BOOK IX.

SPHERICAL GEOMETRY.

DEFINITIONS.

1. A Spherical Angle is the amount of divergence of the arcs of two great circles of a sphere meeting at a point. The arcs are called *sides* of the angle, and their point of intersection is called the *vertex* of the angle.

The measure of a spherical angle is the same as that of the diedral angle included between the planes of its sides. Spherical angles may be *acute*, *right*, or *obtuse*.

2. A Spherical Polygon is a portion of the surface of a sphere bounded by arcs of three or more great circles. The bounding arcs are called *sides* of the polygon, and the points in which the sides meet are called *vertices* of the polygon. Each side is taken less than a semi-circumference.

Spherical polygons are classified in the same manner as plane polygons.

3. A Spherical Triangle is a spherical polygon of three sides.

Spherical triangles are classified in the same manner as plane triangles.

4. A Lune is a portion of the surface of a sphere bounded by semi-circumferences of two great circles.

5. A Spherical Wedge is a portion of a sphere bounded by a lune and two semicircles which intersect in a diameter of the sphere.

6. A Spherical Pyramid is a portion of a sphere bounded by a spherical polygon and sectors of circles whose common centre is the centre of the sphere.

The spherical polygon is called the *base* of the pyramid, and the centre of the sphere is called the *vertex* of the pyramid.

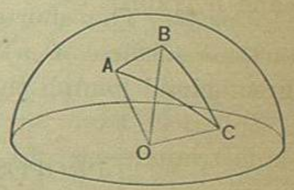
7. A Pole of a Circle is a point, on the surface of the sphere, equally distant from all the points of the circumference of the circle.

8. A Diagonal of a spherical polygon is an arc of a great circle joining the vertices of any two angles which are not consecutive.

PROPOSITION I. THEOREM.

Any side of a spherical triangle is less than the sum of the two others.

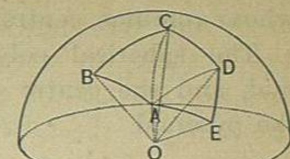
Let ABC be a spherical triangle situated on a sphere whose centre is O: then is any side, as AB, less than the sum of the sides AC and BC.



For, draw the radii OA, OB, and OC: these radii form the edges of a triedral angle whose vertex is O, and the plane angles included between them are measured by the arcs AB, AC, and BC (B. III., P. XVII., Sch.). But any plane angle, as AOB, is less than the sum of the plane angles AOC and BOC (B. VI., P. XIX.): hence, the arc AB is less than the sum of the arcs AC and BC; *which was to be proved.*

Cor. 1. Any side AB, of a spherical polygon ABCDE, is less than the sum of all the other sides.

For, draw the diagonals AC and AD, dividing the polygon into triangles. The arc AB is less than the sum of AC and BC, the arc AC is less than the sum of AD and DC, and the arc AD is less than the sum of DE and EA; hence, AB is less than the sum of BC, CD, DE, and EA.



Cor. 2. The arc of a small circle, on the surface of a sphere, is greater than the arc of a great circle joining its two extremities.

For, divide the arc of the small circle into equal parts, and through the two extremities of each part suppose the arc of a great circle to be drawn. The sum of these arcs, whatever may be their number, will be greater than the arc of the great circle joining the given points (C. 1). But when this number is infinite, each arc of the great circle will coincide with the corresponding arc of the small circle, and their sum is equal to the entire arc of the small circle, which is, consequently, greater than the arc of the great circle.

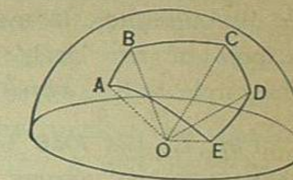
Cor. 3. The shortest distance from one point to another on the surface of a sphere, is measured on the arc of a great circle joining them.

PROPOSITION II. THEOREM.

The sum of the sides of a spherical polygon is less than the circumference of a great circle.

Let ABCDE be a spherical polygon situated on a sphere whose centre is O: then is the sum of its sides less than the circumference of a great circle.

For, draw the radii OA, OB, OC, OD, and OE: these radii form the edges of a polyedral angle whose vertex is at O, and the angles included between them are measured by the arcs AB, BC, CD, DE, and EA. But the sum of these angles is less than four right angles (B. VI., P. XX.): hence, the sum of the arcs which measure them is less than the circumference of a great circle; *which was to be proved.*

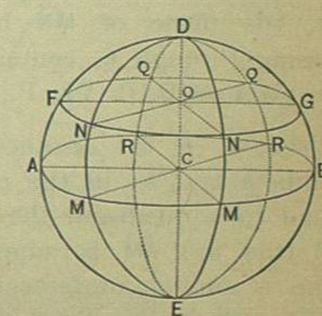


PROPOSITION III. THEOREM.

If a diameter of a sphere is drawn perpendicular to the plane of any circle of the sphere, its extremities are poles of that circle.

Let C be the centre of a sphere, FNG any circle of the sphere, and DE a diameter of the sphere perpendicular to the plane of FNG: then are its extremities, D and E, poles of the circle FNG.

The diameter DE, being perpendicular to the plane of FNG, must pass through the centre O (B. VIII., P. VII., C. 3). If arcs of great circles DN, DF DG, &c., are drawn from D to different points of the circumference FNG, and chords of these arcs are drawn, these chords are equal (B. VI., P. V.), consequently, the arcs themselves are equal. But these arcs are the shortest lines that can be drawn from the point D to the different



points of the circumference (P. I., C. 3): hence, the point D is equally distant from all the points of the circumference, and consequently is a pole of the circle (D. 7). In like manner, it may be shown that the point E is also a pole of the circle: hence, both D and E are poles of the circle FNG; *which was to be proved.*

Cor. 1. Let AMB be a great circle perpendicular to DE: then are the angles DCM, ECM, &c., right angles; and consequently, the arcs DM, EM, &c., are each equal to a quadrant (B. III., P. XVII., S.): hence, the two poles of a great circle are at equal distances from the circumference.

Cor. 2. The two poles of a small circle are at unequal distances from the circumference, the sum of the distances being equal to a semi-circumference.

Cor. 3. If any point, as M, in the circumference of a great circle, is joined with either pole by the arc of a great circle, such arc is perpendicular to the circumference AMB, since its plane passes through CD, which is perpendicular to AMB. Conversely: if MN is perpendicular to the arc AMB, it passes through the poles D and E: for, the plane of MN being perpendicular to AMB and passing through C, contains CD, which is perpendicular to the plane AMB (B. VI., P. XVII., C.).

Cor. 4. If the distance of a point D from each of the points A and M, in the circumference of a great circle, is equal to a quadrant, the point D is the pole of the arc AM (the arc AM is supposed to be either less or greater than a semi-circumference).

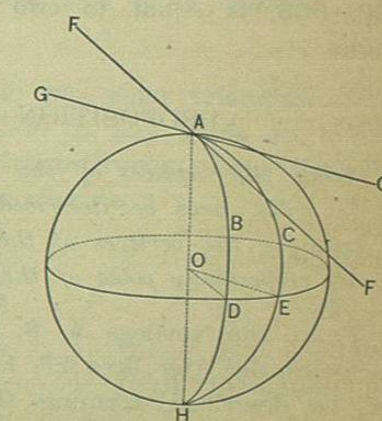
For, let C be the centre of the sphere, and draw the radii CD, CA, CM. Since the angles ACD, MCD, are right angles, the line CD is perpendicular to the two straight lines CA, CM: it is, therefore, perpendicular to their plane

(B. VI., P. IV.): hence, the point D is the pole of the arc AM.

Scholium. The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. For, by turning the arc DF about the point D, the extremity F will describe the small circle FNG; and by turning the quadrant DFA round the point D, its extremity A will describe an arc of a great circle.

PROPOSITION IV. THEOREM.

The angle formed by arcs of two great circles, is equal to that formed by the tangents to these arcs at their point of intersection, and is measured by the arc of a great circle described from the vertex as a pole, and limited by the sides, produced if necessary.

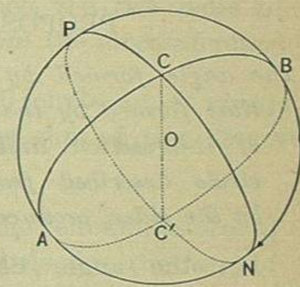


Let the angle BAC be formed by the two arcs AB, AC: then is it equal to the angle FAG formed by the tangents AF, AG, and is measured by the arc DE of a great circle, described about A as a pole.

For, the tangent AF, drawn in the plane of the arc AB, is perpendicular to the radius AO; and the tangent AG, drawn in the plane of the arc AC, is perpendicular to the same radius AO: hence, the angle FAG is equal to the angle contained by the planes ABDH, ACEH (B. VI., D. 4); which is that of the arcs AB, AC. Now, if the arcs AD and AE are both quadrants, the lines OD, OE, are perpendicular to OA, and the angle DOE is equal to the angle of the planes ABDH, ACEH: hence, the arc DE is the measure of the angle contained by these planes, or of the angle CAB; *which was to be proved.*

Cor. 1. The angles of spherical triangles may be compared by means of the arcs of great circles described from their vertices as poles, and included between their sides.

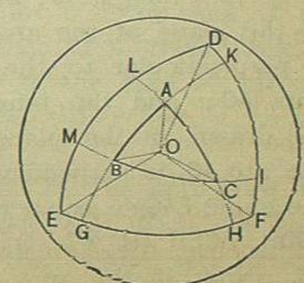
A spherical angle can always be constructed equal to a given spherical angle.



Cor. 2. Vertical angles, such as ACP and BCN, are equal; for either of them is the angle formed by the two planes ACB, PCN. When two arcs ACB, PCN, intersect, the sum of two adjacent angles, as ACP, PCB, is equal to two right angles.

PROPOSITION V. THEOREM.

If from the vertices of the angles of a spherical triangle, as poles, arcs be described forming a second spherical triangle, the vertices of the angles of this second triangle are respectively poles of the sides of the first.



From the vertices A, B, C, as poles, let the arcs EF, FD, DE, be described, forming the triangle DFE: then are the vertices D, E, and F, respectively poles of the sides BC, AC, AB.

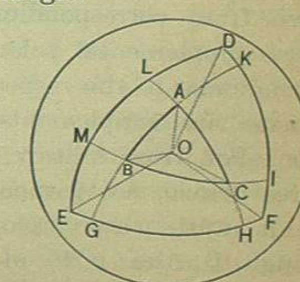
For, the point A being the pole of the arc EF, the distance AE is a quadrant; the point C being the pole of the arc DE, the distance CE is likewise a quadrant: hence, the point E is at a quadrant's distance from the points A and C: hence, it is the pole of the arc AC (P. III., C. 4). It may be shown, in like manner, that D is the pole of the arc BC, and F that of the arc AB; *which was to be proved.*

Cor. The triangle ABC, may be described by means of DEF, as DEF is described by means of ABC. Triangles so related that any vertex of either is the pole of the side opposite it in the other, are called *polar triangles.*

PROPOSITION VI. THEOREM.

Any angle, in one of two polar triangles, is measured by a semi-circumference, minus *the side lying opposite to it in the other triangle.*

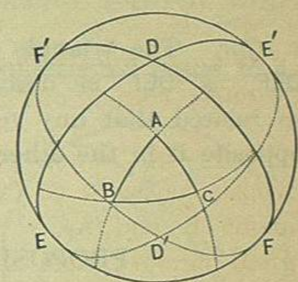
Let ABC, and EFD, be any two polar triangles on a sphere whose centre is O: then is any angle in either triangle measured by a semi-circumference, minus the side lying opposite to it in the other triangle.



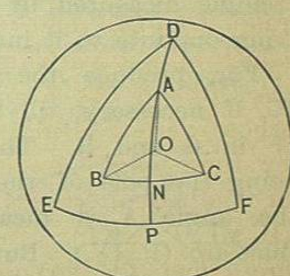
For, produce the sides AB, AC, if necessary, till they meet EF in G and H. The point A being the pole of the arc GH, the angle A is measured by that arc (P. IV.). But, since E is the pole of AH, the arc EH is a quadrant; and since F is the pole of AG, FG is a quadrant: hence, the sum of the arcs EH and GF is equal to a semi-circumference. But, the sum of the arcs EH and

GF is equal to the sum of the arcs EF and GH: hence, the arc GH, which measures the angle A, is equal to a semi-circumference minus the arc EF. In like manner, it may be shown, that any other angle, in either triangle, is measured by a semi-circumference minus the side lying opposite to it in the other triangle; *which was to be proved*

Cor. 1. Beside the triangle DEF, three other triangles, polar to ABC, may be formed by the intersection of the arcs DE, EF, DF, prolonged. But the proposition is applicable only to the central triangle, ABC, which is distinguished from the three others by the circumstance, that the vertices A and D lie on the same side of BC; B and E, on the same side of AC; C and F, on the same side of AB. The polar triangles ABC and DEF are called *supplemental* triangles, any part of either being the supplement of the part opposite it in the other.



Cor. 2. Arcs of great circles, drawn from corresponding vertices of two supplemental polar triangles perpendicular to the respective sides opposite, are supplements of each other. For, from A draw the arc of a great circle, AN, perpendicular to BC; it must, when prolonged, pass through D, the pole of BC, and must also, when prolonged to P, be perpendicular to EF (P. III., C. 3): DN and AP being quadrants (P. III. C. 1), DP and AN are supplements of each other.

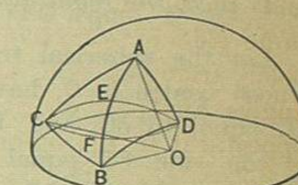


PROPOSITION VII. THEOREM.

If from the vertices of any two angles of a spherical triangle, as poles, arcs of circles are described passing through the vertex of the third angle; and if from the second point in which these arcs intersect, arcs of great circles are drawn to the vertices, used as poles, the parts of the triangle thus formed are equal to those of the given triangle, each to each.

Let ABC be a spherical triangle situated on a sphere whose centre is O, CED and CFD arcs of circles described about B and A as poles, and let DA and DB be arcs of great circles: then are the parts of the triangle ABD equal to those of the given triangle ABC, each to each.

For, by construction, the side AD is equal to AC, the side BD is equal to BC, and the side AB is common: hence, the sides are equal, each to each. Draw the radii OA, OB, OC, and OD. The radii OA, OB, and OC, form the edges of a triedral angle whose vertex is O; and the radii OA, OB, and OD, form the edges of a second triedral angle whose vertex is also at O; and the plane angles formed by these edges are equal, each to each: hence, the planes of the equal angles are equally inclined to each other (B. VI., P. XXI.). But, the angles made by these planes are equal to the corresponding spherical angles; consequently, the angle BAD is equal to BAC, the angle ABD to ABC, and the angle ADB to ACB: hence, the parts of the triangle ABD are equal to the parts of the triangle ACB, each to each; *which was to be proved.*



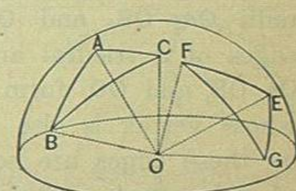
Scholium 1. The triangles ABC and ABD, are not, in general, capable of superposition, but their parts are *symmetrically* disposed with respect to AB. *Triangles which have all the parts of the one equal to all the parts of the other, each to each, but are not capable of superposition, are called symmetrical triangles.*

Scholium 2. If symmetrical triangles are isosceles, they can be so placed as to coincide throughout: hence, they are *equal in area.*

PROPOSITION VIII. THEOREM.

If two spherical triangles, on the same, or on equal spheres, have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the remaining parts are equal, each to each.

Let the spherical triangles ABC and EFG, on the sphere whose centre is O, have the side EF equal to AB, the side EG equal to AC, and the angle FEG equal to BAC: then is the side FG equal to BC, the angle EFG to ABC, and the angle EGF to ACB.

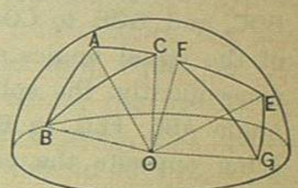


For, draw the radii OE, OF, OG, OA, OB, and OC, forming the triedral angles O–EFG and O–ABC. Since the sides EF and EG are equal, respectively, to the sides AB and AC, the plane angles EOF and EOG are equal, respectively, to the plane angles AOB and AOC; and as the spherical angles FEG and BAC are equal, the inclination of the faces EOF and EOG of the triedral angle O–EFG, is equal to the inclination of the faces AOB and AOC of the triedral angle O–ABC; therefore (B. VI., P. XXI., C.), the angle FOG is equal to BOC, and the side FG equals the side BC: again, since the angle EOF is equal to AOB, FOG to BOC, and GOE to COA, the planes of the equal angles are equally inclined to each other (B. VI., P. XXI.), and, consequently (D. 1), the angle EFG is equal to ABC, and EGF to ACB—hence, the remaining parts of the triangles are equal, each to each; *which was to be proved.*

PROPOSITION IX. THEOREM.

If two spherical triangles on the same, or on equal spheres, have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the remaining parts are equal, each to each.

Let the spherical triangles ABC and EFG, on the sphere whose centre is O, have the angle FEG equal to BAC, the angle EFG equal to ABC, and the side EF equal to AB: then is the side EG equal to AC, the side FG to BC, and the angle FGE to BCA.

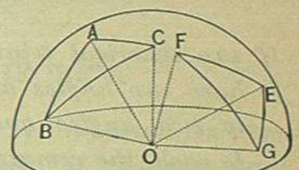


For, draw radii, as before, forming the triedral angles O–EFG and O–ABC. Since the side EF is equal to AB, the plane angle EOF is equal to AOB; as the angle FEG is equal to BAC, and EFG to ABC, the inclination of the face EOF, of the triedral angle O–EFG, to each of the faces EOG and FOG, is equal, respectively, to the inclination of the face AOB, of the triedral angle O–ABC, to each of the faces AOC and BOC, and hence (B. VI., P. XXI., S. 2), the plane angles EOG and GOF are equal, respectively, to AOC and COB; therefore, the sides EG and GF are equal to the sides AC and CB, and the angle FGE to BCA; *which was to be proved.*

PROPOSITION X. THEOREM.

If two spherical triangles on the same, or on equal spheres, have their sides equal, each to each, their angles are equal, each to each, the equal angles lying opposite the equal sides.

Let the spherical triangles EFG and ABC, on the sphere whose centre is O, have the side EF equal to AB, EG equal to AC, and FG equal to BC: then the angle FEG is equal to BAC, EFG to ABC, and EGF to ACB, and the equal angles lie opposite the equal sides.



For, draw the radii, as before, forming the triedral angles O–EFG and O–ABC. Because the sides of the triangles are respectively equal, the plane angle EOF is equal to AOB, FOG to BOC, and GOE to COA. Hence (B. VI., P. XXI.), the planes of the equal angles are equally inclined to each other, and, consequently, the spherical angle EFG is equal to spherical angle ABC, FEG to BAC, and EGF to ACB, the equal angles lying opposite the equal sides; *which was to be proved.*

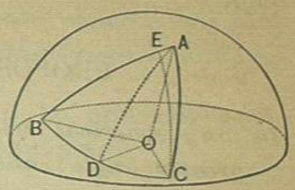
NOTE.—The triangle **EFG** is equal in all respects to either **ABC** or its symmetrical triangle.

PROPOSITION XI. THEOREM.

In any isosceles spherical triangle, the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.

1°. Let ABC be a spherical triangle, on a sphere whose centre is O, having the side AB equal to AC: then is the angle C equal to the angle B.

For, draw the arc of a great circle from the vertex A, to the middle point D, of the base BC: then in the two triangles ADB and ADC, we shall have the side AB equal to AC, by hypothesis, the side BD equal to DC, by construction, and the side AD common; consequently, the triangles have their angles equal, each to each (P. X.): hence, the angle C is equal to the angle B; *which was to be proved.*



2°. Let ABC be a spherical triangle having the angle C equal to the angle B: then is the side AB equal to the side AC, and consequently the triangle is isosceles.

For, suppose that AB and AC are not equal, but that one of them, as AB, is the greater. On AB lay off the arc BE equal to AC, and draw the arc of a great circle from E to C: then in the triangles ACB and EBC, we shall have the side AC equal to EB, by construction, the side BC common, and the included angle ACB equal to the included angle EBC, by hypothesis; hence, the remaining parts of the triangles are equal, each to each, and consequently, the angle ECB is equal to the angle ABC. But, the angle ACB is equal to ABC, by hypothesis, and therefore, the angle ECB is equal to ACB, or a part is equal to the whole, which is impossible: hence, the supposition that AB and AC are unequal, is absurd; they are therefore equal, and consequently, the triangle ABC is isosceles; *which was to be proved.*

Cor. The triangles ADB and ADC, having all of their parts equal, each to each, the angle ADB is equal to ADC, and the angle DAB is equal to DAC; that is, *if an arc of a great circle is drawn from the vertex of an isosceles*