83. To deduce Napier's Analogies.

From equation (1), Art. 80, we have

$$\cos A + \cos B \cos C = \sin B \sin C \cos a$$
$$= \sin C \frac{\sin A}{\sin a} \sin b \cos a; \quad (1.)$$

since, from proportion (1), Art. 78, we have

$$\sin B = \frac{\sin A}{\sin a} \sin b.$$

Also, from equation (2), Art. 80, we have

$$\cos B + \cos A \cos C = \sin A \sin C \cos b$$
$$= \sin C \frac{\sin A}{\sin a} \sin a \cos b. \quad (2.)$$

Adding (1) and (2), and dividing by sin C, we obtain

$$(\cos A + \cos B) \frac{1 + \cos C}{\sin C} = \frac{\sin A}{\sin a} \sin (a + b). \quad (3.)$$

The proportion,

$$\sin A : \sin B :: \sin a : \sin b,$$

taken first by composition, and then by division, gives

$$\sin A + \sin B = \frac{\sin A}{\sin a} (\sin a + \sin b), \quad \cdot \quad \cdot \quad (4.)$$

$$\sin A - \sin B = \frac{\sin A}{\sin a} (\sin a - \sin b). \quad \cdot \quad \cdot \quad (5.)$$

Dividing (4) and (5), in succession, by (3), we obtain

$$\frac{\sin A + \sin B}{\cos A + \cos B} \times \frac{\sin C}{1 + \cos C} = \frac{\sin a + \sin b}{\sin (a + b)}. \quad (6.)$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} \times \frac{\sin C}{1 + \cos C} = \frac{\sin a - \sin b}{\sin (a + b)}. \quad (7.)$$

But, by formulas (2) and (4), Art. 67, and formula (E''), Art. 66, equation (6) becomes

$$\tan \tfrac{1}{2} (A + B) \tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)}; \quad \cdot \quad \cdot \quad (8.)$$

and, by the similar formulas (3) and (5), of Art. 67, equation (7) becomes

$$\tan \tfrac{1}{2} (A - B) \tan \tfrac{1}{2} C = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)}. \quad \cdot \quad \cdot \quad (9.)$$

As $\tan \frac{1}{2} C = \frac{1}{\cot \frac{1}{2} C}$, formulas (8) and (9) may be written

$$\frac{\tan \frac{1}{2} (A + B)}{\cot \frac{1}{2} C} = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)}, \quad \cdot \quad \cdot \quad \cdot \quad (8'.)$$

$$\frac{\tan \frac{1}{2} (A - B)}{\cot \frac{1}{2} C} = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)}. \quad \cdot \quad \cdot \quad \cdot \quad (9'.)$$

These last two formulas give the proportions known as *the first set of Napier's Analogies*; viz.,

$$\cos \tfrac{1}{2} (a+b) : \cos \tfrac{1}{2} (a-b) :: \cot \tfrac{1}{2} C : \tan \tfrac{1}{2} (A+B). \quad (10.)$$

$$\sin \tfrac{1}{2} (a+b) : \sin \tfrac{1}{2} (a-b) :: \cot \tfrac{1}{2} C : \tan \tfrac{1}{2} (A-B). \quad (11.)$$

If in these we substitute the values of a, b, C, A, and B, in terms of the corresponding parts of the supplemental polar triangle, as expressed in Art. 80, we obtain

$$\cos \tfrac{1}{2} (A+B) : \cos \tfrac{1}{2} (A-B) :: \tan \tfrac{1}{2} c : \tan \tfrac{1}{2} (a+b), \quad (12.)$$

$$\sin \tfrac{1}{2} (A+B) : \sin \tfrac{1}{2} (A-B) :: \tan \tfrac{1}{2} c : \tan \tfrac{1}{2} (a-b), \quad (13.)$$

the second set of Napier's Analogies.

In applying logarithms to any of the preceding formulas, they must be made homogeneous in terms of R, as explained in Art. 30.

In all the formulas, the letters may be interchanged at pleasure, provided that, when one large letter is substituted for another, the like substitution is made in the corresponding small letters, and the reverse: for example, C may be substituted for A, provided that at the same time c is substituted for a, &c.

NOTE.—It may be noted that, in formulas (10) and (12), whenever the sign of the first term of the proportion is *minus*, the sign of the last term must, also, be *minus*, *i. e.*, whenever $\frac{1}{2}(a+b)$ is greater than 90°, $\frac{1}{2}(A+B)$ must, also, be greater than 90°, and the reverse; and similarly, whenever $\frac{1}{2}(a+b)$ is less than 90°, $\frac{1}{2}(A+B)$ must, also, be less than 90°, and the reverse.

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

84. In the solution of oblique-angled triangles six different cases may arise: viz., there may be given,

I. Two sides and an angle opposite one of them.
II. Two angles and a side opposite one of them.
III. Two sides and their included angle.
IV. Two angles and their included side.
V. The three sides.
VI. The three angles.

CASE I.

Given two sides and an angle opposite one of them.

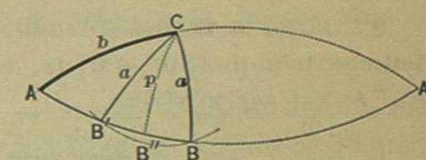
85. The solution, in this case, is commenced by finding the angle opposite the second given side, for which purpose formula (1), Art. 78, is employed.

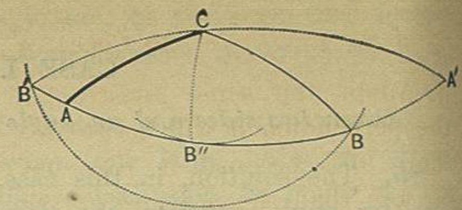
As this angle is found by means of its sine, and because the same sine corresponds to two different arcs, there would seem to be two different solutions. To ascertain when there are *two solutions*, when *one solution*, and when *no solution* at all, it becomes necessary to examine the relations which may exist between the given parts. Two cases may arise, viz., the given angle may be *acute*, or it may be *obtuse*.

We shall consider each case separately (B. IX., Gen. S. 1).

1st Case: $A < 90°$.

Let A be the given acute angle, and let a and b be the given sides. Prolong the arcs AC and AB till they meet at A′, forming the lune AA′; and from C, draw the arc CB″ perpendicular to ABA′. From C, as a pole, and with the arc a, describe the arc of a small circle BB′. If this circle cuts ABA′, in two points between A and A′, there will be *two solutions;* for if C be joined with each point of intersection by the arc of a great circle, we shall have two triangles, ABC and AB′C, both of which will conform to the conditions of the problem.





If only one point of intersection lies between A and A′, or if the small circle is tangent to ABA′, there will be but *one solution*.

If there is no point of intersection, or if there are points of intersection which do not lie between A and A′, there will be *no solution*.

From formula (2), Art. 72, we have

$$\sin \mathrm{CB}'' = \sin b \sin \mathrm{A},$$

from which the perpendicular may be found. This perpendicular will be less than 90°, since it can not exceed the measure of the angle A (B. IX., Gen. S. 2, 1°); denote its value by p. By inspection of the figure, we find the following relations:

1. *When* a *is greater than* p, *and at the same time less than both* b *and* 180° − b, *there will be two solutions.*

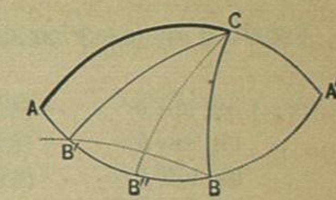
2. *When* a *is greater than* p, *and intermediate in value between* b *and* 180° − b; *or, when* a *is equal to* p, *there will be but one solution.*

If $a = b$, and is also less than $180° - b$, one of the points of intersection will be at A, and there will be but one solution.

3. *When* a *is greater than* p, *and at the same time greater than both* b *and* 180° − b; *or, when* a *is less than* p, *there will be no solution.*

2d Case: A > 90°.

Adopt the same construction as before. In this case, the perpendicular will be greater than 90°, because it can not be less than the measure of the angle A (B. IX., Gen. S. 2, 2°): it will, also, be greater than any other arc CA, CB, CA′, that can be drawn from C to ABA′. By a course of reasoning entirely analogous to that in the preceding case, we have the following principles:



4. *When* a *is less than* p, *and at the same time greater than both* b *and* 180° − b, *there will be two solutions.*

5. *When* a *is less than* p, *and intermediate in value between* b *and* 180° − b; *or, when* a *is equal to* p, *there will be but one solution.*

6. *When* a *is less than* p, *and at the same time less than both* b *and* 180° − b; *or, when* a *is greater than* p, *there will be no solution.*

Having found the angle or angles opposite the second side, the solution may be completed by means of Napier's Analogies.

Examples.

1. Given $a = 43° \ 27' \ 36''$, $b = 82° \ 58' \ 17''$, and $\mathrm{A} = 29° \ 32' \ 29''$, to find B, C, and c.

We see that $a > p$, since p can not exceed A (B. IX., Gen. S. 2, 1°); we see, further, that a is less than both

b and $180° - b$; hence, from the first condition there will be two solutions.

Applying logarithms to formula (1), Art. 78, we have

$$\log\sin B = (\text{a.c.}) \log\sin a + \log\sin b + \log\sin A - 10;$$

(a. c.) log sin a	(43° 27′ 36″)	0.162508
log sin b	(82° 58′ 17″)	9.996724
log sin A	(29° 32′ 29″)	9.692893
log sin B		9.852125

$\therefore$ B = 45° 21′ 01″, and B′ = 134° 38′ 59″.

From the first of Napier's Analogies (10), Art. 83, we find

$$\log\cot \tfrac{1}{2}C = (\text{a.c.}) \log\cos \tfrac{1}{2}(a - b) + \log\cos \tfrac{1}{2}(a + b) + \log\tan \tfrac{1}{2}(A + B) - 10.$$

Taking the first value of B, we have

$$\tfrac{1}{2}(A + B) = 37° \ 26' \ 45'';$$

also, $\tfrac{1}{2}(a + b) = 63° \ 12' \ 56''$;

and $\tfrac{1}{2}(a - b) = 19° \ 45' \ 20''$.

(a. c.) log cos $\frac{1}{2}(a - b)$	(19° 45′ 20″)	0.026344
log cos $\frac{1}{2}(a + b)$	(63° 12′ 56″)	9.653825
log tan $\frac{1}{2}$(A + B)	(37° 26′ 45″)	9.884130
log cot $\frac{1}{2}$C		9.564299

$\therefore$ $\frac{1}{2}$C = 69° 51′ 45″, and C = 139° 43′ 30″.

The side c may be found by means of formula (12), Art. 83, or by means of formula (2), Art. 78.

Applying logarithms to the proportion,

$$\sin A : \sin C :: \sin a : \sin c,$$

we have

$$\log\sin c = (\text{a.c.}) \log\sin A + \log\sin C + \log\sin a - 10;$$

(a. c.) log sin A	(29° 32′ 29″)	0.307107
log sin C	(139° 43′ 30″)	9.810539
log sin a	(43° 27′ 36″)	9.837492
log sin c		9.955138

$\therefore$ c = 115° 35′ 48″

We take the greater value of c, because the angle C, being greater than the angle B, requires that the side c should be greater than the side b. By using the second value of B, we may find, in a similar manner,

C′ = 32° 20′ 28″, and c' = 48° 16′ 18″.

2. Given a = 97° 35′, b = 27° 08′ 22″, and A = 40° 51′ 18″, to find B, C, and c.

Ans. B = 17° 31′ 09″, C = 144° 48′ 10″, c = 119° 08′ 25″.

3. Given a = 115° 20′ 10″, b = 57° 30′ 06″, and A = 126° 37′ 30″, to find B, C, and c.

Ans. B = 48° 29′ 48″, C = 61° 40′ 16″, c = 82° 34′ 04″.

4. Given b = 79° 14′, c = 30° 20′ 45″, and B = 121° 10′ 26″, to find C, A, and a.

Ans. C = 26° 06′ 16″, A = 49° 44′ 16″, a = 61° 11′ 06″.

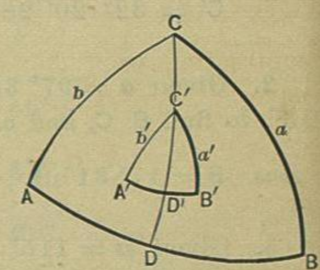
CASE II.

Given two angles and a side opposite one of them.

86. The solution, in this case, is commenced by finding the side opposite the second given angle, by means of formula (1), Art. 78. The solution is completed as in Case I.

Since the second side is found by means of its sine, there may be two solutions. To investigate this case, we pass to the supplemental polar triangle, by substituting for each part its supplement. In this triangle, there will be given two sides and an angle opposite one; it may therefore be discussed as in the preceding case. When the supplemental triangle has *two solutions, one solution,* or *no solution,* the given triangle will, in like manner, have *two solutions, one solution,* or *no solution.*

Let the given parts be A', B', and a', and let p' be the arc, $C'D'$, of a great circle drawn from the extremity of the given side perpendicular to the side opposite: we have



$$\sin p' = \sin a' \sin B'.$$

There will be two cases: a' may be *less* than $90°$; or, a' may be *greater* than $90°$.

1st Case: $a' < 90°$.

Passing to the supplemental polar triangle, we shall have given a, b, A; and since, in the given triangle, $a' < 90°$, in this supplemental triangle $A > 90°$: call the perpendicular CD, p. The conditions determining the number of solutions in this supplemental triangle are given in principles 4, 5, 6, Art. 85.

From principle 4, Art. 85, it appears that, for two solutions, a must be less than p, that is,

$$a < p:$$

subtracting each member of this inequality from $180°$, we have

$$180° - a > 180° - p;$$

but, $180° - a = A'$; and (B. IX., P. VI., C. 2), $180° - p = p'$; hence

$$A' > p':$$

again, it appears from principle 4, that a must be greater than b, that is,

$$a > b;$$

subtracting each member of this inequality from $180°$, we have

$$180° - a < 180° - b;$$

or,

$$A' < B':$$

it further appears from the same principle, that a must be greater than $180° - b$, that is,

$$a > 180° - b;$$

subtracting each member of this inequality from $180°$, we have

$$180° - a < 180° - (180° - b);$$

or.

$$A' < 180° - B'$$

Collecting the results, and, for convenience, omitting the primes, we have the following principle:

Two angles and a side opposite one of them being given, and the given side less than 90°, *i. e.*, A, B, a given, and $a < 90°$;

1. *When* A *is greater than* p, *and at the same time less than both* B *and* $180° - B$, *there will be two solutions.*

In like manner, from principle 5, Art. 85, we have

2. *When* A *is greater than* p, *and intermediate in value between* B *and* $180° - B$; *or, when* A *is equal to* p, *there will be but one solution.*

And from principle 6, Art. 85, we have

3. *When* A *is greater than* p, *and at the same time greater than both* B *and* $180° - B$; *or, when* A *is less than* p, *there will be no solution.*

It is to be noted that, in this case, the perpendicular is less than 90°, and less, also, than the given side; *i. e.*,

$$p < a.$$

2d Case: $a' > 90°$.

Passing to the supplemental polar triangle, we shall have given a, b, A, and $A < 90°$. The conditions determining the number of solutions in this supplemental triangle are given in principles 1, 2, 3, Art. 85.

From principle 1, Art. 85, it appears that, for two solutions, a must be greater than p, that is,

$$a > p;$$

subtracting each member of this inequality from 180°, we have

$$180° - a < 180° - p;$$

or, $$A' < p':$$

in the same manner as before, we may obtain from this principle 1,

$$A' > B';$$

and $$A' > 180° - B'.$$

As before, collecting the results and omitting the primes, we have the following principle:

Two angles and a side opposite one of them being given, the given side greater than 90°, *i. e.*, A, B, a given, and $a > 90°$;

4. *When* A *is less than* p, *and at the same time greater than both* B *and* $180° - B$, *there will be two solutions.*

In like manner, from principle 2, Art. 85, we have

5. *When* A *is less than* p, *and intermediate in value between* B *and* $180° - B$; *or, when* A *is equal to* p, *there will be but one solution.*

And from principle 3, Art. 85, we have

6. *When* A *is less than* p, *and at the same time less than both* B *and* $180° - B$; *or, when* A *is greater than* p, *there will be no solution.*

It is to be noted that, in this case, the perpendicular is greater than 90°, and greater, also, than the given side; *i. e.*, $p > a$.

From the principles deduced in Articles 85 and 86, it is evident that, if the given parts of the spherical triangles considered are named as in the accompanying table, we shall have the following principles, applicable to *all* the cases:

Perpendicular.	*Odd.*	*Adjacent.*	*Opposite.*
p	A	b	a
	a	B	A

7. The sine of p is equal to the rectangle of the sines of the odd part and the adjacent part.

8. p is always of the *same species* as the odd part, and *differs more* from $90°$ than the odd part, *i. e.*, when the odd part is *less* than $90°$, p is *still less;* and when the odd part is *greater* than $90°$, p is *still greater.*

9. There will be *two solutions:*

1°. When (odd part being *less* than 90°) the opposite part is *greater* than p, and *less* than the adjacent part and its supplement.

2°. When (odd part being *greater* than 90°) the opposite part is *less* than p, and *greater* than the adjacent part and its supplement.

10. There will be *one solution:*

1°. When (odd part being *less* than 90°) the opposite part is *greater* than p, and *intermediate in value* between the adjacent part and its supplement.

2°. When (odd part being *greater* than 90°) the

opposite part is *less* than p, and *intermediate in value* between the adjacent part and its supplement.

3°. When the opposite part is *equal* to p.

11. There will be *no solution:*

1°. When (odd part being *less* than 90°) the opposite part is either *less* than p, or *greater* than p and *greater also* than both the adjacent part and its supplement.

2°. When (odd part being *greater* than 90°) the opposite part is either *greater* than p, or *less* than p and *less also* than both the adjacent part and its supplement.

Examples.

1. Given $\mathrm{A} = 95° 16'$, $\mathrm{B} = 80° 42' 10''$, and $a = 57° 38'$, to find c, b, and C.

p might be computed from the formula,

$$\log \sin p = \log \sin \mathrm{B} + \log \sin a - 10;$$

but it is not necessary, as $p < a$ (see principle 8).

Because $\mathrm{A} > p$, and intermediate between $80° 42' 10''$ and $99° 17' 50''$, there will, from the second condition, be but one solution.

Applying logarithms to proportion (1), Art. 78, we have

$$\log \sin b = (\text{a. c.}) \log \sin \mathrm{A} + \log \sin \mathrm{B} + \log \sin a - 10;$$

(a. c.) log sin A	$(95° 16')$	0.001837
log sin B	$(80° 42' 10'')$	9.994257
log sin a	$(57° 38')$	9.926671
log sin b		9.922765 $\therefore b = 56° 49' 57''$.

We take the smaller value of b, for the reason that A, being greater than B, requires that a should be greater than b.

Applying logarithms to proportion (12), Art. 83, we have

$$\log \tan \tfrac{1}{2}c = \text{(a. c.)} \log \cos \tfrac{1}{2}(A - B) + \log \cos \tfrac{1}{2}(A + B) + \log \tan \tfrac{1}{2}(a + b) - 10;$$

we have $\tfrac{1}{2}(A + B) = 87° \, 59' \, 05''$,

$\tfrac{1}{2}(a + b) = 57° \, 13' \, 58''$,

and $\tfrac{1}{2}(A - B) = 7° \, 16' \, 55''$;

(a. c.) log cos ½ (A − B)	(7° 16′ 55″)	0.003517
log cos ½ (A + B)	(87° 59′ 05″)	8.546124
log tan ½ (a + b)	(57° 13′ 58″)	10.191352
log tan ½c		8.740993

$\therefore \; \tfrac{1}{2}c = 3° \, 09' \, 09''$, and $c = 6° \, 18' \, 18''$.

Applying logarithms to the proportion,

$$\sin a \; : \; \sin c \; :: \; \sin A \; : \; \sin C,$$

we have

$$\log \sin C = \text{(a. c.)} \log \sin a + \log \sin c + \log \sin A - 10;$$

(a. c.) log sin a	(57° 38′)	0.073329
log sin c	(6° 18′ 18″)	9.040685
log sin A	(95° 16′)	9.998163
log sin C		9.112177

$\therefore C = 7° \, 26' \, 21''$.

The smaller value of C is taken, for the same reason as before.

2. Given $A = 50° \, 12'$, $B = 58° \, 08'$, and $a = 62° \, 42'$, to find b, c, and C.

$b = 79° \, 12' \, 10''$, $c = 119° \, 03' \, 26''$, $C = 130° \, 54' \, 28''$,

$b' = 100° \, 47' \, 50''$, $c' = 152° \, 14' \, 18''$, $C' = 156° \, 15' \, 06''$.

3. Given $C = 115° \, 20'$, $A = 57° \, 30'$, and $c = 126° \, 38'$, to find a, b, and B.

Ans. $a = 48° \, 29' \, 13''$, $b = 118° \, 20' \, 44''$, $B = 97° \, 35' \, 06''$.

CASE III.

Given two sides and their included angle.

87. The remaining angles are found by means of Napier's Analogies, and the remaining side as in the preceding cases.

Examples.

1. Given $a = 62° \, 38'$, $b = 10° \, 13' \, 19''$, and $C = 150° \, 24' \, 12''$, to find c, A, and B.

Applying logarithms to proportions (10) and (11), Art. 83, we have

$$\log \tan \tfrac{1}{2}(A + B) = \text{(a. c.)} \log \cos \tfrac{1}{2}(a + b) + \log \cos \tfrac{1}{2}(a - b) + \log \cot \tfrac{1}{2}C - 10;$$

$$\log \tan \tfrac{1}{2}(A - B) = \text{(a. c.)} \log \sin \tfrac{1}{2}(a + b) + \log \sin \tfrac{1}{2}(a - b) + \log \cot \tfrac{1}{2}C - 10;$$

we have $\tfrac{1}{2}(a - b) = 26° \, 12' \, 20''$,

$\tfrac{1}{2}C = 75° \, 12' \, 06''$,

and $\tfrac{1}{2}(a + b) = 36° \, 25' \, 39''$.

(a. c.) $\log \cos \frac{1}{2}(a+b)$	·	$(36° \ 25' \ 39'')$	· 0.094415
$\log \cos \frac{1}{2}(a-b)$	·	$(26° \ 12' \ 20'')$	· 9.952897
$\log \cot \frac{1}{2}C$ · · ·	·	$(72° \ 12' \ 06'')$	· 9.421901
$\log \tan \frac{1}{2}(A+B)$ · · · · ·			· 9.469213

$$\therefore \ \tfrac{1}{2}(A+B) = 16° \ 24' \ 51''.$$

(a. c.) $\log \sin \frac{1}{2}(a+b)$	·	$(36° \ 25' \ 39'')$	· 0.226356
$\log \sin \frac{1}{2}(a-b)$	·	$(26° \ 12' \ 20'')$	· 9.645022
$\log \cot \frac{1}{2}C$ · ·	·	$(75° \ 12' \ 06'')$	· 9.421901
$\log \tan \frac{1}{2}(A-B)$ · · · · ·			· 9.293279

$$\therefore \ \tfrac{1}{2}(A-B) = 11° \ 06' \ 53''.$$

The greater angle is equal to the half sum plus the half difference, and the less is equal to the half sum minus the half difference. Hence, we have

$$A = 27° \ 31' \ 44'', \quad \text{and} \quad B = 5° \ 17' \ 58''.$$

Applying logarithms to proportion (13), Art. 83, we have

$$\log \tan \tfrac{1}{2}c = \text{(a. c.)} \log \sin \tfrac{1}{2}(A-B) + \log \sin \tfrac{1}{2}(A+B) + \log \tan \tfrac{1}{2}(a-b) - 10;$$

(a. c.) $\log \sin \frac{1}{2}(A-B)$	·	$(11° \ 06' \ 53'')$	· 0.714952
$\log \sin \frac{1}{2}(A+B)$	·	$(16° \ 24' \ 51'')$	· 9.451139
$\log \tan \frac{1}{2}(a-b)$	·	$(26° \ 12' \ 20'')$	· 9.692125
$\log \tan \frac{1}{2}c$ · · · · · · · ·			· 9.858216

$$\therefore \ \tfrac{1}{2}c = 35° \ 48' \ 33'', \quad \text{and} \quad c = 71° \ 37' \ 06''.$$

2. Given $a = 68° \ 46' \ 02''$, $b = 37° \ 10'$, and $C = 39° \ 23' \ 23''$, to find c, A, and B.

Ans. $A = 120° \ 59' \ 21''$, $B = 33° \ 45' \ 13''$, $c = 43° \ 37' \ 48''$.

3. Given $a = 84° \ 14' \ 29''$, $b = 44° \ 13' \ 45''$, and $C = 36° \ 45' \ 28''$, to find A and B.

Ans. $A = 130° \ 05' \ 22''$, $B = 32° \ 26' \ 06''$.

4. Given $b = 61° \ 12'$, $c = 131° \ 44'$, and $A = 88° \ 40'$, to find B, C, and a. (See Note, Art. 83.)

Ans. $B = 66° \ 55' \ 59''$, $C = 128° \ 25' \ 05''$, $a = 72° \ 12' \ 46''$.

CASE IV.

Given two angles and their included side.

88. The solution of this case is entirely analogous to that of Case III.

Applying logarithms to proportions (12) and (13), Art. 83, and to proportion (11), Art. 83, we have

$$\log \tan \tfrac{1}{2}(a+b) = \text{(a. c.)} \log \cos \tfrac{1}{2}(A+B) + \log \cos \tfrac{1}{2}(A-B) + \log \tan \tfrac{1}{2}c - 10;$$

$$\log \tan \tfrac{1}{2}(a-b) = \text{(a. c.)} \log \sin \tfrac{1}{2}(A+B) + \log \sin \tfrac{1}{2}(A-B) + \log \tan \tfrac{1}{2}c - 10;$$

$$\log \cot \tfrac{1}{2}C = \text{(a. c.)} \log \sin \tfrac{1}{2}(a-b) + \log \sin \tfrac{1}{2}(a+b) + \log \tan \tfrac{1}{2}(A-B) - 10.$$

The application of these formulas is sufficient for the solution of all cases.

Examples.

1. Given $A = 81° \ 38' \ 20''$, $B = 70° \ 09' \ 38''$, and $c = 59° \ 16' \ 22''$, to find C, a, and b.

Ans. $C = 64° \ 46' \ 24''$, $a = 70° \ 04' \ 17''$, $b = 63° \ 21' \ 27''$.

2. Given $A = 34° 15' 03''$, $B = 42° 15' 13''$, and $c = 76° 35' 36''$, to find C, a, and b.

Ans. $C = 121° 36' 12''$, $a = 40° 0' 10''$, $b = 50° 10' 30''$.

3. Given $B = 82° 24'$, $C = 120° 38'$, and $a = 75° 19'$, to find A, b, and c.

Ans. $A = 73° 31' 13''$, $b = 90° 50' 50''$, $c = 119° 46' 22''$.

CASE V.

Given the three sides, to find the remaining parts.

89. The angles may be found by means of formula (3), Art. 81; or, one angle being found by that formula, the two others may be found by means of Napier's Analogies.

Examples.

1. Given $a = 74° 23'$, $b = 35° 46' 14''$, and $c = 100° 39'$, to find A, B, and C.

Applying logarithms to formula (3), Art. 81, we have

$$\log \cos \tfrac{1}{2}A = 10 + \tfrac{1}{2}[\log \sin \tfrac{1}{2}s + \log \sin(\tfrac{1}{2}s - a) + (\text{a. c.}) \log \sin b + (\text{a. c.}) \log \sin c - 20];$$

or,

$$\log \cos \tfrac{1}{2}A = \tfrac{1}{2}[\log \sin \tfrac{1}{2}s + \log \sin(\tfrac{1}{2}s - a) + (\text{a. c.}) \log \sin b + (\text{a. c.}) \log \sin c];$$

we have $\tfrac{1}{2}s = 105° 24' 07''$,

and $\tfrac{1}{2}s - a = 31° 01' 07''$.

$\log \sin \tfrac{1}{2}s$	$(105° 24' 07'')$	9.984116
$\log \sin (\tfrac{1}{2}s - a)$	$(31° 01' 07'')$	9.712074
(a. c.) $\log \sin b$	$(35° 46' 14'')$	0.233185
(a. c.) $\log \sin c$	$(100° 39')$	0.007546
	2)	19.936921
$\log \cos \tfrac{1}{2}A$		9.968460

$\therefore\ \tfrac{1}{2}A = 21° 34' 23''$, and $A = 43° 08' 46''$.

Using the same formula as before, and substituting B for A, b for a, and a for b, and recollecting that $\tfrac{1}{2}s - b = 69° 37' 53''$, we have

$\log \sin \tfrac{1}{2}s$	$(105° 24' 07'')$	9.984116
$\log \sin (\tfrac{1}{2}s - b)$	$(69° 37' 53'')$	9.971958
(a. c.) $\log \sin a$	$(74° 23')$	0.016336
(a. c.) $\log \sin c$	$(100° 39')$	0.007546
	2)	19.979956
$\log \cos \tfrac{1}{2}B$		9.989978

$\therefore\ \tfrac{1}{2}B = 12° 15' 43''$, and $B = 24° 31' 26''$.

Using the same formula, substituting C for A, c for a, and a for c, recollecting that $\tfrac{1}{2}s - c = 4° 45' 07''$, we have

$\log \sin \tfrac{1}{2}s$	$(105° 24' 07'')$	9.984116
$\log \sin (\tfrac{1}{2}s - c)$	$(4° 45' 07'')$	8.918250
(a. c.) $\log \sin a$	$(74° 23')$	0.016336
(a. c.) $\log \sin b$	$(25° 46' 14'')$	9.233185
	2)	19.151887
$\log \cos \tfrac{1}{2}C$		9.575943

$\therefore\ \tfrac{1}{2}C = 67° 52' 25''$, and $C = 135° 44' 50''$.

2. Given $a = 56° 40'$, $b = 83° 13'$, and $c = 114° 30'$, to find A, B, and C.

Ans. $A = 48° 31' 18''$, $B = 62° 55' 44''$, $C = 125° 18' 56''$.

3. Given $a = 115° 15'$, $b = 125° 30'$, and $c = 110° 15'$, to find A, B, and C.

Ans. $A = 145° 15' 04''$, $B = 149° 07' 52$, $C = 143° 45' 10''$.

CASE VI.

The three angles being given, to find the sides.

90. The solution in this case is entirely analogous to the preceding one.

Applying logarithms to formula (2), Art. 82, we have

$$\log \cos \tfrac{1}{2}a = \tfrac{1}{2}[\log \cos(\tfrac{1}{2}S - B) + \log \cos(\tfrac{1}{2}S - C) + (\text{a. c.}) \log \sin B + (\text{a. c.}) \log \sin C].$$

In the same manner as before, we change the letters, to suit each case.

Examples.

1. Given $A = 48° 30'$, $B = 125° 20'$, and $C = 62° 54'$, to find a, b, and c.

Ans. $a = 56° 39' 30''$, $b = 114° 29' 58''$, $c = 83° 12' 06''$.

2. Given $A = 109° 55' 42''$, $B = 116° 38' 33''$, and $C = 120° 43' 37''$, to find a, b, and c.

Ans. $a = 98° 21' 40''$, $b = 109° 50' 22''$, $c = 115° 13' 28''$.

3. Given $A = 160° 20'$, $B = 135° 15'$, and $C = 148° 25'$, to find a, b, and c.

Ans. $a = 155° 56' 10''$, $b = 58° 32' 12''$, $c = 140° 36' 48''$.

MENSURATION.

91. MENSURATION is that branch of Mathematics which treats of the measurement of Geometrical Magnitudes.

92. The measurement of a quantity is the operation of finding how many times it contains another quantity of the same kind, taken as a standard. This standard is called the *unit of measure.*

93. The unit of measure for surfaces is a *square*, one of whose sides is the linear unit. The unit of measure for volumes is a *cube*, one of whose edges is the linear unit.

If the linear unit is *one foot*, the superficial unit is *one square foot*, and the unit of volume is *one cubic foot.* If the linear unit is *one yard*, the superficial unit is *one square yard*, and the unit of volume is *one cubic yard.*

94. In Mensuration, the expression *product of two lines*, is used to denote the product obtained by multiplying the number of linear units in one line by the number of linear units in the other. The expression *product of three lines*, is used to denote the continued product of the number of linear units in each of the three lines.

Thus, when we say that the area of a parallelogram is equal to the product of its base and altitude, we mean that the number of superficial units in the parallelogram is equal to the number of linear units in the base, multiplied by the number of linear units in the altitude. In