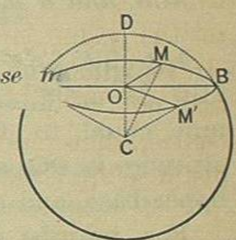
altitudes are equal to that of the frustum: hence, the volume of the frustum of a cone is equal to the sum of the volumes of three cones whose common altitude is that of the frustum, and whose bases are the lower base of the frustum, the upper base of the frustum, and a mean proportional between them; *which was to be proved.*

PROPOSITION VII. THEOREM.

Any section of a sphere made by a plane is a circle.

Let C be the centre of a sphere, CA one of its radii, and AMB any section made by a plane: then is this section a circle.



For, draw a radius CO perpendicular to the cutting plane, and let it pi *its base* the plane of the section at O. I radii of the sphere to any two points M, M′, of the curve which bounds the section, and join these points with O: then, because the radii CM, CM′ are equal, the points M, M′, will be equally distant from O (B. VI., P. V., C.); hence, the section is a circle; *which was to be proved.*

Cor. 1. When the cutting plane passes through the centre of the sphere, the radius of the section is equal to that of the sphere; when the cutting plane does not pass through the centre of the sphere, the radius of the section will be less than that of the sphere.

A section whose plane passes through the centre of the sphere, is called a *great circle* of the sphere. A section whose plane does not pass through the centre of the sphere,

is called a *small circle* of the sphere. All great circles of the same, or of equal spheres, are equal.

Cor. 2. Any great circle divides the sphere, and also the surface of the sphere, into equal parts. For, the parts may be so placed as to coincide, otherwise there would be some points of the surface unequally distant from the centre, which is impossible.

Cor. 3. The centre of a sphere, and the centre of any small circle of that sphere, are in a straight line perpendicular to the plane of the circle.

Cor. 4. The square of the radius of any small circle is equal to the square of the radius of the sphere diminished by the square of the distance from the centre of the sphere to the plane of the circle (B. IV., P. XI., C. 1): hence, circles which are equally distant from the centre, are equal; and of two circles which are unequally distant from the centre, that one is the less whose plane is at the greater distance from the centre.

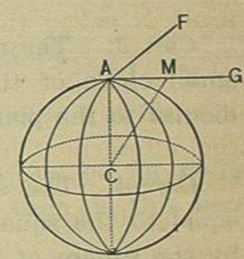
Cor. 5. The circumference of a great circle may always be made to pass through any two points on the surface of a sphere. For, a plane can always be passed through these points and the centre of the sphere (B. VI., P. II.), and its section will be a great circle. If the two points are the extremities of a diameter, an infinite number of planes can be passed through them and the centre of the sphere (B. VI., P. I., S.); in this case, an infinite number of great circles can be made to pass through the two points.

Cor. 6. The bases of a zone are the circumferences of circles (D. 16), and the bases of a segment of a sphere are circles.

PROPOSITION VIII. THEOREM.

Any plane perpendicular to a radius of a sphere at its outer extremity, is tangent to the sphere at that point.

Let C be the centre of a sphere, CA any radius, and FAG a plane perpendicular to CA at A: then is the plane FAG tangent to the sphere at A.



For, from any other point of the plane, as M, draw the line MC: then because CA is a perpendicular to the plane, and CM an oblique line, CM is greater than CA (B. VI., P. V.): hence, the point M lies without the sphere. The plane FAG, therefore, touches the sphere at A, and consequently is tangent to it at that point; *which was to be proved.*

Scholium. It may be shown, by a course of reasoning analogous to that employed in Book III., Propositions XI., XII., XIII., and XIV., that two spheres may have any one of six positions with respect to each other, viz.:

1°. When the distance between their centres is greater than the sum of their radii, *they are external one to the other:*

2°. When the distance is equal to the sum of their radii, *they are tangent externally:*

3°. When this distance is less than the sum, and greater than the difference of their radii, *they intersect each other:*

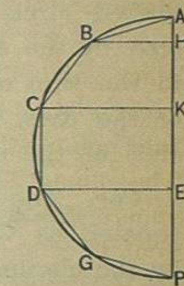
4°. When this distance is equal to the difference of their radii, *they are tangent internally:*

5°. When this distance is less than the difference of their radii, *one is wholly within the other:*

6°. When this distance is equal to zero, *they have a common centre,* or are *concentric.*

DEFINITIONS.

1°. If a semi-circumference is divided into equal arcs, the chords of these arcs form half of the perimeter of a regular inscribed polygon; this half perimeter is called *a regular semi-perimeter.* The figure bounded by the regular semi-perimeter and the diameter of the semi-circumference is called *a regular semi-polygon.* The diameter itself is called the *axis* of the semi-polygon.



2°. If lines are drawn from the extremities of any side perpendicular to the axis, the intercepted portion of the axis is called the *projection* of that side.

The broken line ABCDGP is a regular semi-perimeter; the figure bounded by it and the diameter AP, is a regular semi-polygon, AP is its axis, HK is the projection of the side BC, and the axis, AP, is the projection of the entire semi-perimeter.

PROPOSITION IX. LEMMA.

If a regular semi-polygon is revolved about its axis, the surface generated by the semi-perimeter is equal to the axis multiplied by the circumference of the inscribed circle.

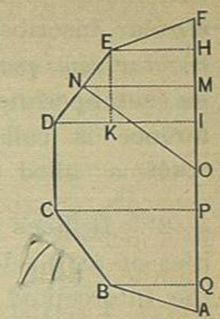
Let ABCDEF be a regular semi-polygon, AF its axis, and ON its apothem: then is the surface generated by the regular semi-perimeter equal to AF $\times$ *circ.* ON.

From the extremities of any side, as DE, draw DI and EH perpendicular to AF; draw also NM perpendicular to AF, and EK perpendicular to DI. Now, the surface generated by DE is equal to DE $\times$ *circ.* NM (P. IV., S.). But, because the triangles EDK and ONM are similar (B. IV., P. XXI.), we have,

$$DE : EK \text{ or } IH :: ON : NM :: circ.\ ON : circ.\ NM;$$

whence,

$$DE \times circ.\ NM = IH \times circ.\ ON;$$

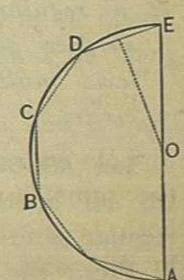


that is, the surface generated by any side is equal to the projection of that side multiplied by the circumference of the inscribed circle: hence, the surface generated by the entire semi-perimeter is equal to the sum of the projections of its sides, or the axis, multiplied by the circumference of the inscribed circle; *which was to be proved.*

Cor. The surface generated by any portion of the perimeter, as CDE, is equal to its projection PH, multiplied by the circumference of the inscribed circle.

PROPOSITION X. THEOREM.

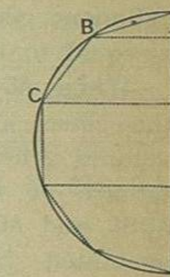
The surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.



Let ABCDE be a semi-circumference, O its centre, and AE its diameter: then is the surface of the sphere generated by revolving the semi-circumference about AE, equal to $AE \times circ.\ OE$.

For, the semi-circumference may be regarded as a regular semi-perimeter with an infinite number of sides, whose axis is AE, and the radius of whose inscribed circle is OE: hence (P. IX.), the surface generated by it is equal to $AE \times circ.\ OE$; *which was to be proved.*

Cor. 1. The circumference of a great circle is equal to $2\pi OE$ (B. V., P. XVI.); hence, the area of the surface of the sphere is equal to $2OE \times 2\pi OE$, or to $4\pi \overline{OE}^2$, that is, *the area of the surface of a sphere is equal to four great circles.*



Cor. 2. The surface generated by any arc of the semicircle, as BC, is a zone, whose altitude is equal to the projection of that arc on the diameter. But, the arc BC is a portion of a semi-perimeter having an infinite number of sides, and the radius of whose inscribed circle is equal to that of the sphere: hence (P. IX., C.), the surface of a zone is equal to its altitude multiplied by the circumference of a great circle of the sphere.

Cor. 3. Zones, on the same sphere, or on equal spheres, are to each other as their altitudes.

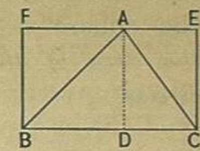
PROPOSITION XI. LEMMA.

If a triangle and a rectangle having the same base and equal altitudes, are revolved about the common base, the volume generated by the triangle is one third of that generated by the rectangle.

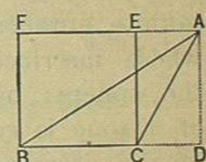
Let ABC be a triangle, and EFBC a rectangle, having the same base BC, and an equal altitude AD, and let them both be revolved about BC: then is the volume generated by ABC one third of that generated by EFBC.

For, the cone generated by the right-angled triangle ADB, is equal to one third of the cylinder generated by the rectangle ADBF (P. V., C. 1), and the cone generated by the triangle ADC, is equal to one third of the cylinder generated by the rectangle ADCE.

When AD falls within the triangle, the sum of the cones generated by ADB and ADC, is equal to the volume generated by the triangle ABC; and the sum of the cylinders generated by ADBF and ADCE, is equal to the volume generated by the rectangle EFBC.



When AD falls without the triangle, the difference of the cones generated by ADB and ADC, is equal to the volume generated by ABC; and the difference of the cylinders generated by ADBF and ADCE, is equal to the volume generated by EFBC: hence, in either case, the volume generated by the triangle ABC, is equal to one third of the volume generated by the rectangle EFBC; *which was to be proved.*



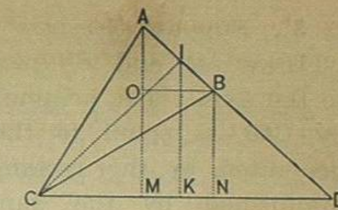
Cor. The volume of the cylinder generated by EFBC, is equal to the product of its base and altitude, or to $\pi\overline{AD}^2 \times BC$: hence, the volume generated by the triangle ABC, is equal to $\frac{1}{3}\pi\overline{AD}^2 \times BC$.

PROPOSITION XII. LEMMA.

If an isosceles triangle is revolved about a straight line passing through its vertex, the volume generated is equal to the surface generated by the base multiplied by one third of the altitude.

Let CAB be an isosceles triangle, C its vertex, AB its base, CI its altitude, and let it be revolved about the line CD, as an axis: then is the volume generated equal to *surf.* $AB \times \frac{1}{3}CI$. There may be three cases:

1°. Suppose the base, when produced, to meet the axis at D; draw AM, IK, and BN, perpendicular to CD, and BO parallel to DC. Now, the volume generated by CAB is equal to the difference of the volumes generated by CAD and CBD; hence (P. XI., C.),



$$vol.\ CAB = \tfrac{1}{3}\pi\overline{AM}^2 \times CD - \tfrac{1}{3}\pi\overline{BN}^2 \times CD = \tfrac{1}{3}\pi(\overline{AM}^2 - \overline{BN}^2) \times CD.$$

But, $\overline{AM}^2 - \overline{BN}^2$ is equal to $(AM + BN)(AM - BN)$ (B. IV., P. X.); and because $AM + BN$ is equal to $2IK$ (P. IV., S.), and $AM - BN$ to AO, we have,

$$vol.\ CAB = \tfrac{2}{3}\pi\, IK \times AO \times CD.$$

But, the right-angled triangles AOB and CDI are similar (B. IV., P. XVIII.); hence,

$$AO : AB :: CI : CD; \quad \text{or,} \quad AO \times CD = AB \times CI.$$

Substituting, and changing the order of the factors, we have,

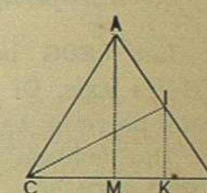
$$vol.\ CAB = AB \times 2\pi IK \times \tfrac{1}{3}CI.$$

But, $AB \times 2\pi IK$ = the surface generated by AB; hence,

$$vol.\ CAB = surf.\ AB \times \tfrac{1}{3}CI.$$

2°. Suppose the axis to coincide with one of the equal sides.

Draw CI perpendicular to AB, and AM and IK, perpendicular to CB. Then,



$$vol.\ CAB = \tfrac{1}{3}\pi\overline{AM}^2 \times CB = \tfrac{1}{3}\pi\, AM \times AM \times CB.$$

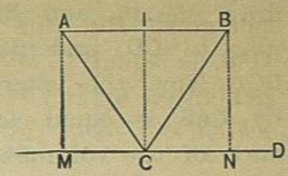
But, since AMB and CIB are similar,

$$AM : AB :: CI : CB; \quad \text{whence,} \quad AM \times CB = AB \times CI.$$

Also, $AM = 2IK$; hence, by substitution, we have,

$$vol.\ CAB = AB \times 2\pi IK \times \tfrac{1}{3}CI = surf.\ AB \times \tfrac{1}{3}CI.$$

3°. Suppose the base to be parallel to the axis.



Draw AM and BN perpendicular to the axis. The volume generated by CAB, is equal to the cylinder generated by the rectangle ABNM, diminished by the sum of the cones generated by the triangles CAM and CBN; hence,

$$vol.\ \mathrm{CAB} = \pi \overline{\mathrm{CI}}^2 \times \mathrm{AB} - \tfrac{1}{3}\pi \overline{\mathrm{CI}}^2 \times \mathrm{AI} - \tfrac{1}{3}\pi \overline{\mathrm{CI}}^2 \times \mathrm{IB}.$$

But the sum of AI and IB is equal to AB: hence, we have, by reducing, and changing the order of the factors,

$$vol.\ \mathrm{CAB} = \mathrm{AB} \times 2\pi\,\mathrm{CI} \times \tfrac{1}{3}\mathrm{CI}.$$

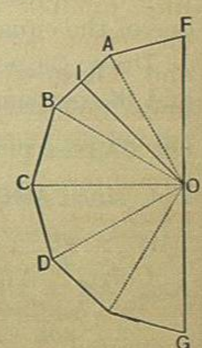
But $\mathrm{AB} \times 2\pi\,\mathrm{CI}$ is equal to the surface generated by AB; consequently,

$$vol.\ \mathrm{CAB} = surf.\ \mathrm{AB} \times \tfrac{1}{3}\mathrm{CI};$$

hence, in all cases, the volume generated by CAB is equal to *surf.* $\mathrm{AB} \times \frac{1}{3}\mathrm{CI}$; *which was to be proved.*

PROPOSITION XIII. LEMMA.

If a regular semi-polygon is revolved about its axis, the volume generated is equal to the surface generated by the semi-perimeter multiplied by one third of the apothem.



Let FBDG be a regular semi-polygon, FG its axis, OI its apothem, and let the semi-polygon be revolved about FG: then is the volume generated equal to *surf.* $\mathrm{FBDG} \times \frac{1}{3}\mathrm{OI}$.

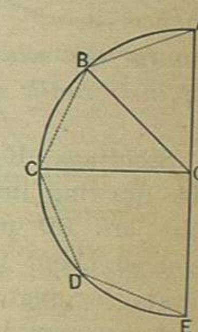
For, draw lines from the vertices to the centre O. These lines will divide the semi-polygon into isosceles triangles whose bases are sides of the semi-polygon, and whose altitudes are each equal to OI.

Now, the sum of the volumes generated by these triangles is equal to the volume generated by the semi-polygon. But, the volume generated by any triangle, as OAB, is equal to *surf.* $\mathrm{AB} \times \frac{1}{3}\mathrm{OI}$ (P. XII.); hence, the volume generated by the semi-polygon is equal to *surf.* $\mathrm{FBDG} \times \frac{1}{3}\mathrm{OI}$; *which was to be proved.*

Cor. The volume generated by a portion of the semi-polygon, OABC, limited by OC, OA, drawn to vertices is equal to *surf.* $\mathrm{ABC} \times \frac{1}{3}\mathrm{OI}$.

PROPOSITION XIV. THEOREM.

The volume of a sphere is equal to its surface multiplied by one third of its radius.



Let ACE be a semicircle, AE its diameter, O its centre, and let the semicircle be revolved about AE: then is the volume generated equal to the surface generated by the semi-circumference multiplied by one third of the radius OA.

For, the semicircle may be regarded as a regular semi-polygon having an infinite number of sides, whose semi-perimeter coincides with the semi-circumference, and whose apothem is equal to the radius: hence (P. XIII.), the volume generated by the semicircle is equal to the surface generated by the semi-circumference multiplied by one third of the radius; *which was to be proved.*

Cor. 1. Any portion of the semicircle, as OBC, bounded by two radii, will generate a volume equal to the surface generated by the arc BC multiplied by one third of the radius (P. XIII., C.). But this portion of the semicircle is a circular sector, the volume which it generates is a spherical sector, and the surface generated by the arc is a zone: hence, the *volume of a spherical sector is equal to the zone which forms its base multiplied by one third of the radius.*

Cor. 2. If we denote the volume of a sphere by V, and its radius by R, the area of the surface will be equal to $4\pi R^2$ (P. X., C. 1), and the volume of the sphere will be equal to $4\pi R^2 \times \frac{1}{3}R$; consequently, we have,

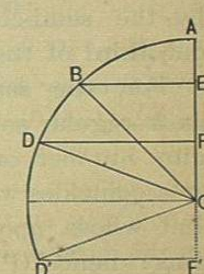
$$V = \tfrac{4}{3}\pi R^3.$$

Again, if we denote the diameter of the sphere by D, we shall have R equal to $\frac{1}{2}D$, and R^3 equal to $\frac{1}{8}D^3$, and consequently,

$$V = \tfrac{1}{6}\pi D^3;$$

hence, *the volumes of spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

Scholium. If the figure EBDF, formed by drawing lines from the extremities of the arc BD perpendicular to CA, be revolved about CA, as an axis, it will generate a segment of a sphere whose volume may be found by adding to the spherical sector generated by CDB, the cone generated by CBE, and subtracting from their sum the cone generated by CDF. If the arc BD is so taken that the points E and F fall on opposite sides of the centre C, the latter cone must be added, instead of subtracted. The area of the zone BD is equal to $2\pi CD \times EF$ (P. X., C. 2); hence,



$$\textit{segment}\ EBDF = \tfrac{1}{3}\pi\,(2\overline{CD}^2 \times EF + \overline{BE}^2 \times CE \mp \overline{DF}^2 \times CF).$$

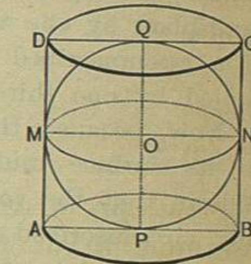
PROPOSITION XV. THEOREM.

The surface of a sphere is to the entire surface of the circumscribed cylinder, including its bases, as 2 is to 3; and the volumes are to each other in the same ratio.

Let PMQ be a semicircle, and PADQ a rectangle, whose sides PA and QD are tangent to the semicircle at P and Q, and whose side AD, is tangent to the semicircle at M. If the semicircle and the rectangle be revolved about PQ, as an axis, the former will generate a sphere, and the latter a circumscribed cylinder.

1°. The surface of the sphere is to the entire surface of the cylinder, as 2 is to 3.

For, the surface of the sphere is equal to four great circles (P. X., C. 1), the convex surface of the cylinder is equal to the circumference of its base multiplied by its altitude (P. I.); that is, it is equal to the circumference of a great circle multiplied by its diameter, or to four great circles (B. V., P. XV.); adding to this the two bases, each of which is equal to a great circle, we have the entire surface of the cylinder equal to six great circles: hence, the surface of the sphere is to the entire surface of the circumscribed cylinder, as 4 is to 6, or as 2 is to 3; *which was to be proved.*



2°. The volume of the sphere is to the volume of the cylinder as 2 is to 3.

For, the volume of the sphere is equal to $\frac{4}{3}\pi R^3$ (P. XIV., C. 2); the volume of the cylinder is equal to its base multiplied by its altitude (P. II.); that is, it is equal to

$\pi R^2 \times 2R$, or to $\frac{4}{3}\pi R^3$: hence, the volume of the sphere is to that of the cylinder as 4 is to 6, or as 2 is to 3; *which was to be proved.*

Cor. The surface of a sphere is to the entire surface of a circumscribed cylinder, as the volume of the sphere is to the volume of the cylinder.

Scholium. Any polyedron which is circumscribed about a sphere, that is, whose faces are all tangent to the sphere, may be regarded as made up of pyramids, whose bases are the faces of the polyedron, whose common vertex is at the centre of the sphere, and each of whose altitudes is equal to the radius of the sphere. But, the volume of any one of these pyramids is equal to its base multiplied by one third of its altitude: hence, the volume of a circumscribed polyedron is equal to its surface multiplied by one third of the radius of the inscribed sphere.

Now, because the volume of the sphere is also equal to its surface multiplied by one third of its radius, it follows that the volume of a sphere is to the volume of any circumscribed polyedron, as the surface of the sphere is to the surface of the polyedron.

Polyedrons circumscribed about the same, or about equal spheres, are proportional to their surfaces.

GENERAL FORMULAS.

If we denote the convex surface of a cylinder by S, its volume by V, the radius of its base by R, and its altitude by H, we have (P. I., II.),

$$S = 2\pi R \times H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$
$$V = \pi R^2 \times H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

If we denote the convex surface of a cone by S, its volume by V, the radius of its base by R, its altitude by H, and its slant height by H', we have (P. III., V.),

$$S = \pi R \times H' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$
$$V = \pi R^2 \times \tfrac{1}{3}H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

If we denote the convex surface of a frustum of a cone by S, its volume by V, the radius of its lower base by R, the radius of its upper base by R', its altitude by H, and its slant height by H', we have (P. IV., VI.),

$$S = \pi(R + R') \times H' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$
$$V = \tfrac{1}{3}\pi(R^2 + R'^2 + R \times R') \times H \quad . \quad . \quad . \quad . \quad (6.)$$

If we denote the surface of a sphere by S, its volume by V, its radius by R, and its diameter by D, we have (P. X., C. 1, XIV., C. 2, XIV., C. 1),

$$S = 4\pi R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$
$$V = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi D^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

If we denote the radius of a sphere by R, the area of any zone of the sphere by S, its altitude by H, and the volume of the corresponding spherical sector by V, we shall have (P. X., C. 2, XIV., C. 1),

$$S = 2\pi R \times H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$
$$V = \tfrac{2}{3}\pi R^2 \times H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

If we denote the volume of the corresponding spherical segment by V, its altitude by H, the radius of its upper base by R', the radius of its lower base by R'', the distance of its upper base from the centre by H', and of its lower base from the centre by H'', we shall have (P. XIV., S.):

$$V = \tfrac{1}{3}\pi(2R^2 \times H + R'^2 H' \mp R''^2 \times H'') \quad . \quad . \quad (11.)$$

EXERCISES.

1. The radius of the base of a cylinder is 2 feet, and its altitude 6 feet; find its entire surface, including the bases.

2. The volume of a cylinder, of which the radius of the base is 10 feet, is 6283.2 cubic feet; find the volume of a similar cylinder of which the diameter of the base is 16 feet, and find also the altitude of each cylinder.

3. Two similar cones have the radii of the bases equal, respectively, to $4\frac{1}{2}$ and 6 feet, and the convex surface of the first is 667.59 square feet; find the convex surface of the second and the volume of both.

4. A line 12 feet long is revolved about another line as an axis; the distance of one extremity of the line from the axis is 4 feet and of the other extremity 6 feet; find the area of the surface generated.

5. Find the convex surface and the volume of the frustum of a cone the altitude of which is 6 feet, the radius of the lower base being 4 feet and that of the upper base 2 feet.

6. Find the surface and the volume of the cone of which the frustum in the preceding example is a frustum.

7. A small circle, the radius of which is 4 feet, is 3 feet from the centre of a sphere; find the circumference of a great circle of the same sphere.

8. The radius of a sphere is 10 feet; find the area of a small circle distant from the centre 6 feet.

9. Find the area of the surface generated by the semi-perimeter of a regular semihexagon revolving about its axis, the radius of the inscribed circle being 5.2 feet and the axis 12 feet.

10. The area of the surface generated by the semi-perimeter of a regular semioctagon revolved about an axis is 178.2426 square feet, and the radius of the inscribed circle is 3.62 feet; find the axis.

11. An isosceles triangle, whose base is 8 feet and altitude 9 feet, is revolved about a line passing through its vertex and parallel to its base; how many cubic feet in the volume generated?

12. The altitude of a zone is 3 feet and the radius of the sphere is 5 feet; find the area of the zone and the volume of the corresponding spherical sector.

13. Find the surface and the volume of a sphere whose radius is 4 feet.

14. The radius of a sphere is 5 feet; how many cubic feet in a spherical segment whose altitude is 7 feet and the distance of whose lower base from the centre of the sphere is 3 feet?

15. A cone such that the diameter of its base is equal to its slant height is circumscribed about a sphere; show that the surface of the sphere is to the entire surface of the cone, including its base, as 4 is to 9, and that the volumes are in the same ratio.

16. The radius of a sphere is 6 feet; find the entire surface and the volume of the circumscribing cylinder.

17. A cone, with the diameter of the base and the slant height equal, is circumscribed about a sphere whose radius is 5 feet; find the entire surface and the volume of the cone.

18. A cone, with the diameter of the base and the slant height equal, and a cylinder, are circumscribed about a sphere; what relation exists between the entire surfaces and the volumes of the cylinder, the cone and the sphere?

19. The edge of a regular octaedron is 10 feet, and the radius of the inscribed sphere is 4.08 feet; find the volume of the octaedron.