3. Given $a = 115° 15'$, $b = 125° 30'$, and $c = 110° 15'$, to find A, B, and C.

Ans. $A = 145° 15' 04''$, $B = 149° 07' 52$, $C = 143° 45' 10''$.

CASE VI.

The three angles being given, to find the sides.

90. The solution in this case is entirely analogous to the preceding one.

Applying logarithms to formula (2), Art. 82, we have

$$\log\cos\tfrac{1}{2}a = \tfrac{1}{2}[\log\cos(\tfrac{1}{2}S - B) + \log\cos(\tfrac{1}{2}S - C) + (a.\,c.)\log\sin B + (a.\,c.)\log\sin C].$$

In the same manner as before, we change the letters, to suit each case.

Examples.

1. Given $A = 48° 30'$, $B = 125° 20'$, and $C = 62° 54'$, to find a, b, and c.

Ans. $a = 56° 39' 30''$, $b = 114° 29' 58''$, $c = 83° 12' 06''$.

2. Given $A = 109° 55' 42''$, $B = 116° 38' 33''$, and $C = 120° 43' 37''$, to find a, b, and c.

Ans. $a = 98° 21' 40''$, $b = 109° 50' 22''$, $c = 115° 13' 28''$.

3. Given $A = 160° 20'$, $B = 135° 15'$, and $C = 148° 25'$, to find a, b, and c.

Ans. $a = 155° 56' 10''$, $b = 58° 32' 12''$, $c = 140° 36' 48''$.

MENSURATION.

91. MENSURATION is that branch of Mathematics which treats of the measurement of Geometrical Magnitudes.

92. The measurement of a quantity is the operation of finding how many times it contains another quantity of the same kind, taken as a standard. This standard is called the *unit of measure.*

93. The unit of measure for surfaces is a *square*, one of whose sides is the linear unit. The unit of measure for volumes is a *cube*, one of whose edges is the linear unit.

If the linear unit is *one foot*, the superficial unit is *one square foot*, and the unit of volume is *one cubic foot.* If the linear unit is *one yard*, the superficial unit is *one square yard*, and the unit of volume is *one cubic yard.*

94. In Mensuration, the expression *product of two lines,* is used to denote the product obtained by multiplying the number of linear units in one line by the number of linear units in the other. The expression *product of three lines,* is used to denote the continued product of the number of linear units in each of the three lines.

Thus, when we say that the area of a parallelogram is equal to the product of its base and altitude, we mean that the number of superficial units in the parallelogram is equal to the number of linear units in the base, multiplied by the number of linear units in the altitude. In like manner, the number of units of volume, in a rectangular parallelopipedon, is equal to the number of superficial units in its base multiplied by the number of linear units in its altitude, and so on.

MENSURATION OF PLANE FIGURES.

To find the area of a parallelogram.

95. From the principle demonstrated in Book IV., Prop. V., we have the following

RULE.—*Multiply the base by the altitude; the product will be the area required.*

Examples.

1. Find the area of a parallelogram, whose base is 12.25, and whose altitude is 8.5. *Ans.* 104.125.

2. What is the area of a square, whose side is 204.3 feet? *Ans.* 41738.49 sq. ft.

3. How many square yards are there in a rectangle whose base is 66.3 feet, and altitude 33.3 feet? *Ans.* 245.31 sq. yds.

4. What is the area of a rectangular board, whose length is $12\frac{1}{2}$ feet, and breadth 9 inches? *Ans.* $9\frac{3}{8}$ sq. ft.

5. What is the number of square yards in a parallelogram, whose base is 37 feet, and altitude 5 feet 3 inches? *Ans.* $21\frac{7}{12}$.

To find the area of a plane triangle.

96. *First Case.* When the base and altitude are given.

From the principle demonstrated in Book IV., Prop. VI., we may write the following

RULE. — *Multiply the base by half the altitude; the product will be the area required.*

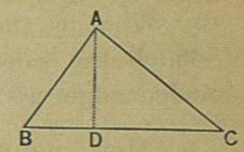
Examples.

1. Find the area of a triangle, whose base is 625, and altitude 520 feet. *Ans.* 162500 sq. ft.

2. Find the area of a triangle, in square yards, whose base is 40, and altitude 30 feet. *Ans.* $66\frac{2}{3}$.

3. Find the area of a triangle, in square yards, whose base is 49, and altitude $25\frac{1}{4}$ feet. *Ans.* 68.7361.

Second Case. When two sides and their included angle are given.



Let ABC represent a plane triangle, in which the side $AB = c$, $BC = a$, and the angle B, are given. From A draw AD perpendicular to BC; this will be the altitude of the triangle. From formula (1), Art. 37, Plane Trigonometry, we have

$$AD = c \sin B.$$

Denoting the area of the triangle by Q, and applying the rule last given, we have

$$Q = \frac{ac \sin B}{2}; \quad \text{or,} \quad 2Q = ac \sin B.$$

Substituting for $\sin B$, $\frac{\sin B}{R}$ (Trig., Art. 30), and applying logarithms, we have

$$\log (2Q) = \log a + \log c + \log \sin B - 10;$$

hence, we may write the following

RULE.—*Add together the logarithms of the two sides and the logarithmic sine of their included angle; from this sum subtract* 10; *the remainder will be the logarithm of double the area of the triangle. Find, from the table, the number corresponding to this logarithm, and divide it by* 2; *the quotient will be the required area.*

Examples.

1. What is the area of a triangle, in which two sides, a and b, are respectively equal to 125.81, and 57.65, and whose included angle C is 57° 25′?
Ans. $2Q = 6111.4$, and $Q = 3055.7$.

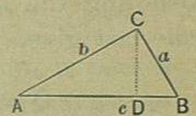
2. What is the area of a triangle, whose sides are 30 and 40, and their included angle 28° 57′?
Ans. 290.427.

3. What is the number of square yards in a triangle, of which the sides are 25 feet and 21.25 feet, and their included angle 45°?
Ans. 20.8694.

LEMMA.

To find half an angle, when the three sides of a plane triangle are given.

97. Let ABC be a plane triangle, the angles and sides being denoted as in the figure.



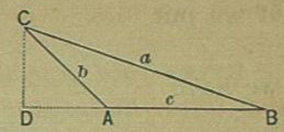
When the angle, A, is *acute*, we have (B. IV., P. XII.),

$$a^2 = b^2 + c^2 - 2c \cdot \text{AD}:$$

but (Art. 37), $\text{AD} = b \cos \text{A}$; hence,

$$a^2 = b^2 + c^2 - 2bc \cos \text{A}.$$

When the angle A is *obtuse*, we have (B. IV., P. XIII.),



$$a^2 = b^2 + c^2 + 2c \cdot \text{AD}:$$

but (Art. 37), $\text{AD} = b \cos \text{CAD}:$

but the angle CAD is the supplement of the angle A of the given triangle, and, therefore (Art. 63),

$$\cos \text{CAD} = -\cos \text{A};$$

hence, $\text{AD} = -b \cos \text{A},$

and, consequently, we have

$$a^2 = b^2 + c^2 - 2bc \cos \text{A}.$$

So that whether the angle, A, is acute or obtuse, we have

$$a^2 = b^2 + c^2 - 2bc \cos \text{A}; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1.)$$

whence, $$\cos \text{A} = \frac{b^2 + c^2 - a^2}{2bc}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2.)$$

If we add 1 to each member, and recollect that $1 + \cos \text{A} = 2\cos^2 \tfrac{1}{2}\text{A}$ (Art. 66) equation (4), we have

$$2\cos^2 \tfrac{1}{2}\text{A} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc}$$

$$= \frac{(b + c + a)\,(b + c - a)}{2bc};$$

or, $$\cos^2 \tfrac{1}{2}\text{A} = \frac{(b + c + a)\,(b + c - a)}{4bc}. \quad \cdot \quad \cdot \quad (3.)$$

If we put $b + c + a = s$,

we have $$\frac{b+c+a}{2} = \tfrac{1}{2}s,$$

and $$\frac{b+c-a}{2} = \tfrac{1}{2}s - a.$$

Substituting in (3), and extracting the square root,

$$\cos \tfrac{1}{2}\mathrm{A} = \sqrt{\frac{\frac{1}{2}s\,(\frac{1}{2}s - a)}{bc}}, \quad . \quad . \quad . \quad . \quad (4.)$$

the plus sign, only, being used, since $\frac{1}{2}\mathrm{A} < 90°$; hence, as A represents any angle,

The cosine of half of any angle of a plane triangle, is equal to the square root of the product of half the sum of the three sides, and half that sum minus *the side opposite the angle, divided by the rectangle of the adjacent sides.*

By applying logarithms, we have

$$\log \cos \tfrac{1}{2}\mathrm{A} = \tfrac{1}{2}\,[\log \tfrac{1}{2}s + \log(\tfrac{1}{2}s - a) + (\text{a. c.}) \log b + (\text{a. c.}) \log c]. \quad \cdot \quad (\text{A.})$$

If we subtract each member of equation (2) from 1, and recollect that $1 - \cos \mathrm{A} = 2 \sin^2 \frac{1}{2}\mathrm{A}$ (Art. 66), we have

$$2 \sin^2 \tfrac{1}{2}\mathrm{A} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a + b - c)\,(a - b + c)}{2bc}. \quad . \quad . \quad (5.)$$

Placing, as before, $a + b + c = s$,

we have $$\frac{a+b-c}{2} = \tfrac{1}{2}s - c,$$

and $$\frac{a-b+c}{2} = \tfrac{1}{2}s - b.$$

Substituting in (5) and reducing, we have

$$\sin \tfrac{1}{2}\mathrm{A} = \sqrt{\frac{(\frac{1}{2}s - b)\,(\frac{1}{2}s - c)}{bc}}; \quad . \quad . \quad . \quad (6.)$$

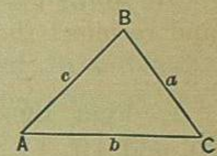
hence,

The sine of half an angle of a plane triangle, is equal to the square root of the product of half the sum of the three sides minus *one of the adjacent sides and half that sum* minus *the other adjacent side, divided by the rectangle of the adjacent sides.*

Applying logarithms, we have

$$\log \sin \tfrac{1}{2}\mathrm{A} = \tfrac{1}{2}\,[\log(\tfrac{1}{2}s - b) + \log(\tfrac{1}{2}s - c) + (\text{a. c.}) \log b + (\text{a. c.}) \log c]. \quad \cdot \quad (\text{B.})$$

Third Case. To find the area of a triangle when the three sides are given.



Let ABC represent a triangle whose sides a, b, and c are given. From the principle demonstrated in the last case, we have

$$\mathrm{Q} = \tfrac{1}{2}bc \sin \mathrm{A}.$$

But, from formula (A′), Trig., Art. 66, we have

$$\sin \mathrm{A} = 2 \sin \tfrac{1}{2}\mathrm{A} \cos \tfrac{1}{2}\mathrm{A};$$

whence, $Q = bc \sin \frac{1}{2}A \cos \frac{1}{2}A.$

Substituting for $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$, their values, taken from Lemma, and reducing, we have

$$Q = \sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s - a)\,(\tfrac{1}{2}s - b)\,(\tfrac{1}{2}s - c)};$$

hence, we may write the following

RULE.—*Find half the sum of the three sides, and from it subtract each side separately. Find the continued product of the half sum and the three remainders, and extract its square root; the result will be the area required.*

It is generally more convenient to employ logarithms; for this purpose, applying logarithms to the last equation, we have

$$\log Q = \tfrac{1}{2}[\log \tfrac{1}{2}s + \log(\tfrac{1}{2}s-a) + \log(\tfrac{1}{2}s-b) + \log(\tfrac{1}{2}s-c)];$$

hence, we have the following

RULE.—*Find the half sum and the three remainders as before, then find the half sum of their logarithms; the number corresponding to the resulting logarithm will be the area required.*

Examples.

1. Find the area of a triangle, whose sides are 20, 30, and 40.

We have $\frac{1}{2}s = 45$, $\frac{1}{2}s - a = 25$, $\frac{1}{2}s - b = 15$, $\frac{1}{2}s - c = 5$. By the first rule,

$$Q = \sqrt{45 \times 25 \times 15 \times 5} = 290.4737, \textit{ Ans.}$$

By the second rule,

$\log \frac{1}{2}s$	(45)	1.653213
$\log(\frac{1}{2}s - a)$	(25)	1.397940
$\log(\frac{1}{2}s - b)$	(15)	1.176091
$\log(\frac{1}{2}s - c)$	(5)	0.698970
	2)	4.926214
$\log Q$		2.463107

$\therefore\ Q = 290.4737$, *Ans.*

2. How many square yards are there in a triangle, whose sides are 30, 40, and 50 feet? *Ans.* $66\frac{2}{3}$.

To find the area of a trapezoid.

98. From the principle demonstrated in Book IV., Prop. VII., we may write the following

RULE.—*Find half the sum of the parallel sides, and multiply it by the altitude; the product will be the area required.*

Examples.

1. In a trapezoid the parallel sides are 750 and 1225, and the perpendicular distance between them is 1540; what is the area? *Ans.* 1520750.

2. How many square feet are contained in a plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches? *Ans.* $13\frac{13}{24}$.

3. How many square yards are there in a trapezoid, whose parallel sides are 240 feet, 320 feet, and altitude 66 feet? *Ans.* $2053\frac{1}{3}$ sq. yd.

To find the area of any quadrilateral.

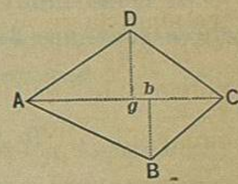
99. From what precedes, we deduce the following

RULE.—*Join the vertices of two opposite angles by a diagonal; from each of the other vertices let fall perpendiculars upon this diagonal; multiply the diagonal by half of the sum of the perpendiculars, and the product will be the area required.*

Examples.

1. What is the area of the quadrilateral ABCD, the diagonal AC being 42, and the perpendiculars Dg, Bb, equal to 18 and 16 feet?

Ans. 714 sq. ft.



2. How many square yards of paving are there in the quadrilateral, whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? *Ans.* $222\frac{1}{12}$.

To find the area of any polygon.

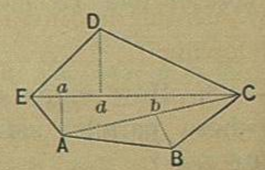
100. From what precedes, we have the following

RULE.—*Draw diagonals dividing the proposed polygon into trapezoids and triangles: then find the area of these figures separately, and add them together for the area of the whole polygon.*

Example.

1. Let it be required to determine the area of the polygon ABCDE, having five sides.

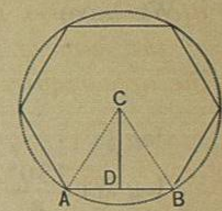
Let us suppose that we have measured the diagonals and perpendiculars,



and found AC = 36.21, EC = 39.11, Bb = 4, Dd = 7.26, Aa = 4.18: required the area. *Ans.* 296.1292.

To find the area of a regular polygon.

101. Let AB, denoted by s, represent one side of a regular polygon whose centre is C. Draw CA and CB, and from C draw CD perpendicular to AB. Then will CD be the apothem, and we shall have AD = BD.



Denote the number of sides of the polygon by n; then will the angle ACB, at the centre, be equal to $\frac{360°}{n}$ (B. V., page 144, D. 2), and the angle ACD, which is half of ACB, will be equal to $\frac{180°}{n}$.

In the right-angled triangle ADC, we shall have, formula (3), Art. 37, Trig.,

$$CD = \tfrac{1}{2}s \tan CAD.$$

But CAD, being the complement of ACD, we have

$$\tan CAD = \cot ACD;$$

hence,

$$CD = \tfrac{1}{2}s \cot \frac{180°}{n},$$

a formula by means of which the apothem may be computed.

But the area is equal to the perimeter multiplied by half the apothem (Book V., Prop. VIII.): hence the following

RULE.—*Find the apothem, by the preceding formula; multiply the perimeter by half the apothem; the product will be the area required.*

Examples.

1. What is the area of a regular hexagon, each of whose sides is 20?

We have $CD = 10 \times \cot 30°$;

or, $\log CD = \log 10 + \log \cot 30° - 10.$

$$\begin{array}{llll} \log \tfrac{1}{2}s & \cdot\ \cdot\ \cdot\ (10) & \cdot & 1.000000 \\ \log \cot \dfrac{180°}{n} & \cdot\ (30°) & \cdot & 10.238561 \\ \log CD & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \overline{1.238561} \quad \therefore\ CD = 17.3205. \end{array}$$

The perimeter is equal to 120: hence, denoting the area by Q,

$$Q = \frac{120 \times 17.3205}{2} = 1039.23, \textit{ Ans.}$$

2. What is the area of an octagon, one of whose sides is 20? *Ans.* 1931.37.

The areas of some of the most important of the regular polygons have been computed by the preceding method, on the supposition that each side is equal to 1, and the results are given in the following

TABLE.

NAMES.	SIDES.	AREAS.	NAMES.	SIDES.	AREAS.
Triangle,	3	0.4330127	Octagon,	8	4.8284271
Square,	4	1.0000000	Nonagon,	9	6.1818242
Pentagon,	5	1.7204774	Decagon,	10	7.6942088
Hexagon,	6	2.5980762	Undecagon,	11	9.3656399
Heptagon,	7	3.6339124	Dodecagon,	12	11.1961524

The areas of similar polygons are to each other as the squares of their homologous sides (Book IV., Prop. XXVII.).

Denoting the area of a regular polygon whose side is s by Q, and that of a similar polygon whose side is 1 by T, the tabular area, we have

$$Q : T :: s^2 : 1^2;$$

$$\therefore\ Q = Ts^2;$$

hence, the following

RULE.—*Multiply the corresponding tabular area by the square of the given side; the product will be the area required.*

Examples.

1. What is the area of a regular hexagon, each of whose sides is 20?

We have $T = 2.5980762$, and $s^2 = 400$: hence,

$$Q = 2.5980762 \times 400 = 1039.23048, \textit{ Ans.}$$

2. Find the area of a pentagon, whose side is 25. *Ans.* 1075.298375.

3. Find the area of a decagon, whose side is 20. *Ans.* 3077.68352.

To find the circumference of a circle, when the diameter is given.

102. From the principle demonstrated in Book V., Prop. XVI., we may write the following

RULE.—*Multiply the given diameter by* 3.1416; *the product will be the circumference required.*

Examples.

1. What is the circumference of a circle, whose diameter is 25? *Ans.* 78.54.

2. If the diameter of the earth is 7921 miles, what is the circumference? *Ans.* 24884.6136.

To find the diameter of a circle, when the circumference is given.

103. From the preceding case, we may write the following

RULE.—*Divide the given circumference by* 3.1416; *the quotient will be the diameter required.*

Examples.

1. What is the diameter of a circle, whose circumference is 11652.1944? *Ans.* 3709.

2. What is the diameter of a circle, whose circumference is 6850? *Ans.* 2180.41.

To find the length of an arc containing any number of degrees.

104. The length of an arc of 1°, in a circle whose diameter is 1, is equal to the circumference, or 3.1416, divided by 360; that is, it is equal to 0.0087266: hence, the length of an arc of n degrees will be $n \times 0.0087266$. To find the length of an arc containing n degrees, when the diameter is d, we employ the principle demonstrated in Book V., Prop. XIII., C. 2: hence, we may write the following

RULE.—*Multiply the number of degrees in the arc by* .0087266, *and the product by the diameter of the circle; the result will be the length required.*

Examples.

1. What is the length of an arc of 30 degrees, the diameter being 18 feet? *Ans.* 4.712364 ft.

2. What is the length of an arc of 12° 10′, or $12\frac{1}{6}$°, the diameter being 20 feet? *Ans.* 2.123472 ft.

To find the area of a circle.

105. From the principle demonstrated in Book V., Prop. XV., we may write the following

RULE.—*Multiply the square of the radius by* 3.1416; *the product will be the area required;*

Examples.

1. Find the area of a circle, whose diameter is 10 and circumference 31.416. *Ans.* 78.54.

2. How many square yards in a circle whose diameter is $3\frac{1}{2}$ feet? *Ans.* 1.069016.

3. What is the area of a circle whose circumference is 12 feet? *Ans.* 11.4595.

To find the area of a circular sector.

106. From the principle demonstrated in Book V., Prop. XIV., C. 1 and 2, we may write the following

RULE.—I. *Multiply half the length of the arc by the radius;* or,

II. *Find the area of the whole circle, by the last rule; then write the proportion, 360 is to the number of degrees in the arc of the sector, as the area of the circle is to the area of the sector.*

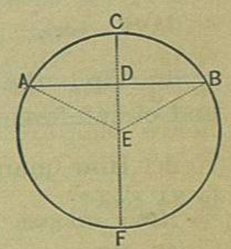
Examples.

1. Find the area of a circular sector, whose arc contains 18°, the diameter of the circle being 3 feet. *Ans.* 0.35343 sq. ft.

2. Find the area of a sector, whose arc is 20 feet, the radius being 10. *Ans.* 100.

3. Required the area of a sector, whose arc is 147° 29′ and radius 25 feet. *Ans.* 804.3986 sq. ft.

To find the area of a circular segment.



107. Let AB represent the chord corresponding to the two segments ACB and AFB. Draw AE and BE. The segment ACB is equal to the sector EACB, *minus* the triangle AEB. The segment AFB is equal to the sector EAFB, *plus* the triangle AEB. Hence, we have the following

Rule.—*Find the area of the corresponding sector, and also of the triangle formed by the chord of the segment and the two extreme radii of the sector; subtract the latter from the former when the segment is less than a semicircle, and add the latter to the former when the segment is greater than a semicircle; the result will be the area required.*

Examples.

1. Find the area of a segment, whose chord is 12 and whose radius is 10.

Solving the triangle AEB, we find the angle AEB is equal to 73° 44′, the area of the sector EACB equal to 64.35, and the area of the triangle AEB equal to 48; hence, the segment ACB is equal to 16.35.

2. Find the area of a segment, whose height is 18, the diameter of the circle being 50. *Ans.* 636.4834.

3. Required the area of a segment, whose chord is 16, the diameter being 20. *Ans.* 44.764.

To find the area of a circular ring contained between the circumferences of two concentric circles.

108. Let R and r denote the radii of the two circles, R being greater than r. The area of the outer circle is $R^2 \times 3.1416$, and that of the inner circle is $r^2 \times 3.1416$; hence, the area of the ring is equal to $(R^2 - r^2) \times 3.1416$. Hence, the following

Rule.—*Find the difference of the squares of the radii of the two circles, and multiply it by* 3.1416; *the product will be the area required.*

Examples.

1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences. *Ans.* 50.2656.

2. What is the area of the ring, when the diameters of the circles are 10 and 20? *Ans.* 235.62.

MENSURATION OF BROKEN AND CURVED SURFACES.

To find the area of the entire surface of a right prism.

109. From the principle demonstrated in Book VII., Prop. I., we may write the following

RULE.—*Multiply the perimeter of the base by the altitude, the product will be the area of the convex surface; to this add the areas of the two bases; the result will be the area required.*

Examples.

1. Find the surface of a cube, the length of each side being 20 feet. *Ans.* 2400 sq. ft.

2. Find the whole surface of a triangular prism, whose base is an equilateral triangle having each of its sides equal to 18 inches, and altitude 20 feet.
Ans. 91.949 sq. ft.

To find the area of the entire surface of a right pyramid.

110. From the principle demonstrated in Book VII., Prop. IV., we may write the following

RULE.—*Multiply the perimeter of the base by half the slant height; the product will be the area of the convex surface; to this add the area of the base; the result will be the area required.*

Examples.

1. Find the convex surface of a right triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet. *Ans.* 90 sq. ft.

2. What is the entire surface of a right pyramid, whose slant height is 27 feet, and the base a pentagon of which each side is 25 feet? *Ans.* 2762.798 sq. ft.

To find the area of the convex surface of a frustum of a right pyramid.

111. From the principle demonstrated in Book VII., Prop. IV., S., we may write the following

RULE.—*Multiply the half sum of the perimeters of the two bases by the slant height; the product will be the area required.*

Examples.

1. How many square feet are there in the convex surface of the frustum of a square pyramid, whose slant height is 10 feet, each side of the lower base 3 feet 4 inches, and each side of the upper base 2 feet 2 inches?
Ans. 110 sq. ft.

2. What is the convex surface of the frustum of a heptagonal pyramid, whose slant height is 55 feet, each side of the lower base 8 feet, and each side of the upper base 4 feet? *Ans.* 2310 sq. ft.

112. Since a cylinder may be regarded as a prism whose base has an infinite number of sides, and a cone as a pyramid whose base has an infinite number of sides, the rules just given may be applied to find the areas of the surfaces of right cylinders, cones, and frustums of cones, by simply changing the term *perimeter* to circumference.

Examples.

1. What is the convex surface of a cylinder, the diameter of whose base is 20, and whose altitude 50?
Ans. 3141.6.

2. What is the entire surface of a cylinder, the altitude being 20, and diameter of the base 2 feet?
Ans. 131.9472 sq. ft.

3. Required the convex surface of a cone, whose slant height is 50 feet, and the diameter of its base $8\frac{1}{2}$ feet.
Ans. 667.59 sq. ft.

4. Required the entire surface of a cone, whose slant height is 36, and the diameter of its base 18 feet.
Ans. 1272.348 sq. ft.

5. Find the convex surface of the frustum of a cone, the slant height of the frustum being $12\frac{1}{2}$ feet, and the circumferences of the bases 8.4 feet and 6 feet.
Ans. 90 sq. ft.

6. Find the entire surface of the frustum of a cone, the slant height being 16 feet, and the radii of the bases 3 feet and 2 feet. *Ans.* 292.1688 sq. ft.

To find the area of the surface of a sphere.

113. From the principle demonstrated in Book VIII., Prop. X., C. 1, we may write the following

RULE.—*Find the area of one of its great circles, and multiply it by 4; the product will be the area required.*

Examples.

1. What is the area of the surface of a sphere, whose radius is 16? *Ans.* 3216.9984.

2. What is the area of the surface of a sphere, whose radius is 27.25? *Ans.* 9331.3374.

To find the area of a zone.

114. From the principle demonstrated in Book VIII., Prop. X., C. 2, we may write the following

RULE.—*Find the circumference of a great circle of the sphere, and multiply it by the altitude of the zone; the product will be the area required.*

Examples.

1. The diameter of a sphere being 42 inches, what is the area of the surface of a zone whose altitude is 9 inches?
Ans. 1187.5248 sq. in.

2. If the diameter of a sphere is $12\frac{1}{2}$ feet, what will be the surface of a zone whose altitude is 2 feet?
Ans. 78.54 sq. ft.

To find the area of a spherical polygon.

115. From the principle demonstrated in Book IX., Prop. XIX., we may write the following

RULE.—*From the sum of the angles of the polygon, subtract* 180° *taken as many times, less two, as the polygon has sides, and divide the remainder by* 90°; *the quotient will be the spherical excess. Find the area of a great circle of the sphere, and divide it by* 2; *the quotient will be the area of a tri-rectangular triangle. Multiply the area of the tri-rectangular triangle by the spherical excess, and the product will be the area required.*

This rule applies to the spherical triangle, as well as to any other spherical polygon.

Examples.

1. Required the area of a triangle, described on a sphere whose diameter is 30 feet, the angles being 140°, 92°, and 68°. *Ans.* 471.24 sq. ft.

2. What is the area of a polygon of seven sides, described on a sphere whose diameter is 17 feet, the sum of the angles being 1080°? *Ans.* 226.98.

3. What is the area of a regular polygon of eight sides, described on a sphere whose diameter is 30 yards, each angle of the polygon being 140°? *Ans.* 157.08 sq. yds.

MENSURATION OF VOLUMES.

To find the volume of a prism.

116. From the principle demonstrated in Book VII., Prop. XIV., we may write the following

RULE.—*Multiply the area of the base by the altitude; the product will be the volume required.*

Examples.

1. What is the volume of a cube, whose side is 24 inches? *Ans.* 13824 cu. in.

2. How many cubic feet in a block of marble, of which the length is 3 feet 2 inches, breadth 2 feet 8 inches, and height or thickness 2 feet 6 inches? *Ans.* $21\frac{1}{9}$ cu. ft.

3. Required the volume of a triangular prism, whose height is 10 feet, and the three sides of its triangular base 3, 4, and 5 feet. *Ans.* 60.

To find the volume of a pyramid.

117. From the principle demonstrated in Book VII., Prop. XVII., we may write the following

RULE.—*Multiply the area of the base by one third of the altitude; the product will be the volume required.*

Examples.

1. Required the volume of a square pyramid, each side of its base being 30, and the altitude 25. *Ans.* 7500.

2. Find the volume of a triangular pyramid, whose altitude is 30, and each side of the base 3 feet. *Ans.* 38.9711 cu. ft.

3. What is the volume of a pentagonal pyramid, its altitude being 12 feet, and each side of its base 2 feet? *Ans.* 27.5276 cu. ft.

4. What is the volume of a hexagonal pyramid, whose altitude is 6.4 feet, and each side of its base 6 inches? *Ans.* 1.38564 cu. ft.

To find the volume of a frustum of a pyramid.

118. From the principle demonstrated in Book VII., Prop. XVIII., C., we may write the following

RULE.—*Find the sum of the upper base, the lower base, and a mean proportional between them; multiply the result by one third of the altitude; the product will be the volume required.*