This rule applies to the spherical triangle, as well as to any other spherical polygon.

Examples.

1. Required the area of a triangle, described on a sphere whose diameter is 30 feet, the angles being 140°, 92°, and 68°. *Ans.* 471.24 sq. ft.

2. What is the area of a polygon of seven sides, described on a sphere whose diameter is 17 feet, the sum of the angles being 1080°? *Ans.* 226.98.

3. What is the area of a regular polygon of eight sides, described on a sphere whose diameter is 30 yards, each angle of the polygon being 140°? *Ans.* 157.08 sq. yds.

MENSURATION OF VOLUMES.

To find the volume of a prism.

116. From the principle demonstrated in Book VII., Prop. XIV., we may write the following

RULE.—*Multiply the area of the base by the altitude; the product will be the volume required.*

Examples.

1. What is the volume of a cube, whose side is 24 inches? *Ans.* 13824 cu. in.

2. How many cubic feet in a block of marble, of which the length is 3 feet 2 inches, breadth 2 feet 8 inches, and height or thickness 2 feet 6 inches? *Ans.* $21\frac{1}{9}$ cu. ft.

3. Required the volume of a triangular prism, whose height is 10 feet, and the three sides of its triangular base 3, 4, and 5 feet. *Ans.* 60.

To find the volume of a pyramid.

117. From the principle demonstrated in Book VII., Prop. XVII., we may write the following

RULE.—*Multiply the area of the base by one third of the altitude; the product will be the volume required.*

Examples.

1. Required the volume of a square pyramid, each side of its base being 30, and the altitude 25. *Ans.* 7500.

2. Find the volume of a triangular pyramid, whose altitude is 30, and each side of the base 3 feet. *Ans.* 38.9711 cu. ft.

3. What is the volume of a pentagonal pyramid, its altitude being 12 feet, and each side of its base 2 feet? *Ans.* 27.5276 cu. ft.

4. What is the volume of a hexagonal pyramid, whose altitude is 6.4 feet, and each side of its base 6 inches? *Ans.* 1.38564 cu. ft.

To find the volume of a frustum of a pyramid.

118. From the principle demonstrated in Book VII., Prop. XVIII., C., we may write the following

RULE.—*Find the sum of the upper base, the lower base, and a mean proportional between them; multiply the result by one third of the altitude; the product will be the volume required.*

Examples.

1. Find the number of cubic feet in a piece of timber, whose bases are squares, each side of the lower base being 15 inches, and each side of the upper base 6 inches, the altitude being 24 feet. *Ans.* 19.5.

2. Required the volume of a pentagonal frustum, whose altitude is 5 feet, each side of the lower base 18 inches, and each side of the upper base 6 inches. *Ans.* 9.31925 cu. ft.

119. Since cylinders and cones are limiting cases of prisms and pyramids, the three preceding rules are equally applicable to them.

Examples.

1. Required the volume of a cylinder whose altitude is 12 feet, and the diameter of its base 15 feet. *Ans.* 2120.58 cu. ft.

2. Required the volume of a cylinder whose altitude is 20 feet, and the circumference of whose base is 5 feet 6 inches. *Ans.* 48.144 cu. ft.

3. Required the volume of a cone whose altitude is 27 feet, and the diameter of the base 10 feet. *Ans.* 706.86 cu. ft.

4. Required the volume of a cone whose altitude is $10\frac{1}{2}$ feet, and the circumference of its base 9 feet. *Ans.* 22.56 cu. ft.

5. Find the volume of the frustum of a cone, the altitude being 18, the diameter of the lower base 8, and that of the upper base 4. *Ans.* 527.7888.

6. What is the volume of the frustum of a cone, the altitude being 25, the circumference of the lower base 20, and that of the upper base 10? *Ans.* 464.216.

7. If a cask, which is composed of two equal conic frustums joined together at their larger bases, have its bung diameter 28 inches, the head diameter 20 inches, and the length 40 inches, how many gallons of wine will it contain, there being 231 cubic inches in a gallon? *Ans.* 79.0613.

To find the volume of a sphere.

120. From the principle demonstrated in Book VIII., Prop. XIV., we may write the following

RULE.—*Cube the diameter of the sphere, and multiply the result by* $\frac{1}{6}\pi$, *that is, by* 0.5236; *the product will be the volume required.*

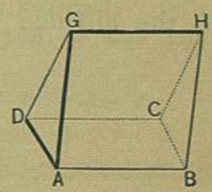
Examples.

1. What is the volume of a sphere, whose diameter is 12? *Ans.* 904.7808.

2. What is the volume of the earth, if the mean diameter is taken equal to 7918.7 miles? *Ans.* 259992792082 cu. miles.

To find the volume of a wedge.

121. A WEDGE is a volume bounded by a rectangle ABCD, called the *back*, two trapezoids ABHG, DCHG, called *faces*, and two triangles ADG, CBH, called *ends*. The line GH, in which the faces meet, is called the *edge*.



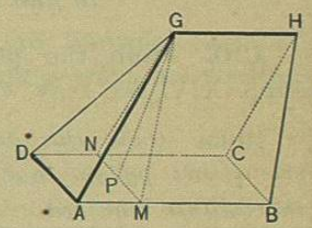
There are three cases;

1st, When the length of the edge is equal to the length of the back;

2d, When it is less; and

3d, When it is greater.

In the first case, the wedge is equal in volume to a right prism, whose base is the triangle ADG, and altitude GH or AB: hence, its volume is equal to ADG multiplied by AB.



In the second case, through H, a point of the edge, pass a plane HCB perpendicular to the back, and intersecting it in the line BC parallel to AD. This plane will divide the wedge into two parts, one of which is represented by the figure.

Through G, draw the plane GNM parallel to HCB, and it will divide the part of the wedge represented by the figure into the right triangular prism GNM–B, and the quadrangular pyramid ADNM–G. Draw GP perpendicular to NM: it will also be perpendicular to the back of the wedge (B. VI., P. XVII.), and hence, will be equal to the altitude of the wedge.

Denote AB by L, the breadth AD by b, the edge GH by l, the altitude by h, and the volume by V; then,

$$AM = L - l,$$

$$MB = GH = l,$$

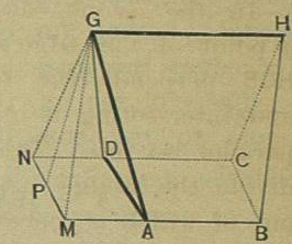
and $$\text{area } NGM = \tfrac{1}{2}bh:$$

then $$\text{Prism} = \tfrac{1}{2}bhl;$$

$$\text{Pyramid} = b(L - l)\,\tfrac{1}{3}h = \tfrac{1}{3}bh(L - l),$$

and
$$\begin{aligned} V &= \tfrac{1}{2}bhl + \tfrac{1}{3}bh(L - l) \\ &= \tfrac{1}{2}bhl + \tfrac{1}{3}bhL - \tfrac{1}{3}bhl \\ &= \tfrac{1}{6}bh(l + 2L). \end{aligned}$$

We can find a similar expression for the remaining part of the wedge, and by adding, the factor within the parenthesis becomes the entire length of the edge plus twice the length of the back.



In the third case, l is greater than L; the volume of each part is equal to *the difference* of the prism and pyramid, and is of the same form as before. Hence, in either case, we have the following

RULE.—*Add twice the length of the back to the length of the edge; multiply the sum by the breadth of the back, and that result by one sixth of the altitude; the final product will be the volume required.*

Examples.

1. If the back of a wedge is 40 by 20 feet, the edge 35 feet, and the altitude 10 feet, what is the volume?

Ans. 3833.33 cu. ft.

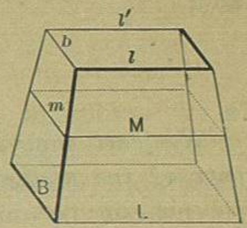
2. What is the volume of a wedge, whose back is 18 feet by 9, edge 20 feet, and altitude 6 feet?

Ans. 504 cu. ft.

To find the volume of a prismoid.

122. A PRISMOID is a frustum of a wedge.

Let L and B denote the length and breadth of the lower base, l and b the length and breadth of the upper base, M and m the length and breadth of the section equidistant from the bases, and h the altitude of the prismoid.



Through the edges L and l', let a plane be passed, and it will divide the prismoid into two wedges, having for bases the bases of the prismoid, and for edges the lines L and l'.

The volume of the prismoid, denoted by V, will be equal to the sum of the volumes of the two wedges; hence,

$$V = \tfrac{1}{6}Bh(l + 2L) + \tfrac{1}{6}bh(L + 2l);$$

or,

$$V = \tfrac{1}{6}h(2BL + 2bl + Bl + bL);$$

which may be written under the form,

$$V = \tfrac{1}{6}h[(BL + bl + Bl + bL) + BL + bl]. \quad \cdot \quad (A.)$$

Because the auxiliary section is midway between the bases, we have

$$2M = L + l, \quad \text{and} \quad 2m = B + b;$$

hence, $4Mm = (L + l)(B + b) = BL + Bl + bL + bl.$

Substituting in (A), we have

$$V = \tfrac{1}{6}h(BL + bl + 4Mm).$$

But BL is the area of the lower base, or lower section, bl is the area of the upper base, or upper section, and Mm is the area of the middle section; hence, the following

RULE.—*To find the volume of a prismoid, find the sum of the areas of the extreme sections and four times the middle section; multiply the result by one sixth of the distance between the extreme sections; the result will be the volume required.*

This rule is used in computing volumes of earth-work in railroad cutting and embankment, and is of very extensive application. It may be shown that the same rule holds for every one of the volumes heretofore discussed in this work. Thus, in a pyramid, we may regard the base as one extreme section, and the vertex (whose area is 0), as the other extreme; their sum is equal to the area of the base. The area of a section midway between them is equal to one fourth of the base: hence, four times the middle section is equal to the base. Multiplying the sum of these by one sixth of the altitude, gives the same result as that already found. The application of the rule to the case of cylinders, frustums of cones, spheres, &c., is left as an exercise for the student.

Examples.

1. One of the bases of a rectangular prismoid is 25 feet by 20, the other 15 feet by 10, and the altitude 12 feet: required the volume. *Ans.* 3700 cu. ft.

2. What is the volume of a stick of hewn timber, whose ends are 30 inches by 27, and 24 inches by 18, its length being 24 feet? *Ans.* 102 cu. ft.