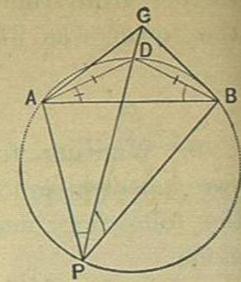
8. The distances AB, AC, and BC, between the points A, B, and C, are known; viz.: AB = 800 yds., AC = 600 yds., and BC = 400 yds. From a fourth point P, the angles APC and BPC are measured; viz.:



$$\text{APC} = 33^\circ\ 45',$$

and $$\text{BPC} = 22^\circ\ 30'.$$

Required the distances AP, BP, and CP.

$$\textit{Ans.} \begin{cases} \text{AP} = 710.198 \text{ yds.} \\ \text{BP} = 934.289 \text{ yds.} \\ \text{CP} = 1042.524 \text{ yds.} \end{cases}$$

This problem is used in locating the position of buoys in maritime surveying, as follows. Three points, A, B, and C, on shore are known in position. The surveyor stationed at a buoy P, measures the angles APC and BPC. The distances AP, BP, and CP, are then found as follows:

Suppose the circumference of a circle to be described through the points A, B, and P. Draw CP, cutting the circumference in D, and draw the lines DB and DA.

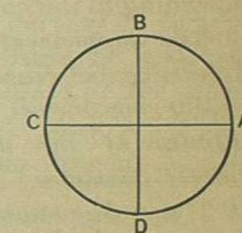
The angles CPB and DAB, being inscribed in the same segment, are equal (B. III., P. XVIII., C. 1); for a like reason, the angles CPA and DBA are equal: hence, in the triangle ADB, we know two angles and one side; we may, therefore, find the side DB. In the triangle ACB, we know the three sides, and we may compute the angle B. Subtracting from this the angle DBA, we have the angle DBC. Now, in the triangle DBC, we have two sides and their included angle, and we can find the angle DCB. Finally, in the triangle CPB, we have two angles and one side, from which data we can find CP and BP. In like manner, we can find AP.

ANALYTICAL TRIGONOMETRY.

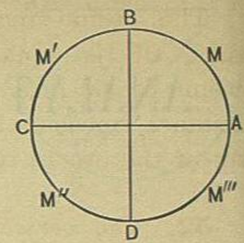
47. Analytical Trigonometry is that branch of Mathematics which treats of the general properties and relations of trigonometrical functions.

DEFINITIONS AND GENERAL PRINCIPLES.

48. Let ABCD represent a circle whose radius is 1, and suppose its circumference to be divided into four equal parts, by the diameters AC and BD drawn perpendicular to each other. The horizontal diameter AC is called the *initial diameter*; the vertical diameter BD is called the *secondary diameter*; the point A, from which arcs are usually reckoned, is called the *origin of arcs*, and the point B, 90° distant, is called the *secondary origin*. Arcs estimated from A, around toward B, that is, in a direction contrary to that of the motion of the hands of a watch, are considered *positive*; consequently, those reckoned in a contrary direction must be regarded as *negative*.



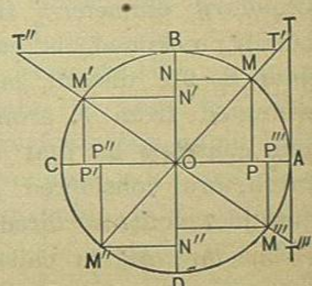
The arc AB, is called the *first quadrant*; the arc BC, the *second quadrant*; the arc CD, the *third quadrant*; and the arc DA, the *fourth quadrant*. The point at which an arc terminates, is called its *extremity,* and an arc is said to be in that quadrant in which its extremity is situated. Thus, the arc AM is in the *first quadrant,* the arc AM′ in the *second,* the arc AM″ in the *third,* and the arc AM‴ in the *fourth.*



49. The *complement* of an arc has been defined to be the difference between that arc and 90° (Art. 23); geometrically considered, the *complement* of an arc is *the arc included between the extremity of the arc and the secondary origin.* Thus, MB is the complement of AM; M′B, the complement of AM′; M″B, the complement of AM″, and so on. When the arc is greater than a quadrant, the complement is negative, according to the conventional principle agreed upon (Art. 48).

The *supplement* of an arc has been defined to be the difference between that arc and 180° (Art. 24); geometrically considered, it is *the arc included between the extremity of the arc and the left-hand extremity of the initial diameter.* Thus, MC is the supplement of AM, and M″C the supplement of AM″. The supplement is negative, when the arc is greater than two quadrants.

50. *The sine of an arc is the distance from the initial diameter to the extremity of the arc.* Thus, PM is the sine of AM, and P″M″ is the sine of the arc AM″. The term *distance* is used in the sense of *shortest* or *perpendicular distance.*



51. *The cosine of an arc is the distance from the secondary diameter to the extremity of the arc:* thus, NM is the cosine of AM, and N′M′ is the cosine of AM′.

The cosine may be measured on the initial diameter: thus, OP is equal to the cosine of AM, and OP′ to the cosine of AM′; that is, the cosine of an arc is equal to the distance, measured on the initial diameter, from the centre of the arc to the foot of the sine.

52. *The versed-sine of an arc is the distance from the sine to the origin of arcs:* thus, PA is the versed-sine of AM, and P′A is the versed-sine of AM′.

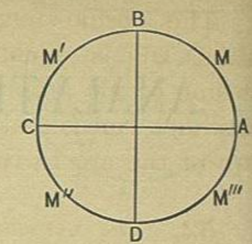
53. *The co-versed-sine of an arc is the distance from the cosine to the secondary origin:* thus, NB is the co-versed-sine of AM, and N″B is the co-versed-sine of AM″.

54. *The tangent of an arc is that part of a perpendicular to the initial diameter, at the origin of arcs, included between the origin and the prolongation of the diameter drawn to the extremity of the arc:* thus, AT is the tangent of AM, or of AM″, and AT‴ is the tangent of AM′, or of AM‴.

55. *The cotangent of an arc is that part of a perpendicular to the secondary diameter, at the secondary origin, included between the secondary origin and the prolongation of the diameter drawn to the extremity of the arc:* thus, BT′ is the cotangent of AM, or of AM″, and BT″ is the cotangent of AM, or of AM‴.

56. *The secant of an arc is the distance from the centre of the arc to the extremity of the tangent:* thus, OT is the secant of AM, or of AM″, and OT‴ is the secant of AM′. or of AM‴.

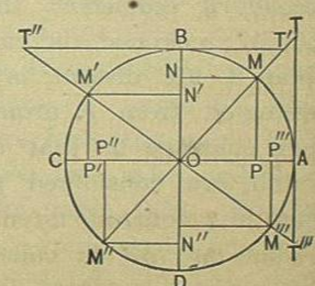
an arc terminates, is called its *extremity*, and an arc is said to be in that quadrant in which its extremity is situated. Thus, the arc AM is in the *first quadrant*, the arc AM′ in the *second*, the arc AM″ in the *third*, and the arc AM‴ in the *fourth*.



49. The *complement* of an arc has been defined to be the difference between that arc and 90° (Art. 23); geometrically considered, the *complement* of an arc is *the arc included between the extremity of the arc and the secondary origin.* Thus, MB is the complement of AM; M′B, the complement of AM′; M″B, the complement of AM″, and so on. When the arc is greater than a quadrant, the complement is negative, according to the conventional principle agreed upon (Art. 48).

The *supplement* of an arc has been defined to be the difference between that arc and 180° (Art. 24); geometrically considered, it is *the arc included between the extremity of the arc and the left-hand extremity of the initial diameter.* Thus, MC is the supplement of AM, and M″C the supplement of AM″. The supplement is negative, when the arc is greater than two quadrants.

50. *The sine of an arc is the distance from the initial diameter to the extremity of the arc.* Thus, PM is the sine of AM, and P″M″ is the sine of the arc AM″. The term *distance* is used in the sense of *shortest* or *perpendicular distance.*



51. *The cosine of an arc is the distance from the secondary diameter to the extremity of the arc:* thus, NM is the cosine of AM, and N′M′ is the cosine of AM′.

The cosine may be measured on the initial diameter: thus, OP is equal to the cosine of AM, and OP′ to the cosine of AM′; that is, the cosine of an arc is equal to the distance, measured on the initial diameter, from the centre of the arc to the foot of the sine.

52. *The versed-sine of an arc is the distance from the sine to the origin of arcs:* thus, PA is the versed-sine of AM, and P′A is the versed-sine of AM′.

53. *The co-versed-sine of an arc is the distance from the cosine to the secondary origin:* thus, NB is the co-versed-sine of AM, and N″B is the co-versed-sine of AM″.

54. *The tangent of an arc is that part of a perpendicular to the initial diameter, at the origin of arcs, included between the origin and the prolongation of the diameter drawn to the extremity of the arc:* thus, AT is the tangent of AM, or of AM″, and AT‴ is the tangent of AM′, or of AM‴.

55. *The cotangent of an arc is that part of a perpendicular to the secondary diameter, at the secondary origin, included between the secondary origin and the prolongation of the diameter drawn to the extremity of the arc:* thus, BT′ is the cotangent of AM, or of AM″, and BT″ is the cotangent of AM, or of AM‴.

56. *The secant of an arc is the distance from the centre of the arc to the extremity of the tangent:* thus, OT is the secant of AM, or of AM″, and OT‴ is the secant of AM′. or of AM‴.

57. *The cosecant of an arc is the distance from the centre of the arc to the extremity of the cotangent:* thus, OT′ is the cosecant of AM, or of AM″, and OT″ is the cosecant of AM′, or of AM‴.

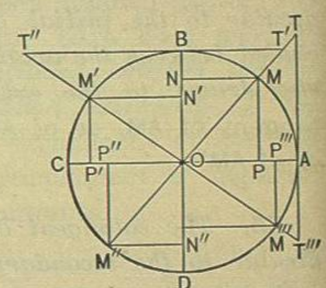
The prefix *co*, as used here, is equivalent to *complement;* thus, the *cosine* of an arc is the "*complement sine,*" that is, the *sine of the complement*, of that arc, and so on, as explained in Art. 27.

The eight *trigonometrical functions* above defined are also called *circular functions.*

RULES FOR DETERMINING THE ALGEBRAIC SIGNS OF CIRCULAR FUNCTIONS.

58. All distances estimated *upward* are regarded as *positive;* consequently, all distances estimated *downward* must be considered *negative.*

Thus, AT, PM, NB, P′M′, are positive, and AT‴, P‴M‴, P″M″, &c., are negative.



All distances estimated *toward the right* are regarded as *positive;* consequently, all distances estimated *toward the left* must be considered *negative.*

Thus, NM, BT′, PA, &c., are positive, and N′M′, BT″, &c., are negative.

These two rules are sufficient for determining the algebraic signs of all the circular functions, except the secant and cosecant. For the secant and cosecant, the following is the rule:

All distances estimated from the centre in a direction *toward the extremity* of the arc are regarded as *positive;* consequently, all distances estimated in a direction *away from the extremity* of the arc must be considered *negative.*

Thus, OT, regarded as the secant of AM, is estimated in a direction *toward* M, and is *positive;* but OT, regarded as the secant of AM″, is estimated in a direction *away from* M″, and is *negative.*

These conventional rules enable us to give at once the proper sign to any function of an arc in any quadrant.

59. In accordance with the above rules, and the definitions of the circular functions, we have the following principles:

The sine is positive in the first and second quadrants, and negative in the third and fourth.

The cosine is positive in the first and fourth quadrants, and negative in the second and third.

The versed-sine and the co-versed-sine are always positive.

The tangent and cotangent are positive in the first and third quadrants, and negative in the second and fourth.

The secant is positive in the first and fourth quadrants, and negative in the second and third.

The cosecant is positive in the first and second quadrants, and negative in the third and fourth.

LIMITING VALUES OF THE CIRCULAR FUNCTIONS.

60. The limiting values of the circular functions are those values which they have at the beginning and the end of the different quadrants. Their numerical values are discovered by following them as the arc increases from 0° around to 360°, and so on around through 450°, 540°, &c. The signs of these values are determined by the principle, that *the sign of a varying magnitude up to the limit, is the sign at the limit.* For illustration, let us examine the limiting values of the sine and the tangent.

If we suppose the arc to be 0, the sine will be 0; as the arc increases, the sine increases until the arc becomes equal to 90°, when the sine becomes equal to $+1$, which is its greatest possible value; as the arc increases from 90°, the sine diminishes until the arc becomes equal to 180°, when the sine becomes equal to $+0$; as the arc increases from 180°, the sine becomes negative, and increases numerically, but *decreases algebraically,* until the arc becomes equal to 270°, when the sine becomes equal to -1, which is its least *algebraical* value; as the arc increases from 270°, the sine decreases numerically, but *increases algebraically,* until the arc becomes 360°, when the sine becomes equal to -0. It is -0, for this value of the arc, in accordance with the principle of limits.

The tangent is 0 when the arc is 0, and increases till the arc becomes 90°, when the tangent is $+\infty$; in passing through 90°, the tangent changes from $+\infty$ to $-\infty$, and as the arc increases the tangent decreases numerically, but increases algebraically, till the arc becomes equal to 180°, when the tangent becomes equal to -0; from 180° to 270° the tangent is again positive, and at 270° it becomes equal to $+\infty$; from 270° to 360°, the tangent is again negative, and at 360° it becomes equal to -0.

If we still suppose the arc to increase after reaching 360°, the functions will again go through the same changes, that is, the functions of an arc are the same as the functions of that arc increased by 360°, 720°, &c.

By discussing the limiting values of all the circular functions we may form the following table:

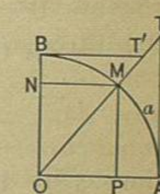
TABLE I.

Arc = 0°.	Arc = 90°.	Arc = 180°.	Arc = 270°.	Arc = 360°.
sin $= 0$	sin $= 1$	sin $= 0$	sin $= -1$	sin $= -0$
cos $= 1$	cos $= 0$	cos $= -1$	cos $= -0$	cos $= 1$
v-sin $= 0$	v-sin $= 1$	v-sin $= 2$	v-sin $= 1$	v-sin $= 0$
co-v-sin $= 1$	co-v-sin $= 0$	co-v-sin $= 1$	co-v-sin $= 2$	co-v-sin $= 1$
tan $= 0$	tan $= \infty$	tan $= -0$	tan $= \infty$	tan $= -0$
cot $= \infty$	cot $= 0$	cot $= -\infty$	cot $= 0$	cot $= -\infty$
sec $= 1$	sec $= \infty$	sec $= -1$	sec $= -\infty$	sec $= 1$
cosec $= \infty$	cosec $= 1$	cosec $= \infty$	cosec $= -1$	cosec $= -\infty$

RELATIONS BETWEEN THE CIRCULAR FUNCTIONS OF ANY ARC.

61. Let AM, denoted by a, represent any arc whose radius is 1. Draw the lines as represented in the figure. Then we shall have,

$$\begin{aligned} OM &= OA = 1; & PM &= ON = \sin a; \\ NM &= OP = \cos a; & PA &= \text{ver-sin}\ a; \\ NB &= \text{co-ver-sin}\ a; & AT &= \tan a; \\ BT' &= \cot a; & OT &= \sec a; \\ & \text{and} & OT' &= \text{cosec}\ a. \end{aligned}$$



From the right-angled triangle OPM, we have,

$$\overline{PM}^2 + \overline{OP}^2 = \overline{OM}^2, \quad \text{or,} \quad \sin^2 a + \cos^2 a = 1. \quad \cdot \quad \cdot \quad (1.)$$

The symbols $\sin^2 a$, $\cos^2 a$, &c., denote the square of the sine of a, the square of the cosine of a, &c.

From formula (1) we have, by transposition,

$$\sin^2 a = 1 - \cos^2 a; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2.)$$

$$\cos^2 a = 1 - \sin^2 a. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3.)$$

We have, from the figure,

$$\mathsf{PA} = \mathsf{OA} - \mathsf{OP},$$

or, $$\text{ver-sin}\, a = 1 - \cos a; \quad \cdots \quad (4.)$$

and, $$\mathsf{NB} = \mathsf{OB} - \mathsf{ON},$$

or, $$\text{co-ver-sin}\, a = 1 - \sin a. \quad \cdots \quad (5.)$$

From the similar triangles OAT and OPM, we have,

$$\mathsf{OP} : \mathsf{PM} :: \mathsf{OA} : \mathsf{AT}, \quad \text{or,} \quad \cos a : \sin a :: 1 : \tan a;$$

whence, $$\tan a = \frac{\sin a}{\cos a}. \quad \cdots \quad (6.)$$

From the similar triangles ONM and OBT′, we have,

$$\mathsf{ON} : \mathsf{NM} :: \mathsf{OB} : \mathsf{BT'}, \quad \text{or,} \quad \sin a : \cos a :: 1 : \cot a;$$

whence, $$\cot a = \frac{\cos a}{\sin a}. \quad \cdots \quad (7.)$$

Multiplying (6) and (7), member by member, we have,

$$\tan a \cot a = 1; \quad \cdots \quad (8.)$$

whence, by division, $$\tan a = \frac{1}{\cot a}; \quad \cdots \quad (9.)$$

and $$\cot a = \frac{1}{\tan a}. \quad \cdots \quad (10.)$$

From the similar triangles OPM and OAT, we have,

$$\mathsf{OP} : \mathsf{OM} :: \mathsf{OA} : \mathsf{OT}, \quad \text{or,} \quad \cos a : 1 :: 1 : \sec a;$$

whence, $$\sec a = \frac{1}{\cos a}. \quad \cdots \quad (11.)$$

From the similar triangles ONM and OBT′, we have,

$$\mathsf{ON} : \mathsf{OM} :: \mathsf{OB} : \mathsf{OT'}, \quad \text{or,} \quad \sin a : 1 :: 1 : \text{cosec}\, a;$$

whence, $$\text{cosec}\, a = \frac{1}{\sin a}. \quad \cdots \quad (12.)$$

From the right-angled triangle OAT, we have,

$$\overline{\mathsf{OT}}^2 = \overline{\mathsf{OA}}^2 + \overline{\mathsf{AT}}^2; \quad \text{or,} \quad \sec^2 a = 1 + \tan^2 a. \quad \cdots \quad (13.)$$

From the right-angled triangle OBT′, we have,

$$\overline{\mathsf{OT}}'^2 = \overline{\mathsf{OB}}^2 + \overline{\mathsf{BT}}'^2; \quad \text{or,} \quad \text{cosec}^2 a = 1 + \cot^2 a. \quad \cdot \quad (14.)$$

It is to be observed that formulas (5), (7), (12), and (14), may be deduced from formulas (4), (6), (11), and (13), by substituting $90° - a$, for a, and then making the proper reductions.

Collecting the preceding formulas, we have the following table:

TABLE II.

(1.)	$\sin^2 a + \cos^2 a$	=	$1.$	(9.)	$\tan a$	=	$\frac{1}{\cot a}.$
(2.)	$\sin^2 a$	=	$1 - \cos^2 a.$	(10.)	$\cot a$	=	$\frac{1}{\tan a}.$
(3.)	$\cos^2 a$	=	$1 - \sin^2 a.$	(11.)	$\sec a$	=	$\frac{1}{\cos a}.$
(4.)	$\text{ver-sin}\, a$	=	$1 - \cos a.$	(12.)	$\text{cosec}\, a$	=	$\frac{1}{\sin a}.$
(5.)	$\text{co-ver-sin}\, a$	=	$1 - \sin a.$	(13.)	$\sec^2 a$	=	$1 + \tan^2 a.$
(6.)	$\tan a$	=	$\frac{\sin a}{\cos a}.$	(14.)	$\text{cosec}^2 a$	=	$1 + \cot^2 a.$
(7.)	$\cot a$	=	$\frac{\cos a}{\sin a}.$				
(8.)	$\tan a \cot a$	=	$1.$				