPOSITION V. THEOREM.

If from a point without a plane, a perpendicular is drawn to the plane, and oblique lines drawn to different points of the plane:

1°. *The perpendicular is shorter than any oblique line:*

2°. *Oblique lines which meet the plane at equal distances from the foot of the perpendicular, are equal:*

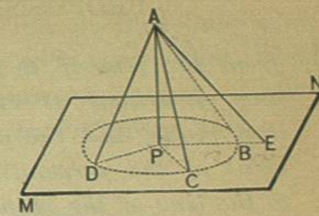
3°. *Of two oblique lines which meet the plane at unequal distances from the foot of the perpendicular, the one which meets it at the greater distance is the longer.*

Let A be a point without the plane MN; let AP be perpendicular to the plane; let AC, AD, be any two oblique lines meeting the plane at equal distances from the foot of the perpendicular; and let AC and AE be any

two oblique lines meeting the plane at unequal distances from the foot of the perpendicular:

1°. AP is shorter than any oblique line AC.

For, draw PC; then is AP less than AC (B. I., P. XV.); *which was to be proved.*



2°. AC and AD are equal.

For, draw PD; then the right-angled triangles APC, APD, have the side AP common, and the sides PC, PD, equal: hence, the triangles are equal in all respects, and consequently, AC and AD are equal; *which was to be proved.*

3°. AE is greater than AC.

For, draw PE, and take PB equal to PC; draw AB: then is AE greater than AB (B. I., P. XV.); but AB and AC are equal: hence, AE is greater than AC; *which was to be proved.*

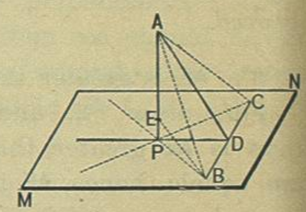
Cor. The equal oblique lines AB, AC, AD, meet the plane MN in the circumference of a circle whose centre is P, and whose radius is PB: hence, to draw a perpendicular to a given plane MN, from a point A, without that plane, find three points B, C, D, of the plane equally distant from A, and then find the centre, P, of the circle whose circumference passes through these points: then AP is the perpendicular required.

Scholium. The angle ABP is called *the inclination of the oblique line* AB to the plane MN. The equal oblique lines AB, AC, AD, are all equally inclined to the plane MN. The inclination of AE is less than the inclination of any shorter line AB.

PROPOSITION VI. THEOREM.

If from the foot of a perpendicular to a plane, a straight line is drawn at right angles to any straight line of that plane, and the point of intersection joined with any point of the perpendicular, the last line is perpendicular to the line of the plane.

Let AP be perpendicular to the plane MN, P its foot, BC the given line, and A any point of the perpendicular; draw PD at right angles to BC, and join the point D with A: then is AD perpendicular to BC.



For, lay off DB equal to DC, and draw PB, PC, AB, and AC. Because PD is perpendicular to BC, and DB equal to DC, we have, PB equal to PC (B. I., P. XV.); and because AP is perpendicular to the plane MN, and PB equal to PC, we have AB equal to AC (P. V.). The line AD has, therefore, two of its points A and D, each equally distant from B and C: hence, it is perpendicular to BC (B. I., P. XVI., C.); *which was to be proved.*

Cor. 1. The line BC is perpendicular to the plane of the triangle APD; because it is perpendicular to AD and PD, at D (P. IV.).

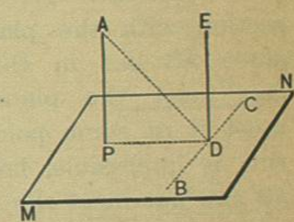
Cor. 2. The shortest distance between AP and BC is measured on PD, perpendicular to both. For, draw BE between any other points of the lines: then BE is greater than PB, and PB greater than PD: hence, PD is less than BE.

Scholium. The lines AP and BC, though not in the same plane, are considered perpendicular to each other. In general, any two straight lines not in the same plane are considered as making an angle with each other, which angle is equal to that formed by drawing, through a given point, two lines respectively parallel to the given lines.

PROPOSITION VII. THEOREM.

If one of two parallels is perpendicular to a plane, the other one is also perpendicular to the same plane.

Let AP and ED be two parallels, and let AP be perpendicular to the plane MN: then is ED also perpendicular to the plane MN.



For, pass a plane through the parallels; its intersection with MN is PD; draw AD, and in the plane MN draw BC perpendicular to PD at D. Now, BD is perpendicular to the plane APDE (P. VI., C. 1); the angle BDE is consequently a right angle; but the angle EDP is a right angle, because ED is parallel to AP (B. I., P. XX., C. 1): hence, ED is perpendicular to BD and PD, at their point of intersection, and consequently, to their plane MN (P. IV.); *which was to be proved.*

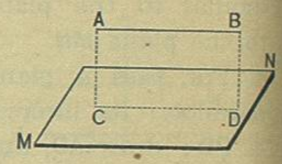
Cor. 1. If the lines AP and ED are perpendicular to the plane MN, they are parallel to each other. For, if not, conceive a line drawn through D parallel to PA; it would be perpendicular to the plane MN, from what has just been proved; we would, therefore, have two perpendiculars to the plane MN, at the same point; which is impossible (P. IV., C. 2).

Cor. 2. If two straight lines, A and B, are parallel to a third line C, they are parallel to each other. For, pass a plane perpendicular to C; it will be perpendicular to both A and B: hence, A and B are parallel.

PROPOSITION VIII. THEOREM.

If a straight line is parallel to a line of a plane, it is parallel to that plane.

Let the line AB be parallel to the line CD of the plane MN; then is AB parallel to the plane MN.

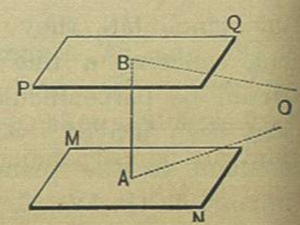


For, through AB and CD pass a plane (P. II., C. 4); CD is its intersection with the plane MN. Now, since AB lies in this plane, if it can meet the plane MN, it will meet it at some point of CD; but this is impossible, because AB and CD are parallel: hence, AB can not meet the plane MN, and consequently, it is parallel to it; *which was to be proved.*

PROPOSITION IX. THEOREM.

If two planes are perpendicular to the same straight line, they are parallel to each other.

Let the planes MN and PQ be perpendicular to the line AB, at the points A and B: then are they parallel to each other.

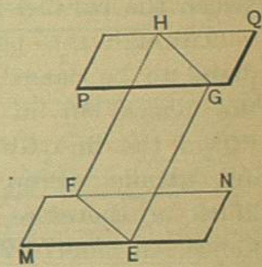


For, if they are not parallel, they will meet; and let O be a point common to both. From O draw the lines OA and OB: then, since OA lies in the plane MN, it is perpendicular to BA at A (D. 1). For a like reason, OB is perpendicular to AB at B: hence, the triangle OAB has two right angles, which is impossible; consequently, the planes can not meet, and are therefore parallel; *which was to be proved.*

PROPOSITION X. THEOREM.

If a plane intersects two parallel planes, the lines of intersection are parallel.

Let the plane EH intersect the parallel planes MN and PQ, in the lines EF and GH: then are EF and GH parallel.



For, if they are not parallel, they will meet if sufficiently prolonged, because they lie in the same plane; but if the lines meet, the planes MN and PQ, in which they lie, also meet; but this is impossible, because these planes are parallel: hence, the lines EF and GH can not meet; they are, therefore, parallel; *which was to be proved.*

PROPOSITION XI. THEOREM.

If a straight line is perpendicular to one of two parallel planes, it is also perpendicular to the other.

Let MN and PQ be two parallel planes, and let the line AB be perpendicular to PQ: then is it also perpendicular to MN.