12. Show that the volume of a triangular spherical pyramid is equal to its base multiplied by one third the radius of the sphere.

13. Show that the volume of any spherical pyramid is equal to its base multiplied by one third the radius of the sphere.

14. Find the volume of a spherical pyramid whose base is a tri-rectangular triangle, the diameter of the sphere being 8 feet.

15. The angles of a triangle, on a sphere whose radius is 9 feet, are 100°, 115°, and 120°; find the area of the triangle and the volume of the corresponding spherical pyramid.

16. A spherical pyramid, of a sphere whose diameter is 10 feet, has for its base a triangle of which the angles are 60°, 80°, and 85°; what is its ratio to a pyramid whose base is a tri-rectangular triangle of the same sphere?

17. The sum of the angles of a regular spherical octagon is 1140°, and the radius of the sphere is 12 feet; find the area of the octagon.

18. The volume of a spherical pyramid, whose base is an equiangular triangle, is 84.8232 cubic feet, and the radius of the sphere is 6 feet; find one of the angles of the base.

19. Given a spherical angle of 40°; what is the number of degrees in the longest arc of a great circle that can be drawn perpendicular to either side of the angle and included between the two sides?

20. Given a spherical angle of 115°; what is the number of degrees in the shortest arc of a great circle that can be drawn perpendicular to either side of the angle and included between the two sides?

APPENDIX.

GRADED EXERCISES IN PLANE GEOMETRY.

ADDITIONAL DEFINITIONS.

1. The Distance of a point from a line is measured on a perpendicular to that line.

2. The Bisectrix of an angle is a line that divides the angle into two equal parts.

3. A Median is a line drawn from any vertex of a triangle to the middle of the opposite side.

4. The Projection of a point, on a line, is the foot of a perpendicular drawn from the point to the line.

5. The Projection of one straight line on another, is that part of the second line which is contained between the projections of the two extreme points of the first line, upon the second.

PROPOSITIONS.

I. Theorem.—Show that the bisectrices of two adjacent angles are perpendicular to each other.

II. Theorem.—Show that the perimeter of any triangle is greater than the sum of the distances from any point within the triangle to its three vertices, and less than twice that sum.

III. Theorem.—Show that the angle between the bisectrices of two consecutive angles of any quadrilateral, is equal to one half the sum of the other two angles.

IV. Theorem.—Show that any point in the bisectrix of an angle is equally distant from the sides of the angle.

V. Theorem.—If two sides of a triangle are prolonged beyond the third side, show that the bisectrices of this included angle and of the exterior angles all meet in the same point.

VI. Theorem.—Show that the projection of a line on a parallel line, is equal to the line itself; and that the projection of a line on a line to which it is oblique, is less than the line itself.

VII. Theorem.—If a line is drawn through the point of intersection of the diagonals of a parallelogram and limited by the sides of the parallelogram, show that the line is bisected at the point.

VIII. Theorem.—The bisectrices of the four angles of any parallelogram form, by their intersection, a rectangle whose diagonals are parallel to the sides of the given parallelogram.

IX. Theorem.—Show that the sum of the distances from any point in the base of an isosceles triangle to the two other sides, is equal to the distance from the vertex of either angle at the base to the opposite side.

X. Theorem.—Show that the middle point of the hypothenuse of any right-angled triangle is equally distant from the three vertices of the triangle.

XI. Problem.—Draw two lines that shall divide a given right angle into three equal parts.

XII. Theorem.—Draw a line AP through the vertex A of a triangle ABF and perpendicular to the bisectrix of the angle A; construct a triangle PBF, having its vertex P on AP, and its base coinciding with that of the given triangle: then show that the perimeter of PBF is greater than that of ABF.

XIII. Theorem.—Let an altitude of the triangle ABC be drawn from the vertex A; and also the bisectrix of the angle A; then show that their included angle is equal to half the difference of the angles B and C.

XIV. Problem.—Given two lines that would meet, if sufficiently prolonged: then draw the bisectrix of their included angle, without finding its vertex.

XV. Problem.—From two points on the same side of a given line, to draw two lines that shall meet each other at some point of the given line, and make equal angles with that line.

XVI. Theorem.—Show that the sum of the lines drawn to a point of a given line, from two given points, is the least possible when these lines are equally inclined to the given line.

XVII. Problem.—From two given points, on the same side of a given line, draw two lines meeting on the given line and equal to each other.

XVIII. PROBLEM.—Through a given point A, draw a line that shall be equally distant from two given points, B and C.

XIX. PROBLEM.—Through a given point, draw a line cutting the sides of a given angle and making the interior angles equal to each other.

XX. PROBLEM.—Draw a line PQ parallel to the base BC of a triangle ABC, so that PQ shall be equal to the sum of BP and CQ.

XXI. PROBLEM.—In a given isosceles triangle, draw a line that shall cut off a trapezoid whose base is the base of the given triangle and whose three other sides shall be equal to each other.

XXII. THEOREM.—If two opposite sides of a parallelogram are bisected, and lines are drawn from the points of bisection to the vertices of the opposite angles, show that these lines divide the diagonal, which they intersect, into three equal parts.

XXIII. PROBLEM.—Construct a triangle, having given the two angles at the base and the sum of the three sides.

XXIV. PROBLEM.—Construct a triangle, having given one angle, one of its including sides, and the sum of the two other sides.

XXV. PROBLEM.—Construct an equilateral triangle, having given one of its altitudes.

XXVI. THEOREM.—Show that the three altitudes of a triangle all intersect in a common point.

XXVII. THEOREM.—If one of the acute angles of a right-angled triangle is double the other, show that the hypothenuse is double the smaller side about the right angle.

XXVIII. THEOREM.—Let a median be drawn from the vertex of any angle A of a triangle ABC: then show that the angle A is a right angle when the median is equal to half the side BC, an acute angle when the median is greater than half of BC, and an obtuse angle when the median is less than half of BC.

XXIX. THEOREM.—Let any quadrilateral be circumscribed about a circle: then show that the sum of two opposite sides is equal to the sum of the other two opposite sides.

XXX. PROBLEM.—Draw a straight line tangent to two given circles.

XXXI. PROBLEM.—Through a given point P, draw a circle that shall be tangent to a given line CB, at a given point B.

XXXII. THEOREM.—Let two circles intersect each other, and through either point of intersection let diameters of the circles be drawn: then show that the other extremities of these diameters and the other point of intersection lie in the same straight line.

XXXIII. PROBLEM.—Through two given points A and B, draw a circle that shall be tangent to a given line CP.

XXXIV. PROBLEM.—Draw a circle that shall be tangent to a given circle C, and also to a given line DP, at a given point P.

XXXV. PROBLEM.—Draw a circle that shall be tangent to a given line TP, and also to a given circle C, at a given point Q.

XXXVI. PROBLEM.—Draw a circle that shall pass through a given point Q, and be tangent to a given circle C, at a given point P.

XXXVII. PROBLEM.—Draw a circle, with a given radius, that shall be tangent to a given line DP, and to a given circle C.

XXXVIII. PROBLEM.—Find a point in the prolongation of any diameter of a given circle, such that a tangent from it to the circumference shall be equal to the diameter of the circle.

XXXIX. THEOREM.—Show that when two circles intersect each other, the longest common secant that can be drawn through either point of intersection, is parallel to the line joining the centres of the circles.

XL. PROBLEM.—Construct the greatest possible equilateral triangle whose sides shall pass through three given points A, B, and C, not in the same straight line.

XLI. THEOREM.—Show that the bisectrices of the four angles of any quadrilateral intersect in four points, all of which lie on the circumference of the same circle.

XLII. THEOREM.—If two circles touch each other externally, and if two common secants are drawn through the point of contact and terminating in the concave arcs, show that the lines joining the extremities of these secants, in the two circles, are parallel.

XLIII. THEOREM.—Let an equilateral triangle be inscribed in a circle, and let two of the subtended arcs be bisected by a chord of the circle: then show that the sides of the triangle divide the chord into three equal parts.

XLIV. PROBLEM.—Find a point, within a triangle, such that the angles formed by drawing lines from it to the three vertices of the triangle shall be equal to each other.

XLV. PROBLEM.—Inscribe a circle in a quadrant of a given circle.

XLVI. PROBLEM.—Through a given point P, within a given angle ABC, draw a circle that shall be tangent to both sides of that angle.

XLVII. THEOREM.—Show that the middle points of the sides of any quadrilateral are the vertices of an inscribed parallelogram.

XLVIII. PROBLEM.—Inscribe in a given triangle, a triangle whose sides shall be parallel to the sides of a second given triangle.

XLIX. PROBLEM.—Through a point P, within a given angle, draw a line such that it and the parts of the sides that are intercepted shall contain a given area.

L. PROBLEM.—Construct a parallelogram whose area and perimeter are respectively equal to the area and perimeter of a given triangle.

LI. PROBLEM.—Inscribe a square in a semicircle; that is, a square two of whose vertices are in the diameter, and the other two in the semi-circumference.

LII. Problem.—Through a given point P draw a line cutting a triangle, so that the sum of the perpendiculars to it, from the two vertices on one side of the line, shall be equal to the perpendicular to it from the vertex, on the other side of the line.

LIII. Theorem.—Show that the line which joins the middle points of two opposite sides of any quadrilateral, bisects the line joining the middle points of the two diagonals.

LIV. Theorem.—If from the extremities of one of the oblique sides of a trapezoid, lines are drawn to the middle point of the opposite side, show that the triangle thus formed is equal to one half the given trapezoid.

LV. Problem.—Find a point in the base of a triangle, such that the lines drawn from it, parallel to and limited by the other sides of the triangle, shall be equal to each other.

LVI. Theorem.—Show that the line drawn from the middle of the base of any triangle to the middle of any line of the triangle parallel to the base, will pass through the opposite vertex, if sufficiently produced.

LVII. Theorem.—Show that the three medians of any triangle meet in a common point.

LVIII. Theorem.—On the sides AB and AC of any triangle ABC, construct any two parallelograms ABDE and ACFG; prolong the sides DE and FG till they meet in H; draw HA, and on the third side BC of the triangle, construct a parallelogram two of whose sides are parallel and equal to HA: then show that the parallelogram on BC is equal to the sum of the parallelograms on AB and AC.

LIX. Theorem.—Assuming the principle demonstrated in the last proposition, deduce from it the truth that the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the two other sides.

LX. Theorem.—If from the middle of the base of a right-angled triangle, a line is drawn perpendicular to the hypothenuse dividing it into two segments, show that the difference of the squares of these segments is equal to the square of the other side about the right angle.

LXI. Theorem.—If lines are drawn from any point P to the four vertices of a rectangle, show that the sum of the squares of the two lines drawn to the extremities of one diagonal, is equal to the sum of the squares of the two lines drawn to the extremities of the other diagonal.

LXII. Theorem.—Let a line be drawn from the centre of a circle to any point of any chord; then show that the square of this line, plus the rectangle of the segments of the chord, is equal to the square of the radius.

LXIII. Problem.—Draw a line from the vertex of any scalene triangle to a point in the base, such that this line shall be a mean proportional between the segments into which it divides the base.

LXIV. Theorem.—Show that the sum of the squares of the diagonals of any quadrilateral is equal to the sum of the squares of the four sides of the quadrilateral, diminished by four times the square of the distance between the middle points of the diagonals.

LXV. Problem.—Construct an equilateral triangle equal in area to any given isosceles triangle.

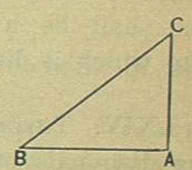
LXVI. THEOREM.—In a triangle ABC, let two lines be drawn from the extremities of the base BC, intersecting at any point P on the median through A, and meeting the opposite sides in the points E and D: show that DE is parallel to BC.

APPLICATION OF ALGEBRA TO GEOMETRY.

To solve a geometrical problem by means of algebra, draw a figure which shall contain all the given and required parts and also such other lines as may be necessary to establish the relations between them; then denote the given parts by leading letters, and the required parts by final letters of the alphabet: next consider the relations between the given and required parts and express these relations by equations, taking care to have as many independent equations as there are parts to be determined (Bourdon, Art. 92). The solution of these equations will give the values of the required parts.

To indicate the method of proceeding, the solution of the first problem is given.

LXVII. PROBLEM.—In a right-angled triangle ABC, given the base BA and the sum of the hypothenuse and the perpendicular, to find the hypothenuse and the perpendicular.



Solution. Denote BA by c, BC by x, AC by y, and the sum of BC and AC by s.

Then, $\quad x + y = s. \quad \cdots \quad (1.)$

From B. IV., P. XI., $\quad x^2 = y^2 + c^2. \quad \cdots \quad (2.)$

From (1), we have, $\quad x = s - y.$

Squaring, $\quad x^2 = s^2 - 2sy + y^2. \quad \cdots \quad (3.)$

Subtracting (2) from (3), $\quad 0 = s^2 - 2sy - c^2.$

Transposing and dividing, $\quad y = \frac{s^2 - c^2}{2s};$

whence, $\quad x = s - \frac{s^2 - c^2}{2s} = \frac{s^2 + c^2}{2s}.$

If $c = 3$ and $s = 9$, we have $x = 5$ and $y = 4$.

LXVIII. PROBLEM.—In a right-angled triangle, given the hypothenuse and the sum of the sides about the right angle, to find these sides.

LXIX. PROBLEM.—In a rectangle, given the diagonal and the perpendicular, to find the sides.

LXX. PROBLEM.—Given the base and perpendicular of a triangle, to find the side of an inscribed square.

LXXI. PROBLEM.—In an equilateral triangle, given the distances from a point within the triangle to each of the three sides, to find one of the equal sides.

LXXII. PROBLEM.—In a right-angled triangle, given the base and the difference between the hypothenuse and the perpendicular, to find the sides.

LXXIII. PROBLEM.—In a right-angled triangle, given the hypothenuse and the difference between the base and the perpendicular, to determine the triangle.

LXXIV. PROBLEM.—Having given the area of a rectangle inscribed in a given triangle, to determine the sides of the rectangle.

LXXV. PROBLEM.—In a triangle, having given the ratio of the two sides together with both segments of the base made by a perpendicular from the vertex, to determine the triangle.

LXXVI. PROBLEM.—In a triangle, having given the base, the sum of the two other sides, and the length of a line drawn from the vertex to the middle of the base; to find the sides of the triangle.

LXXVII. PROBLEM.—In a triangle, having given the two sides about the vertical angle, together with the line bisecting that angle and terminating in the base; to find the base.

LXXVIII. PROBLEM.—To determine a right-angled triangle, having given the lengths of two lines drawn from the vertices of the acute angles to the middle points of the opposite sides.

LXXIX. PROBLEM.—To determine a right-angled triangle, having given the perimeter and the radius of the inscribed circle.

LXXX. PROBLEM.—To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.

LXXXI. PROBLEM.—To determine a right-angled triangle, having given the hypothenuse and the side of the inscribed square.

LXXXII. PROBLEM.—To determine the radii of three equal circles, described within and tangent to a given circle, and also tangent to each other.

LXXXIII. PROBLEM.—In a right-angled triangle, having given the perimeter and the perpendicular let fall from the right angle on the hypothenuse, to determine the triangle.

LXXXIV. PROBLEM.—To determine a right-angled triangle, having given the hypothenuse and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.

LXXXV. PROBLEM.—To determine a triangle, having given the base, the perpendicular, and the difference of the two other sides.

LXXXVI. PROBLEM.—To determine a triangle, having given the base, the perpendicular, and the rectangle of the two sides.

LXXXVII. PROBLEM.—To determine a triangle, having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.

LXXXVIII. PROBLEM.—In a triangle, having given the three sides, to find the radius of the inscribed circle.

LXXXIX. PROBLEM.—To determine a right-angled triangle, having given the side of the inscribed square and the radius of the inscribed circle.

XC. PROBLEM.—To determine a right-angled triangle, having given the hypothenuse and the radius of the inscribed circle.