11. Show that the inclination of a line to a plane—that is, the angle which the line makes with its own projection on the plane—is the least angle made by the line with any line of the plane.

12. Show that if three lines are perpendicular to a fourth at the same point, the first three are in the same plane.

13. Show that when a plane is perpendicular to a given line at its middle point, every point of the plane is equally distant from the extremities of the line, and that every point out of the plane is unequally distant from the extremities of the line.

14. Show that through a line parallel to a given plane, but one plane can be passed perpendicular to the given plane.

15. Show that if two planes which intersect contain two lines parallel to each other, the intersection of the planes is parallel to the lines.

16. Show that when a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

17. Draw a perpendicular to two lines not in the same plane.

18. Show that the three planes which bisect the diedral angles formed by the consecutive faces of a triedral angle, meet in the same line.

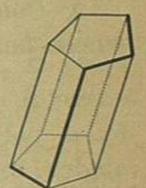
BOOK VII.

POLYEDRONS.

DEFINITIONS.

1. A Polyedron is a volume bounded by polygons.

The bounding polygons are called *faces* of the polyedron; the lines in which the faces meet, are called *edges* of the polyedron; the points in which the edges meet, are called *vertices* of the polyedron.



2. A Prism is a polyedron in which two of the faces are polygons equal in all respects, and having their homologous sides parallel. The other faces are parallelograms (B. I., P. XXX.).

The equal polygons are called *bases* of the prism; one the *upper*, and the other the *lower base*; the parallelograms taken together make up the *lateral* or *convex surface* of the prism; the lines in which the lateral faces meet, are called *lateral edges*, and the lines in which the lateral faces meet either base are called *basal edges* of the prism.

3. The Altitude of a prism is the perpendicular distance between the planes of its bases.

4. A Right Prism is one whose lateral edges are perpendicular to the planes of the bases.

In this case, any lateral edge is equal to the altitude.

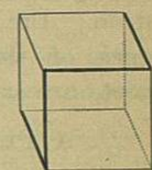
5. An Oblique Prism is one whose lateral edges are oblique to the planes of the bases.

In this case, any lateral edge is greater than the altitude.

6. Prisms are named from the number of sides of their bases; a *triangular prism* is one whose bases are triangles; a *pentagonal* prism is one whose bases are pentagons, &c.

7. A Parallelopipedon is a prism whose bases are parallelograms.

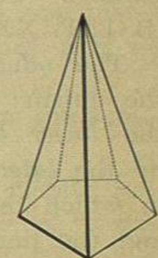
A *Right Parallelopipedon* is one whose lateral edges are perpendicular to the planes of the bases.



A *Rectangular Parallelopipedon* is one whose faces are all rectangles.

A *Cube* is a rectangular parallelopipedon whose faces are squares.

8. A Pyramid is a polyedron bounded by a polygon called the *base*, and by triangles meeting at a common point, called the *vertex* of the pyramid.



The triangles taken together make up the *lateral* or *convex surface* of the pyramid; the lines in which the lateral faces meet, are called the *lateral edges*, and the lines in which the lateral faces meet the base are called *basal edges* of the pyramid.

9. Pyramids are named from the number of sides of their bases; a *triangular pyramid* is one whose base is a triangle; a *quadrangular* pyramid is one whose base is a quadrilateral, and so on.

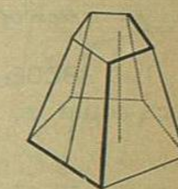
10. The Altitude of a pyramid is the perpendicular distance from the vertex to the plane of its base.

11. A Right Pyramid is one whose base is a regular polygon, and in which the perpendicular, drawn from the vertex to the plane of the base, passes through the centre of the base.

This perpendicular is called the axis of the pyramid.

12. The Slant Height of a right pyramid, is the perpendicular distance from the vertex to any side of the base.

13. A Truncated Pyramid is that portion of a pyramid included between the base and any plane which cuts the pyramid.



When the cutting plane is parallel to the base, the truncated pyramid is called a frustum of a pyramid, and the intersection of the cutting plane with the pyramid, is called the *upper base* of the frustum; the base of the pyramid is called the *lower* base of the frustum.

14. The Altitude of a frustum of a pyramid, is the perpendicular distance between the planes of its bases.

15. The Slant Height of a frustum of a right pyramid, is that portion of the slant height of the pyramid which lies between the planes of its upper and lower bases.

16. Similar Polyedrons are those which are bounded by the same number of similar polygons, similarly placed.

Parts which are similarly placed, whether faces, edges, or angles, are called *homologous*.

17. A Diagonal of a polyedron, is a straight line joining the vertices of two polyedral angles not in the same face.

18. The VOLUME OF A POLYEDRON is its numerical value expressed in terms of some other polyedron taken as a unit.

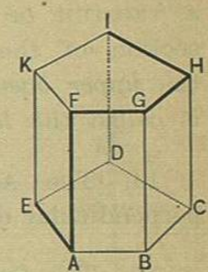
The unit generally employed is a cube constructed on the linear unit as an edge.

PROPOSITION I. THEOREM.

The convex surface of a right prism is equal to the perimeter of either base multiplied by the altitude.

Let ABCDE–K be a right prism: then is its convex surface equal to,

$$(AB + BC + CD + DE + EA) \times AF.$$



For, the convex surface is equal to the sum of all the rectangles AG, BH, CI, DK, EF, which compose it. Now, the altitude of each of the rectangles AF, BG, CH, &c., is equal to the altitude of the prism, and the area of each rectangle is equal to its base multiplied by its altitude (B. IV., P. V.): hence, the sum of these rectangles, or the convex surface of the prism, is equal to,

$$(AB + BC + CD + DE + EA) \times AF;$$

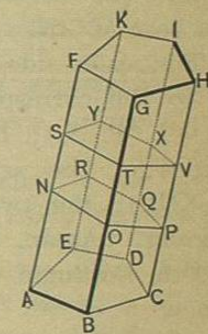
that is, to the perimeter of the base multiplied by the altitude; *which was to be proved.*

Cor. If two right prisms have the same altitude, their convex surfaces are to each other as the perimeters of their bases.

PROPOSITION II. THEOREM.

In any prism, the sections made by parallel planes are polygons equal in all respects.

Let the prism AH be intersected by the parallel planes NP, SV: then are the sections NOPQR, STVXY, equal polygons.



For, the sides NO, ST, are parallel, being the intersections of parallel planes with a third plane ABGF; these sides, NO, ST, are included between the parallels NS, OT: hence, NO is equal to ST (B. I., P. XXVIII., C. 2). For like reasons, the sides OP, PQ, QR, &c., of NOPQR, are equal to the sides TV, VX, &c., of STVXY, each to each; and since the equal sides are parallel, each to each, it follows that the angles NOP, OPQ, &c., of the first section, are equal to the angles STV, TVX, &c., of the second section, each to each (B. VI., P. XIII.): hence, the two sections NOPQR, STVXY, are equal in all respects; *which was to be proved.*

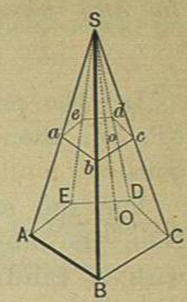
Cor. The bases of a prism and any section of a prism parallel to the bases, are equal in all respects.

PROPOSITION III. THEOREM.

If a pyramid is cut by a plane parallel to the base:
1°. *The edges and the altitude are divided proportionally:*
2°. *The section is a polygon similar to the base.*

Let the pyramid S–ABCDE, whose altitude is SO, be cut by the plane *abcde*, parallel to the base ABCDE.

1°. The edges and altitude are divided proportionally. For, let a plane be passed through the vertex S, parallel to the base AC; then the edges and the altitude are cut by three parallel planes, and are consequently divided proportionally (B. VI., P. XV., C. 2); *which was to be proved.*



2°. The section *abcde* is similar to the base ABCDE.

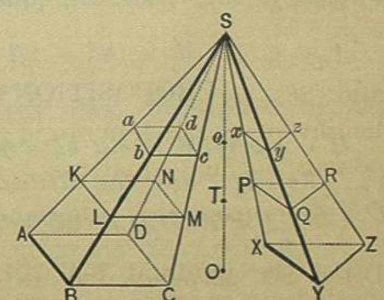
For, each side of the section is parallel to the corresponding side of the base (B. VI., P. X.); hence, the corresponding angles of the section and of the base are equal (B. VI., P. XIII.); the two polygons are therefore mutually equiangular. Again, because *ab* is parallel to AB, and *bc* to BC, the triangle S*ba* is similar to SBA, and S*bc* to SBC; hence,

$$ab : AB :: Sb : SB, \quad \text{and} \quad bc : BC :: Sb : SB,$$

whence (B. II., P. IV.), $ab : AB :: bc : BC.$

In like manner, it may be shown that the remaining sides of *abcde* are proportional to the corresponding sides of ABCDE; hence (B. IV., D. 1), the polygons are similar; *which was to be proved.*

Cor. 1. If two pyramids S–ABCD and S–XYZ, having a common vertex S and their bases in the same plane, are cut by a plane *aoz* parallel to the plane of their bases, the sections are to each other as the bases.



For the polygons *abcd* and ABCD, being similar, are to each other as the squares of any homologous sides (B. IV., P. XXVII.); but

$$\overline{ab}^2 : \overline{AB}^2 :: \overline{Sa}^2 : \overline{SA}^2 :: \overline{So}^2 :: \overline{SO}^2;$$

hence (B. II., P. IV.), we have, $\quad abcd : ABCD :: \overline{So}^2 : \overline{SO}^2.$

In like manner, we have, $\quad xyz : XYZ :: \overline{So}^2 : \overline{SO}^2;$

hence, $\quad abcd : ABCD :: xyz : XYZ.$

Cor. 2. If the bases are equal, any sections at equal distances from the vertex, or from the bases, are equal.

Cor. 3. The area of any section parallel to the base is proportional to the square of its distance from the vertex.

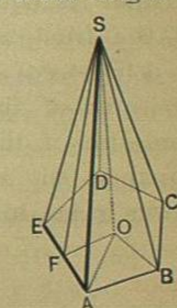
Cor. 4. If the two pyramids are cut by a plane KTR, so that ST is a mean proportional between S*o* and SO, that is, so that $\overline{ST}^2$ is a mean proportional between $\overline{So}^2$ and $\overline{SO}^2$, the section KLMN is a mean proportional between *abcd* and ABCD, and also PQR is a mean proportional between *xyz* and XYZ.

PROPOSITION IV. THEOREM.

The convex surface of a right pyramid is equal to the perimeter of its base multiplied by half the slant height.

Let S be the vertex, ABCDE the base, and SF, perpendicular to EA, the slant height of a right pyramid: then is the convex surface equal to,

$$(AB + BC + CD + DE + EA) \times \tfrac{1}{2}SF.$$



Draw SO perpendicular to the plane of the base.

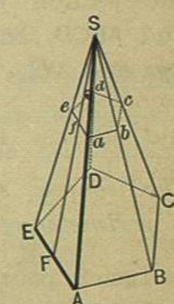
From the definition of a right pyramid, the point O is the centre of the base (D. 11): hence, the lateral edges, SA, SB, &c., are all equal (B. VI., P. V.); but the sides of the base are all equal, being sides of a regular polygon: hence, the lateral faces are all equal, and consequently their altitudes are all equal, each being equal to the slant height of the pyramid.

Now, the area of any lateral face, as SEA, is equal to its base EA, multiplied by half its altitude SF: hence, the sum of the areas of the lateral faces, or the convex surface of the pyramid, is equal to,

$$(AB + BC + CD + DE + EA) \times \tfrac{1}{2}SF;$$

which was to be proved.

Scholium. The convex surface of a frustum of a right pyramid is equal to half the sum of the perimeters of its upper and lower bases, multiplied by the slant height.

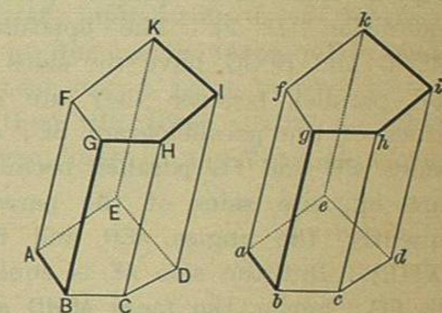


Let ABCDE-*e* be a frustum of a right pyramid, whose vertex is S: then the section *abcde* is similar to the base ABCDE, and their homologous sides are parallel (P. III.). Any lateral face of the frustum, as AE*ea*, is a trapezoid, whose altitude is equal to F*f*, the slant height of the frustum; hence, its area is equal to $\frac{1}{2}(EA + ea) \times Ff$ (B. IV., P. VII.). But the area of the convex surface of the frustum is equal to the sum of the areas of its lateral faces; it is, therefore, equal to the half sum of the perimeters of its upper and lower bases, multiplied by the slant height.

PROPOSITION V. THEOREM.

If the three faces which include a triedral angle of a prism are equal in all respects to the three faces which include a triedral angle of a second prism, each to each, and are like placed, the two prisms are equal in all respects.

Let B and *b* be the vertices of two triedral angles, included by faces respectively equal to each other, and similarly placed: then the prism ABCDE–K is equal to the prism *abcde–k* in all respects.



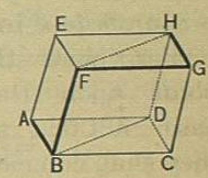
For, place the base *abcde* upon the equal base ABCDE, so that they shall coincide; then because the triedral angles whose vertices are *b* and B, are equal, the parallelogram *bh* will coincide with BH, and the parallelogram *bf* with BF: hence, the two sides *fg* and *gh*, of one upper base, will coincide with the homologous sides FG and GH, of the other upper base; and because the upper bases are equal in all respects, and have been shown to coincide in part, they must coincide throughout; consequently, each of the lateral faces of one prism will coincide with the corresponding lateral face of the other prism; the prisms, therefore, coincide throughout, and are therefore equal in all respects; *which was to be proved.*

Cor. If two right prisms have their bases equal in all respects, and have also equal altitudes, the prisms themselves are equal in all respects. For, the faces which include any triedral angle of the one, are equal in all respects to the faces which include the corresponding triedral angle of the other, each to each, and they are similarly placed.

PROPOSITION VI. THEOREM.

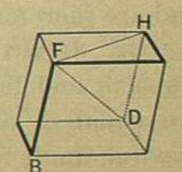
In any parallelopipedon, the opposite faces are equal in all respects, each to each, and their planes are parallel.

Let ABCD–H be a parallelopipedon: then its opposite faces are equal and their planes are parallel.



For, the bases, ABCD and EFGH are equal, and their planes parallel by definition (D. 7). The opposite faces AEHD and BFGC, have the sides AE and BF parallel, because they are opposite sides of the parallelogram BE; and the sides EH and FG parallel, because they are opposite sides of the parallelogram EG; and consequently, the angles AEH and BFG are equal (B. VI., P. XIII.). But the side AE is equal to BF, and the side EH to FG; hence, the faces AEHD and BFGC are equal; and because AE is parallel to BF, and EH to FG, the planes of the faces are parallel (B. VI., P. XIII.). In like manner, it may be shown that the parallelograms ABFE and DCGH, are equal and their planes parallel: hence, the opposite faces are equal, each to each, and their planes are parallel; *which was to be proved.*

Cor. 1. Any two opposite faces of a parallelopipedon may be taken as bases.

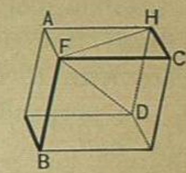


Cor. 2. In a rectangular parallelopipedon, the square of any of the diagonals is equal to the sum of the squares of the three edges which meet at the same vertex.

For, let FD be one of the diagonals, and draw FH.

Then, in the right-angled triangle FHD, we have,

$$\overline{FD}^2 = \overline{DH}^2 + \overline{FH}^2.$$



But DH is equal to FB, and $\overline{FH}^2$ is equal to $\overline{FA}^2$ plus $\overline{AH}^2$ or $\overline{FC}^2$: hence,

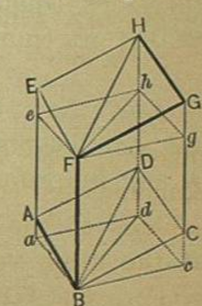
$$\overline{FD}^2 = \overline{FB}^2 + \overline{FA}^2 + \overline{FC}^2.$$

Cor. 3. A parallelopipedon may be constructed on three straight lines AB, AD, and AE, intersecting in a common point A, and not lying in the same plane. For, pass through the extremity of each line, a plane parallel to the plane of the two others; then will these planes, together with the planes of the given lines, be the faces of a parallelopipedon.

PROPOSITION VII. THEOREM.

If a plane is passed through the diagonally opposite edges of a parallelopipedon, it divides the parallelopipedon into two equal triangular prisms.

Let ABCD–H be a parallelopipedon, and let a plane be passed through the edges BF and DH; then are the prisms ABD–H and BCD–H equal in volume.



For, through the vertices F and B let planes be passed perpendicular to FB, the former cutting the other lateral edges in the points *e*, *h*, *g*, and the latter cutting those edges produced, in the points *a*, *d*, and *c*. The sections F*ehg* and B*adc* are parallelograms, because their opposite sides are parallel, each to each (B. VI., P. X.); they are also equal (P. II.): hence, the polyedron B*adc*–*g* is a right prism (D. 2, 4), as are also the polyedrons B*ad*–*h* and B*cd*–*h*.

Place the triangle F*eh* upon B*ad*, so that F shall coincide with B, *e* with *a*, and *h* with *d*; then, because *e*E, *h*H, are perpendicular to the plane F*eh*, and *a*A, *d*D, to the plane B*ad*, the line *e*E takes the direction *a*A, and the line *h*H the direction *d*D. The lines AE and *ae* are equal, because each is equal to BF (B. I., P. XXVIII.). If we take away from the line *a*E the part *ae*, there remains the part *e*E; and if from the same line, we take away the part AE, there remains the part A*a*: hence, *e*E and *a*A are equal (A. 3); for a like reason *h*H is equal to *d*D: hence, the point E coincides with A, and the point H with D, and consequently, the polyedrons F*eh*–H and B*ad*–D coincide throughout, and are therefore equal.

If from the polyedron B*ad*–H, we take away the part B*ad*–D, there remains the prism BAD–H; and if from the same polyedron we take away the part F*eh*–H, there remains the prism B*ad*–*h*: hence, these prisms are equal in volume. In like manner, it may be shown that the prisms BCD–H and B*cd*–*h* are equal in volume.

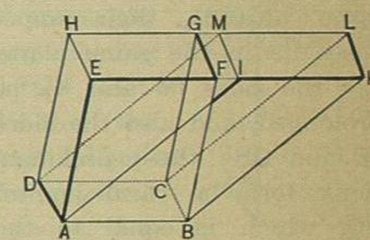
The prisms B*ad*–*h*, and B*cd*–*h*, have equal bases, because these bases are halves of equal parallelograms (B. I., P. XXVIII., C. 1); they have also equal altitudes; they are therefore equal (P. V., C.): hence, the prisms BAD–H and BCD–H are equal (A. 1); *which was to be proved.*

Cor. Any triangular prism ABD–H, is equal to half of the parallelopipedon AG, which has the same triedral angle A, and the same edges AB, AD, and AE.

PROPOSITION VIII. THEOREM.

If two parallelopipedons have a common lower base, and their upper bases between the same parallels, they are equal in volume.

Let the parallelopipedons AG and AL have the common lower base ABCD, and their upper bases EFGH and IKLM, between the same parallels EK and HL: then are they equal in volume.



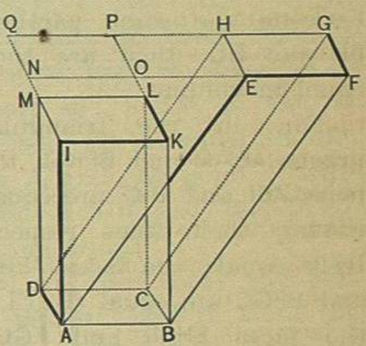
For, in the triangular prisms AEI–M and BFK–L, the faces AEI and BKF are equal, having their sides respectively equal; the faces AEHD and BFGC are equal (P. VI.); the faces EHMI and FGLK are equal, as they consist, respectively, of the common part FGMI and the equal parts EHGF and IMLK: hence, the triangular prisms AEI–M and BFK–L are equal (P. V.).

If from the polyedron ABKE–H, we take away the prism BFK–L, there remains the parallelopipedon AG; and if from the same polyedron we take away the prism AEI–M, there remains the parallelopipedon AL: hence, these parallelopipedons are equal in volume (A. 3); *which was to be proved.*

PROPOSITION IX. THEOREM.

If two parallelopipedons have a common lower base and the same altitude, they are equal in volume.

Let the parallelopipedons AG and AL have the common lower base ABCD and the same altitude: then are they equal in volume.



Because they have the same altitude, their upper bases lie in the same plane. Let the sides IM and KL be prolonged, and also the sides FE and GH; these prolongations form a parallelogram OQ, which is equal to the common base of the given parallelopipedons, because its sides are respectively parallel and equal to the corresponding sides of that base.

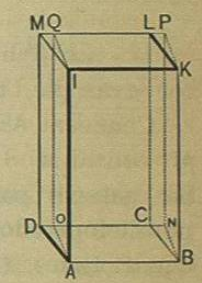
Now, if a third parallelopipedon be constructed, having for its lower base the parallelogram ABCD, and for its upper base NOPQ, this third parallelopipedon will be equal in volume to the parallelopipedon AG, since they will have the same lower base, and their upper bases between the same parallels, QG, NF (P. VIII.). For a like reason, this third parallelopipedon will also be equal in volume to the parallelopipedon AL: hence, the two parallelopipedons AG, AL, are equal in volume; *which was to be proved.*

Cor. Any oblique parallelopipedon is equal in volume to a right parallelopipedon having the same base and the same altitude.

PROPOSITION X. PROBLEM.

To construct a rectangular parallelopipedon equal in volume to a right parallelopipedon whose base is any parallelogram.

Let ABCD–M be a right parallelopipedon, having for its base the parallelogram ABCD.



Through the edges AI and BK pass the planes AQ and BP, respectively perpendicular to the plane AK, the former meeting the face DL in OQ, and the latter meeting that face produced in NP: then the polyedron AP is a rectangular parallelopipedon equal to the given parallelopipedon. It is a rectangular parallelopipedon, because all of its faces are rectangles, and it is equal to the given parallelopipedon, because the two may be regarded as having the common base AK (P. VI., C. 1), and an equal altitude AO (P. IX.).

Cor. 1. Since any oblique parallelopipedon is equal in volume to a right parallelopipedon, having the same base and altitude (P. IX., Cor.); and since any right parallelopipedon is equal in volume to a rectangular parallelopipedon having an equal base and altitude; it follows, that any oblique parallelopipedon is equal in volume to a rectangular parallelopipedon, having an equal base and an equal altitude.

Cor. 2. Any two parallelopipedons are equal in volume when they have equal bases and equal altitudes.