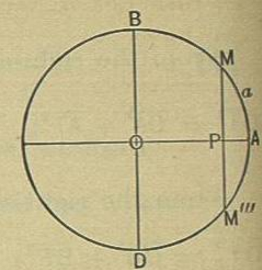
FUNCTIONS OF NEGATIVE ARCS.

62. Let AM‴, estimated from A toward D, be numerically equal to AM; then, if we denote the arc AM by a, the arc AM‴ will be denoted by $-a$ (Art. 48).



A being the middle point of the arc M‴AM, the radius OA bisects the chord M‴M at right angles (B. III., P. VI.); therefore, PM‴ is numerically equal to PM, but PM‴ being measured downward from the initial diameter is negative, while PM being measured upward is positive, and, therefore, PM‴ = − PM; OP is equal to the cosine of both AM‴ and AM (Art. 61); hence, we have,

$$\sin(-a) = -\sin a, \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (1.)$$

$$\cos(-a) = \cos a. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (2.)$$

Dividing (1) by (2), member by member, and then dividing (2) by (1), member by member, we have (formulas 6 and 7, Art. 61),

$$\tan(-a) = -\tan(a); \qquad \cot(-a) = -\cot a.$$

Taking the reciprocals of the members of (2), and then the reciprocals of the members of (1), we have (formulas 11 and 12, Art. 61),

$$\sec(-a) = \sec a; \qquad \operatorname{cosec}(-a) = -\operatorname{cosec} a.$$

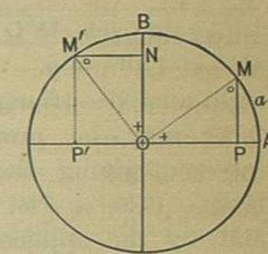
FUNCTIONS OF ARCS

FORMED BY ADDING AN ARC TO, OR SUBTRACTING IT FROM, ANY NUMBER OF QUADRANTS.

63. Let a denote any arc less than 90°. By definition, we have,

$$\sin(90° - a) = \cos a; \qquad \cos(90° - a) = \sin a.$$
$$\tan(90° - a) = \cot a; \qquad \cot(90° - a) = \tan a.$$
$$\sec(90° - a) = \operatorname{cosec} a; \qquad \operatorname{cosec}(90° - a) = \sec a.$$

Let the arc BM′ = AM = a; then AM′ = $90° + a$. Draw lines, as in the figure. Then PM = $\sin a$; OP = $\cos a$; ON = P′M′ = $\sin(90° + a)$; NM′ = $\cos(90° + a)$.



The right-angled triangles ONM′ and OPM have the angles NOM′ and POM equal (B. III., P. XV.), the angles ONM′ and OPM equal, both being right angles, and therefore (B. I., P. XXV., C. 2), the angles OM′N and OMP equal; they have, also, the sides OM′ and OM equal, and are, consequently (B. I., P. VI.), equal in all respects: hence, ON = OP, and NM′ = PM. These are *numerical* relations; by the rules for signs, Art. 58, ON and OP are both positive, NM′ is negative, and PM positive; and hence, *algebraically*, ON = OP, and NM′ = − PM; therefore, we have,

$$\sin(90° + a) = \cos a; \quad \cdot \cdot \cdot \cdot \cdot \quad (1.)$$

$$\cos(90° + a) = -\sin a. \quad \cdot \cdot \cdot \cdot \cdot \quad (2.)$$

Dividing (1) by (2), member by member, we have,

$$\frac{\sin(90° + a)}{\cos(90° + a)} = \frac{\cos a}{-\sin a};$$

or (formulas 6 and 7, Art. 61),

$$\tan(90° + a) = -\cot a.$$

In like manner, dividing (2) by (1), member by member, we have,

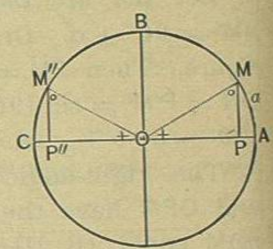
$$\cot(90^\circ + a) = -\tan a.$$

Taking the reciprocals of both members of (2), we have (formulas 11 and 12, Art. 61),

$$\sec(90^\circ + a) = -\operatorname{cosec} a.$$

In like manner, taking the reciprocals of both members of (1), we have,

$$\operatorname{cosec}(90^\circ + a) = \sec a.$$



Again, let $M''C = AM = a$; then $AM'' = 180^\circ - a$. As before, the right-angled triangles $OP''M''$ and OPM may be proved equal in all respects, giving the *numerical* relations, $P''M'' = PM$, and $OP'' = OP$, and, by the application of the rules for signs, Art. 58, may be obtained, $P''M'' = PM$, and $OP'' = -OP$; hence,

$$\sin(180^\circ - a) = \sin a; \quad \cdots\cdots \quad (1.)$$

$$\cos(180^\circ - a) = -\cos a. \quad \cdots\cdots \quad (2.)$$

From these equations (1) and (2), and formulas (6), (7), (11), and (12), Art. 61, may be obtained, as before,

$$\tan(180^\circ - a) = -\tan a;$$

$$\cot(180^\circ - a) = -\cot a;$$

$$\sec(180^\circ - a) = -\sec a;$$

$$\operatorname{cosec}(180^\circ - a) = \operatorname{cosec} a.$$

In like manner, the values of the several functions of the remaining arcs in question may be obtained in terms of functions of the arc a. Tabulating the results, we have the following

TABLE III.

Arc = $90^\circ + a$.		**Arc = $270^\circ - a$.**	
sin = cos a,	cos = − sin a,	sin = − cos a,	cos = − sin a,
tan = − cot a,	cot = − tan a,	tan = cot a,	cot = tan a,
sec = − cosec a,	cosec = sec a.	sec = − cosec a,	cosec = − sec a.
Arc = $180^\circ - a$.		**Arc = $270^\circ + a$.**	
sin = sin a,	cos = − cos a,	sin = − cos a,	cos = sin a,
tan = − tan a,	cot = − cot a,	tan = − cot a,	cot = − tan a,
sec = − sec a,	cosec = cosec a.	sec = cosec a,	cosec = − sec a.
Arc = $180^\circ + a$.		**Arc = $360^\circ - a$.**	
sin = − sin a,	cos = − cos a,	sin = − sin a,	cos = cos a,
tan = tan a,	cot = cot a,	tan = − tan a,	cot = − cot a,
sec = − sec a,	cosec = − cosec a.	sec = sec a,	cosec = − cosec a.

It will be observed that, when the arc is added to, or subtracted from, an *even* number of quadrants, the name of the function is the *same* in both columns; and when the arc is added to, or subtracted from, an *odd* number of quadrants, the names of the functions in the two columns are *contrary:* in all cases, the algebraic sign is determined by the rules already given (Art. 58).

By means of this table, we may find the functions of any arc in terms of the functions of an arc less than 90°. Thus,

$$\sin 115^\circ = \sin(90^\circ + 25^\circ) = \cos 25^\circ,$$

$$\sin 284^\circ = \sin(270^\circ + 14^\circ) = -\cos 14^\circ,$$

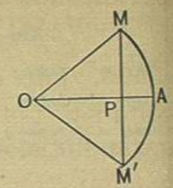
$$\sin 400^\circ = \sin(360^\circ + 40^\circ) = \sin 40^\circ,$$

$$\tan 210^\circ = \tan(180^\circ + 30^\circ) = \tan 30^\circ.$$

&c. &c. &c.

PARTICULAR VALUES OF CERTAIN FUNCTIONS.

64. Let MAM′ be any arc, denoted by $2a$, M′M its chord, and OA a radius drawn perpendicular to M′M: then will $PM = \frac{1}{2}M'M$, and $AM = \frac{1}{2}M'AM$ (B. III., P. VI.). But PM is the sine of AM, or, $PM = \sin a$: hence,



$$\sin a = \tfrac{1}{2}M'M;$$

that is, *the sine of an arc is equal to one half the chord of twice the arc.*

Let $M'AM = 60°$; then will $AM = 30°$, and M′M will equal the radius, or 1 (B. V., P. IV.): hence, we have

$$\sin 30° = \tfrac{1}{2};$$

that is, *the sine of 30° is equal to half the radius.*

Also, $$\cos 30° = \sqrt{1 - \sin^2 30°} = \tfrac{1}{2}\sqrt{3};$$

hence, $$\tan 30° = \frac{\sin 30°}{\cos 30°} = \sqrt{\tfrac{1}{3}}.$$

Again, let $M'AM = 90°$: then will $AM = 45°$, and $M'M = \sqrt{2}$ (B. V., P. III.): hence, we have

$$\sin 45° = \tfrac{1}{2}\sqrt{2};$$

Also, $$\cos 45° = \sqrt{1 - \sin^2 45°} = \tfrac{1}{2}\sqrt{2};$$

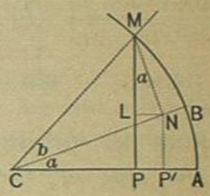
hence, $$\tan 45° = \frac{\sin 45°}{\cos 45°} = 1.$$

Many other numerical values might be deduced.

FORMULAS

EXPRESSING RELATIONS BETWEEN THE CIRCULAR FUNCTIONS OF DIFFERENT ARCS.

65. Let AB and BM represent two arcs, having the common radius 1; denote the first by a, and the second by b; then, $AM = a + b$. From M draw PM perpendicular to CA, and NM perpendicular to CB; from N draw NP′ perpendicular, and NL parallel, to CA.



Then, by definition, we have

$$PM = \sin(a + b), \quad NM = \sin b, \quad \text{and} \quad CN = \cos b.$$

From the figure, we have

$$PM = PL + LM. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (1.)$$

From the right-angled triangle CP′N (Art. 37), we have

$$P'N = CN \sin a;$$

or, since $$P'N = PL,$$

$$PL = \cos b \sin a = \sin a \cos b.$$

Since the triangle MLN is similar to CP′N (B. IV., P. XXI.), the angle LMN is equal to the angle P′CN; hence, from the right-angled triangle MLN, we have

$$LM = NM \cos a = \sin b \cos a = \cos a \sin b.$$

Substituting the values of PM, PL, and LM, in equation (1), we have

$$\sin(a + b) = \sin a \cos b + \cos a \sin b; \quad \cdot \quad (A.)$$

that is, *the sine of the sum of two arcs is equal to the sine of the first into the cosine of the second, plus the cosine of the first into the sine of the second.*

Since the above formula is true for any values of a and b, we may substitute $-b$ for b; whence,

$$\sin(a-b) = \sin a \cos(-b) + \cos a \sin(-b);$$

but (Art. 62),

$$\cos(-b) = \cos b, \quad \text{and} \quad \sin(-b) = -\sin b;$$

hence, $\sin(a-b) = \sin a \cos b - \cos a \sin b;$. (B.)

that is, *the sine of the difference of two arcs is equal to the sine of the first into the cosine of the second, minus the cosine of the first into the sine of the second.*

If, in formula (B), we substitute $(90° - a)$, for a, we have

$$\sin(90°-a-b) = \sin(90°-a)\cos b - \cos(90°-a)\sin b; \quad (2.)$$

but (Art. 63),

$$\sin(90° - a - b) = \sin[90° - (a+b)] = \cos(a+b),$$

and,

$$\sin(90° - a) = \cos a,$$

$$\cos(90° - a) = \sin a;$$

hence, by substitution in equation (2), we have

$$\cos(a+b) = \cos a \cos b - \sin a \sin b; \quad . \quad (C.)$$

that is, *the cosine of the sum of two arcs is equal to the rectangle of their cosines, minus the rectangle of their sines.*

If, in formula (C), we substitute $-b$, for b, we find

$$\cos(a-b) = \cos a \cos(-b) - \sin a \sin(-b),$$

or, $\cos(a-b) = \cos a \cos b + \sin a \sin b;$ . . (D.)

that is, *the cosine of the difference of two arcs is equal to the rectangle of their cosines, plus the rectangle of their sines.*

If we divide formula (A) by formula (C), member by member, we have

$$\frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b}.$$

Dividing both terms of the second member by $\cos a \cos b$, recollecting that the sine divided by the cosine is equal to the tangent, we find

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}; \quad . \quad . \quad . \quad (E.)$$

that is, *the tangent of the sum of two arcs, is equal to the sum of their tangents, divided by* 1 *minus the rectangle of their tangents.*

If, in formula (E), we substitute $-b$ for b, recollecting that $\tan(-b) = -\tan b$, we have

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}; \quad . \quad . \quad . \quad (F.)$$

that is, *the tangent of the difference of two arcs is equal to the difference of their tangents, divided by* 1 *plus the rectangle of their tangents.*

In like manner, dividing formula (C) by formula (A), member by member, and reducing, we have

$$\cot(a+b) = \frac{\cot a \cot b - 1}{\cot a + \cot b}; \quad . \quad . \quad . \quad (G.)$$

and thence, by the substitution of $-b$ for b,

$$\cot(a-b) = \frac{\cot a \cot b + 1}{\cot b - \cot a} \quad . \quad . \quad . \quad \text{(H.)}$$

FUNCTIONS OF DOUBLE ARCS AND HALF ARCS.

66. If, in formulas (A), (C), (E), and (G), we make $b = a$, we find

$$\sin 2a = 2 \sin a \cos a\,; \quad . \quad . \quad . \quad . \quad \text{(A}'.\text{)}$$

$$\cos 2a = \cos^2 a - \sin^2 a\,; \quad . \quad . \quad . \quad \text{(C}'.\text{)}$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(E}'.\text{)}$$

$$\cot 2a = \frac{\cot^2 a - 1}{2 \cot a}. \quad . \quad . \quad . \quad . \quad . \quad \text{(G}'.\text{)}$$

Substituting in (C′) for $\cos^2 a$, its value, $1 - \sin^2 a$; and afterwards for $\sin^2 a$, its value, $1 - \cos^2 a$, we have

$$\cos 2a = 1 - 2 \sin^2 a,$$

$$\cos 2a = 2 \cos^2 a - 1\,;$$

whence, by solving these equations,

$$\sin a = \sqrt{\frac{1 - \cos 2a}{2}}\,; \quad . \quad . \quad . \quad . \quad (1.)$$

$$\cos a = \sqrt{\frac{1 + \cos 2a}{2}}. \quad . \quad . \quad . \quad . \quad (2.)$$

We also have, from the same equations,

$$1 - \cos 2a = 2 \sin^2 a\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$1 + \cos 2a = 2 \cos^2 a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Dividing equation (A′), first by equation (4), and then by equation (3), member by member, we have

$$\frac{\sin 2a}{1 + \cos 2a} = \tan a\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

$$\frac{\sin 2a}{1 - \cos 2a} = \cot a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

Substituting $\frac{1}{2}a$ for a, in equations (1), (2), (5), and (6), we have

$$\sin \tfrac{1}{2}a = \sqrt{\frac{1 - \cos a}{2}}\,; \quad . \quad . \quad . \quad . \quad \text{(A}''.\text{)}$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{1 + \cos a}{2}}\,; \quad . \quad . \quad . \quad . \quad \text{(C}''.\text{)}$$

$$\tan \tfrac{1}{2}a = \frac{\sin a}{1 + \cos a}\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(E}''.\text{)}$$

$$\cot \tfrac{1}{2}a = \frac{\sin a}{1 - \cos a}. \quad . \quad . \quad . \quad . \quad . \quad \text{(G}''.\text{)}$$

Taking the reciprocals of both members of the last two formulas, we have also,

$$\cot \tfrac{1}{2}a = \frac{1 + \cos a}{\sin a}, \quad \text{and} \quad \tan \tfrac{1}{2}a = \frac{1 - \cos a}{\sin a}.$$

ADDITIONAL FORMULAS.

67. If formulas (A) and (B) are first added, member to member, and then subtracted, member from member, and the same operations are performed upon (C) and (D), we obtain

$$\sin(a+b) + \sin(a-b) = 2\sin a\cos b;$$

$$\sin(a+b) - \sin(a-b) = 2\cos a\sin b;$$

$$\cos(a+b) + \cos(a-b) = 2\cos a\cos b;$$

$$\cos(a-b) - \cos(a+b) = 2\sin a\sin b.$$

If in these we make

$$a+b = p, \qquad \text{and} \qquad a-b = q,$$

whence,

$$a = \tfrac{1}{2}(p+q), \qquad b = \tfrac{1}{2}(p-q);$$

and then substitute in the above formulas, we obtain

$$\sin p + \sin q = 2\sin\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q). \quad \cdot \quad \text{(K.)}$$

$$\sin p - \sin q = 2\cos\tfrac{1}{2}(p+q)\sin\tfrac{1}{2}(p-q). \quad \cdot \quad \text{(L.)}$$

$$\cos p + \cos q = 2\cos\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q). \quad \cdot \quad \text{(M.)}$$

$$\cos q - \cos p = 2\sin\tfrac{1}{2}(p+q)\sin\tfrac{1}{2}(p-q). \quad \cdot \quad \text{(N.)}$$

From formulas (L) and (K), by division, we obtain

$$\frac{\sin p - \sin q}{\sin p + \sin q} = \frac{\cos\frac{1}{2}(p+q)\sin\frac{1}{2}(p-q)}{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}$$

$$= \frac{\tan\frac{1}{2}(p-q)}{\tan\frac{1}{2}(p+q)}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(1.)}$$

Hence, since p and q represent any arcs whatever, *the sum of the sines of two arcs is to their difference, as the tangent of one half the sum of the arcs is to the tangent of one half their difference.*

Also, in like manner, we obtain

$$\frac{\sin p+\sin q}{\cos p+\cos q} = \frac{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)} = \tan\tfrac{1}{2}(p+q), \qquad (2.)$$

$$\frac{\sin p-\sin q}{\cos p+\cos q} = \frac{\cos\frac{1}{2}(p+q)\sin\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)} = \tan\tfrac{1}{2}(p-q), \qquad (3.)$$

$$\frac{\sin p+\sin q}{\sin(p+q)} = \frac{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)} = \frac{\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)}, \qquad (4.)$$

$$\frac{\sin p-\sin q}{\sin(p+q)} = \frac{\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)} = \frac{\sin\frac{1}{2}(p-q)}{\sin\frac{1}{2}(p+q)}, \qquad (5.)$$

$$\frac{\sin(p-q)}{\sin p-\sin q} = \frac{\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p-q)}{\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)} = \frac{\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)}, \qquad (6.)$$

all of which give proportions analogous to that deduced from formula (1).

Since the second members of (6) and (4) are the same, we have

$$\frac{\sin p - \sin q}{\sin(p-q)} = \frac{\sin(p+q)}{\sin p + \sin q}; \quad \cdot \quad \cdot \quad \cdot \quad (7.)$$

that is, *the sine of the difference of two arcs is to the difference of the sines, as the sum of the sines is to the sine of the sum.*

All of the preceding formulas may be made homogeneous in terms of R, R being any radius, as explained in Art. 30; or, we may simply introduce R, as a factor, into each term as many times as may be necessary to render all of its terms of the same degree.

METHOD OF COMPUTING A TABLE OF NATURAL SINES.

68. Since the length of the semi-circumference of a circle whose radius is 1, is equal to the number 3.14159265 . . . , if we divide this number by 10800, the number of minutes in 180°, the quotient, .0002908882 . . . , will be the length of the arc of *one minute;* and since this arc is so small that it does not differ materially from its sine or tangent, this may be placed in the table as *the sine of one minute.*

Formula (3) of Table II., gives

$$\cos 1' = \sqrt{1 - \sin^2 1'} = .9999999577. \quad (1.)$$

Having thus determined, to a near degree of approximation, the sine and cosine of one minute, we take the first formula of Art. 67, and put it under the form,

$$\sin (a + b) = 2 \sin a \cos b - \sin (a - b),$$

and make in this, $b = 1'$, and then in succession,

$$a = 1', \quad a = 2', \quad a = 3', \quad a = 4', \quad \&c.,$$

and obtain,

$$\sin 2' = 2 \sin 1' \cos 1' - \sin 0 = .0005817764 \ldots$$

$$\sin 3' = 2 \sin 2' \cos 1' - \sin 1' = .0008726646 \ldots$$

$$\sin 4' = 2 \sin 3' \cos 1' - \sin 2' = .0011635526 \ldots$$

$$\sin 5' = \quad \&c.,$$

thus obtaining the sine of every number of degrees and minutes from 1′ to 45°.

The cosines of the corresponding arcs may be computed by means of equation (1).

Having found the sines and cosines of arcs less than 45°, those of the arcs between 45° and 90° may be deduced, by considering that the sine of an arc is equal to the cosine of its complement, and the cosine equal to the sine of its complement. Thus,

$$\sin 50° = \sin (90° - 40°) = \cos 40°, \qquad \cos 50° = \sin 40°,$$

in which the second members are known from the previous computations.

To find the tangent of any arc, divide its sine by its cosine. To find the cotangent, take the reciprocal of the corresponding tangent.

As the accuracy of the calculation of the sine of any arc, by the above method, depends upon the accuracy of each previous calculation, it would be well to verify the work, by calculating the sines of the degrees separately (after having found the sines of one and two degrees), by the last proportion of Art. 67. Thus,

$$\sin 1° : \sin 2° - \sin 1° :: \sin 2° + \sin 1° : \sin 3°;$$

$$\sin 2° : \sin 3° - \sin 1° :: \sin 3° + \sin 1° : \sin 4°; \ \&c.$$