MENSURATION OF REGULAR POLYEDRONS.

123. A REGULAR POLYEDRON is a polyedron bounded by equal regular polygons.

The polyedral angles of any regular polyedron are all equal.

124. There are five regular polyedrons (Book VII., page 219).

To find the diedral angle contained between two consecutive faces of a regular polyedron.

125. As in the figure, let the vertex, O, of a polyedral angle of a tetraedron be taken as the centre of a sphere whose radius is 1: then will the three faces of this polyedral angle, by their intersections with the surface of the sphere, determine the spherical triangle FAB. The plane angles FOA, FOB, and AOB, being equal to each other, the arcs FA, FB, and AB, which measure these angles, are also equal to each other, and the spherical triangle FAB is equilateral. The angle FAB of the triangle is equal to the diedral angle of the planes FOA and AOB, that is, to the diedral angle between the faces of the tetraedron.
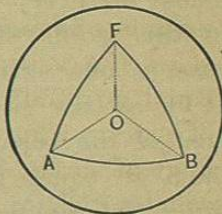
In like manner, if the vertex of a polyedral angle of any one of the regular polyedrons be taken as the centre of a sphere whose radius is 1, the faces of this polyedral angle will, by their intersections with the surface of the sphere, determine a regular spherical polygon; the *number of sides* of this spherical polygon will be equal to the

number of faces of the polyedral angle; *each side* of the polygon will be the measure of one of the plane angles formed by the edges of the polyedral angle; and *each angle* of the polygon will be equal to the diedral angle contained between two consecutive faces of the regular polyedron.

To find the required diedral angle, therefore, it only remains to deduce a formula for finding one angle of a regular spherical polygon when the sides are given.

Let ABCDE represent a regular spherical polygon, and let P be the pole of a small circle passing through its vertices. Suppose P to be connected with each of the vertices by arcs of great circles; there will thus be formed as many equal isosceles triangles as the polygon has sides, the vertical angle in each being equal to 360° divided by the number of sides. Through P draw the arc of a great circle, PQ, perpendicular to AB: then will AQ be equal to BQ, and the angle APQ to the angle QPB (B. IX., P. XI., C.). If we denote the number of sides of the spherical polygon by n', the angle APQ will be equal to $\frac{360°}{2n'}$, or $\frac{180°}{n'}$.
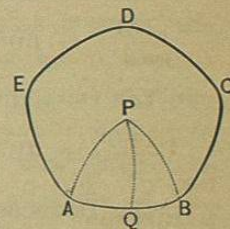
In the right-angled spherical triangle AQP, we know the base AQ, and the vertical angle APQ; hence, by Napier's rules for circular parts, we have

$$\sin (90° - APQ) = \cos (90° - PAQ) \cos AQ,$$

or, $$\cos APQ = \sin PAQ \cos AQ;$$

denoting the side AB of the polygon by s', and the angle PAQ, which is half the angle EAB of the polygon, by $\frac{1}{2}A$, we have

$$\cos \frac{180°}{n'} = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}s';$$

whence, $$\sin \tfrac{1}{2}A = \frac{\cos \dfrac{180°}{n'}}{\cos \tfrac{1}{2}s'}.$$

Examples.

In the Tetraedron,

$$\frac{180°}{n'} = 60°, \text{ and } \tfrac{1}{2}s' = 30°; \therefore A = 70° \; 31' \; 42''.$$

In the Hexaedron,

$$\frac{180°}{n'} = 60°, \text{ and } \tfrac{1}{2}s' = 45°; \therefore A = 90°.$$

In the Octaedron,

$$\frac{180°}{n'} = 45°, \text{ and } \tfrac{1}{2}s' = 30°; \therefore A = 109° \; 28' \; 19''.$$

In the Dodecaedron,

$$\frac{180°}{n'} = 60°, \text{ and } \tfrac{1}{2}s' = 54°; \therefore A = 116° \; 63' \; 54''.$$

In the Icosaedron,

$$\frac{180°}{n'} = 36°, \text{ and } \tfrac{1}{2}s' = 30°; \therefore A = 138° \; 11' \; 23''.$$

To find the volume of a regular polyedron.

126. If planes be passed through the centre of the polyedron and each of the edges, they will divide the polyedron into as many equal right pyramids as the polyedron has faces. The common vertex of these pyramids will be at the centre of the polyedron, their bases will be the faces of the polyedron, and their lateral faces will bisect the diedral angles of the polyedron. The volume of each pyramid will be equal to the product of its base and one third of its altitude, and this product multiplied

by the number of faces, will be the volume of the polyedron.

It only remains to deduce a formula for finding the altitude of the several pyramids, i. e., the distance from the centre to one face of the polyedron.

Conceive a perpendicular OC to be drawn from O, the centre of the polyedron, to one face; the foot of this perpendicular will be the centre of the face. From C, the foot of this perpendicular, draw a perpendicular to one side of the face in which it lies, and connect the point D with the centre of the polyedron. There will thus be formed a right-angled triangle, OCD, whose base, CD, is the apothem of the face, whose angle ODC is half the angle CDL contained between two consecutive faces of the polyedron, and whose altitude OC is the required altitude of the pyramid, or, in other words, the radius of the inscribed sphere. This will be true for any one of the regular polyedrons—the hexaedron is taken here for simplicity of illustration.
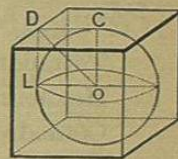
Denote the line CD by p, the angle ODC by $\tfrac{1}{2}A$, and the perpendicular OC by R. p may be found by the formula, given in Art. 101, for finding the apothem of a regular polygon; $\tfrac{1}{2}A$ may be found from the formula for sin $\tfrac{1}{2}A$, given in Art. 125; then, in the right-angled triangle OCD, we have, formula (3), Art. 37,

$$R = p \tan \tfrac{1}{2}A.$$

Compute the area of one of the faces of the given polyedron and multiply it by $\tfrac{1}{3}R$, as determined by the formula just given, and multiply the result thus obtained by the number of faces of the polyedron; the final product will be the volume of the given regular polyedron.

The volumes of all the regular polyedrons have been computed on the supposition that their edges are each equal to 1, and the results are given in the following

TABLE.

NAMES.	NO. OF FACES.	VOLUMES.
Tetraedron,	4	0.1178513
Hexaedron,	6	1.0000000
Octaedron,	8	0.4714045
Dodecaedron,	12	7.6631189
Icosaedron,	20	2.1816950

From the principles demonstrated in Book VII., we may write the following

RULE.—*To find the volume of any regular polyedron, multiply the cube of its edge by the corresponding tabular volume; the product will be the volume required.*

Examples.

1. What is the volume of a tetraedron, whose edge is 15?   *Ans.* 397.75.

2. What is the volume of a hexaedron, whose edge is 12?   *Ans.* 1728.

3. What is the volume of an octaedron, whose edge is 20?   *Ans.* 3771.236.

4. What is the volume of a dodecaedron, whose edge is 25?   *Ans.* 119736.2328.

5. What is the volume of an icosaedron, whose edge is 20?   *Ans.* 17453.56.

A TABLE

OF

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806181	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

REMARKS. In the following table, in the nine right-hand columns of each page, where the first or leading figures change from 9's to 0's, points or dots are introduced instead of the 0's, to catch the eye, and to indicate that from thence the two figures of the Logarithm to be taken from the second column, stand in the next line below.