La geometría analítica es la rama de las matemáticas que se ocupa de resolver los problemas de geometría por medio del cálculo.
(análisis algebraico)

PART I.

ANALYTIC GEOMETRY OF TWO DIMENSIONS.

CHAPTER I.

THE POINT.

1. Analytic Geometry is that branch of Mathematics in which the magnitudes considered are represented by letters, and the properties and relations of these magnitudes are investigated by the aid of Algebraic Analysis.

2. All the points of the magnitudes to be considered are referred to fixed objects, by means of elements called co-ordinates, and hence this method is sometimes known as **Co-ordinate Geometry.** It was introduced by Descartes in 1637, and hence is also called the **Cartesian System.**

3. Analytic Geometry is divided into two parts: **Analytic Geometry of two Dimensions,** which treats of lines lying wholly in a single plane, and requires but two co-ordinates to determine the position of a point; and **Analytic Geometry of three Dimensions,** which treats of lines and surfaces lying in any manner in space, and requires three co-ordinates to determine the position of a point.

4. There are two systems of co-ordinates in common use for determining the position of a point in a plane. The first is by means of its distances from any two given right lines of the plane which intersect each other. The second is by means of its distance and direction from a given point in the plane. The first is called the **Rectilinear System**, and the second is called the **Polar System.**

5. Let us suppose that we have given the position of two fixed right lines XX′, YY′, intersecting in the point O, and let the plane of the two lines be represented by the surface of the paper. Now, if through any point P we draw PM parallel to OY, and PN parallel to OX, it is plain that the position of P is known if the lengths of PM and PN are known. For example, if we have given PN $= a$, PM $= b$, we can determine the position of the point P with regard to the lines OX and OY: we need only measure OM ($= a$) along OX, and ON ($= b$) along OY, and draw the parallels PM, PN: P will be the point whose position we wished to determine.

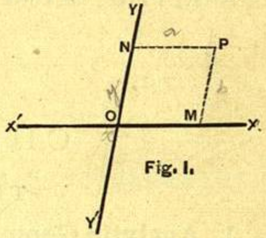

Fig. I.

6. The line PM, or its equal ON, is usually denoted by the letter y, and is called the **Ordinate** of the point P. OM, or its equal NP, is denoted by the letter x, and is called the **Abscissa** of the same point; and the two lines, when spoken of together, are called the **Co-ordinates** of P.

The lines XX′ and YY′ are called the **Axes of Co-ordinates**, or the **Co-ordinate Axes**, and the point O in which they intersect is called the **Origin.** The line XX′ is called the **Axis of Abscissas**, or the **Axis of x.** It may have any direction, but it is usually assumed to be horizontal. The line YY′ is called the **Axis of Ordinates**, or the **Axis of y.** The axes are said to be rectangular or oblique, according as the angle at which they intersect is a right or an oblique angle. The rectangular axes are the

most simple, and, in this work, will always be employed, unless otherwise specified.

The abscissa of any point is its distance from the axis of ordinates, measured on a line parallel to the axis of abscissas.

The ordinate of any point is its distance from the axis of abscissas, measured on a line parallel to the axis of ordinates.

The point P is said to be determined when the values of its co-ordinates, x and y, are given, as by the two equations $x = a$, $y = b$. For example, if we have given that $x = 5$ feet, $y = 3$ feet, we shall determine the position of the point of which x and y are the co-ordinates, by measuring, from the origin O, on the axis of x, a distance OM equal to 5 feet; then through M draw a line parallel to the axis of y, and on this line measure a distance MP equal to 3 feet. P will be the position of the point required.

Hence, in order to determine the position of a point, it is sufficient to have the two equations, $x = a$, $y = b$, in which a and b are given. These equations are the analytic representatives of the point, and are called the **Equations of a Point.**

It will easily be seen that the equations of the point M, in the preceding figure, are $x = a$, $y = 0$; that those of the point N are $x = 0$, $y = b$; and of the origin itself are $x = 0$, $y = 0$. The point whose position is defined by the equations $x = a$, $y = b$, is commonly spoken of as the point (a, b).

7. In order that the equations $x = a$, $y = b$, should be satisfied by only one point, it is necessary to pay attention, not only to the absolute values of the co-ordinates, but also to the signs of these quantities.

If no attention were given to the signs of the co-ordinates, we might measure OM $= a$, and ON $= b$ (Fig. 2), on either side of the origin, and any of the four points, P, P′, P″, P‴, would satisfy the equations $x = a$, $y = b$. This ambiguity, however, may be avoided by distinguishing algebraically between the lines OM and OM′, by giving them different signs. If lines measured in one direction be considered as positive, lines measured in the opposite direction must be considered negative. It is, of course, arbitrary in which direction we measure positive lines; but it is customary to regard OM measured towards the right, and ON measured upwards, as positive; and hence, OM′ and ON′, measured in the opposite directions, must be considered negative, as in Trigonometry.

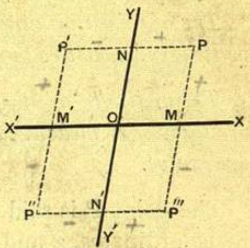

Fig. 2.

The four angles into which the plane is divided by the axes are distinguished thus: The angle YOX is called the **First Angle**; YOX′, the **Second Angle**; Y′OX′, the **Third Angle**; and Y′OX the **Fourth Angle**. If P, P′, P″, P‴ be points situated in the four angles, they will be represented by the following equations:

$$\text{P}\begin{cases} x = a, \\ y = b. \end{cases} \qquad \text{P}''\begin{cases} x = -a, \\ y = -b. \end{cases}$$

$$\text{P}'\begin{cases} x = -a, \\ y = b. \end{cases} \qquad \text{P}'''\begin{cases} x = a, \\ y = -b. \end{cases}$$

Or, by (a, b), $(-a, b)$, $(-a, -b)$, $(a, -b)$ respectively.

8. *To determine a point whose co-ordinates are given.*

Lay off from the origin, on the axis of x, a distance equal to the given abscissa, to the right if the abscissa is $+$, and to the left if it is $-$. Through the point thus found, draw a line parallel to the axis of y, and lay off on it a distance from the axis of x equal to the given ordinate, above if the

EXAMPLES.

1. Find the distance from the point $(-8, -2)$ to the point $(3, 7)$.

Ans. $d = \sqrt{(3+8)^2+(7+2)^2} = \sqrt{121+81} = 14.21.$

2. Find the distance between the two points $(2, -3)$ and $(-5, 6)$, the axes being inclined at an angle of 60°.

Here $x' - x'' = 2 + 5 = 7$; $y' - y'' = -3 - 6 = -9$; and $\cos\omega = \frac{1}{2}$. Hence, in (2) we get

$$d = \sqrt{49+81-2\cdot7\cdot9\cdot\tfrac{1}{2}} = \sqrt{49+81-63} = \sqrt{67}.$$

3. Find the lengths of the sides of a triangle, the co-ordinates of whose vertices are $(2, 3)$, $(4, -5)$, $(-3, -6)$.

Ans. $\sqrt{68}$, $\sqrt{50}$, $\sqrt{106}$.

4. Find the lengths of the sides of a triangle, the co-ordinates of whose vertices are the same as in the last example, the axes being inclined at an angle of 60°.

Ans. $\sqrt{52}$, $\sqrt{57}$, $\sqrt{151}$.

5. Find the lengths of the three sides of the triangle whose vertices are $(2, 5)$, $(-4, 1)$, $(-2, -6)$.

Ans. $\sqrt{52}$, $\sqrt{53}$, $\sqrt{137}$.

6. Express algebraically that the distance of the point (x, y) from the point $(2, 3)$ is equal to 4.

Ans. $\sqrt{(x-2)^2+(y-3)^2} = 4.$

7. Express algebraically that the point (x, y) is equidistant from the points $(2, 3)$ and $(4, 5)$.

Ans. $(x-2)^2 + (y-3)^2 = (x-4)^2 + (y-5)^2$, or $x + y = 7$.

8. Find the point equidistant from the points $(2, 3)$, $(4, 5)$, $(6, 1)$.

Here we have two equations, formed as in Ex. 7, to determine the two unknown quantities.

Ans. $x = \frac{13}{3}$, $y = \frac{8}{3}$, and the common distance is $\frac{\sqrt{50}}{3}$.

11. *To find the co-ordinates of the point which divides in a given ratio,* $m:n$, *the right line joining two given points,* (x', y') *and* (x'', y'').

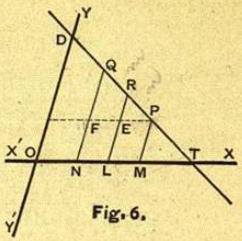

Fig. 6.

Let P and Q be the two given points, (x', y') and (x'', y''), and R the required point, whose co-ordinates we denote by x and y*. Then we have,

$$PR : RQ :: m : n.$$

Draw the ordinates PM, RL, QN, and the line PEF parallel to OX ; then we have,

$$\frac{PR}{RQ} = \frac{PE}{EF} = \frac{ML}{LN} = \frac{m}{n},$$

or,

$$\frac{m}{n} = \frac{x' - x}{x - x''};$$

hence,

$$x = \frac{mx'' + nx'}{m + n}.$$

Similarly we have,

$$y = \frac{my'' + ny'}{m + n}.$$

If the line were to be cut *externally* in the given ratio, we should have (Geom. Art. 302)

$$m : n :: x - x' : x - x'',$$

$$\therefore x = \frac{mx'' - nx'}{m - n}, \qquad y = \frac{my'' - ny'}{m - n}.$$

If $m = n$, or PQ is bisected in R, we have,

$$x = \frac{x'' + x'}{2}, \qquad y = \frac{y'' + y'}{2},$$

a result which is of frequent use. In this article the axes may be oblique or rectangular, the result being the same.

* (x, y) is generally used to denote an unknown point, while (x', y'), (x'', y''), etc., denote given (or known) points.

EXAMPLES.

1. Find the co-ordinates of the middle points of the sides of the triangle whose vertices are (2, 3), (4, −5), (−3, −6).
Ans. $(\frac{1}{2}, -\frac{11}{2})$, $(-\frac{1}{2}, -\frac{3}{2})$, $(3, -1)$.

2. The line joining the points (2, 3), (4, − 5), is trisected ; to find the co-ordinates of the point of trisection nearest to the former point. *Ans.* $x = \frac{8}{3}$, $y = \frac{1}{3}$.

3. The co-ordinates of P are (2, 3), and of Q (3, 4) ; find the co-ordinates of R, so that PR : RQ :: 3 : 4.
Ans. $x = 2\frac{3}{7}$, $y = 3\frac{3}{7}$.

4. The point (x, y) is midway between (3, 4) and (−5, −8); find its distance from the origin. *Ans.* $\sqrt{5}$.

POLAR CO-ORDINATES.

12. Let O be a given point, and OA a fixed line through it ; it is evident that we shall know the position of any point P, if we know the length OP and the angle POA. The line OA is called the **Initial Line** (called also the **Prime Radius** and the **Polar Axis**), the fixed point is called the **Pole**, the line OP is called the **Radius Vector**, and the variable angle AOP is called the **Direction Angle**, or **Vectorial Angle***. This method is called the method of **Polar Co-ordinates.** The *initial line* may have any position in the plane, but it is usually drawn through O horizontally to the right. The angle AOP and the distance OP are the polar co-ordinates of P.

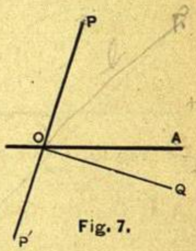

Fig. 7.

If the *direction angle* of any point be denoted by θ, and its *radius vector* by ρ, the point may be called the point (ρ, θ). When the *direction angle* is estimated from A *upwards* towards P, as in Trigonometry, it is called + ;

* It may be expressed either in degrees or in circular measure, but should *never* be expressed partly in one measure and partly in the other, as $2\pi + 40°$.

when estimated in the *opposite* direction from A *downwards* towards Q, it is called —. The *radius vector* is + when estimated from the pole in the direction of the extremity of the arc which measures the direction angle; and it is — when estimated in the *opposite* direction.

The following example* will make this clear. Let a be any distance OP, measured from O towards P, θ being the angle which OP makes with OA; then

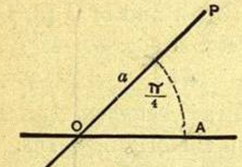

Fig. 8.

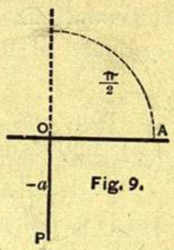

Fig. 9.

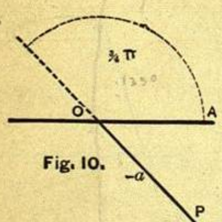

Fig. 10.

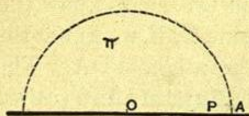

Fig. 11.

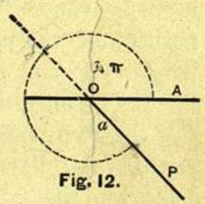

Fig. 12.

$\theta = \frac{1}{4}\pi,$	$\rho = \ \ a,$	represents P in Fig. 8;
$\theta = \frac{1}{2}\pi,$	$\rho = -a,$	" " " 9;
$\theta = \frac{3}{4}\pi,$	$\rho = -a,$	" " " 10;
$\theta = \ \pi,$	$\rho = -a,$	" " " 11;
$\theta = \frac{7}{4}\pi,$	$\rho = \ \ a,$	" " " 12;

We observe that the direction in which ρ is measured depends, not only on its sign, but also on the value of θ; thus, when $\theta = \frac{3}{4}\pi$, and $\rho = -a$, ρ must be measured from O to P, as in Fig. 10; and when $\theta = \frac{7}{4}\pi$, $\rho = a$, ρ must be measured in exactly the same direction.

* Puckle's Conic Sections, p. 9. Also, O'Brien's Co-ordinate Geometry, p. 37.

13. *To locate a point whose polar co-ordinates are given.*

Draw the initial line, and lay off, at any point taken for the pole, an angle equal to the given angle θ; then measure the distance ρ from the pole, in the direction of the extremity of the arc which measures the direction angle, or in the opposite direction, according as ρ is + or —, and the required point is obtained.

EXAMPLES.

1. Locate the point $\rho = 7$, $\theta = \frac{1}{4}\pi$.

The radius of the measuring arc being 1, π is the semi-circumference. Hence, $\frac{1}{4}\pi = 45°$. Now draw the initial line OA, and, at the point O taken for the pole, lay off AOP = 45°, and measure OP = + 7; P is the point required.

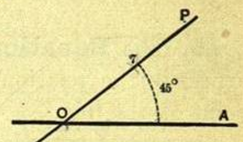

Fig. 13.

2. Represent the points $\rho = -8$, $\theta = \pi$, and $\rho = 15$, $\theta = \frac{1}{2}\pi$.

3. Represent the points $\rho = 15$, $\theta = \frac{3}{2}\pi$, and $\rho = -6$, $\theta = \frac{5}{4}\pi$.

4. Represent the points $\rho = -6$, $\theta = -\frac{5}{6}\pi$, and $\rho = 10$, $\theta = \frac{1}{4}\pi$.

5. Represent the points $\rho = 5$, $\theta = \frac{2}{3}\pi$, and $\rho = 6$, $\theta = \frac{1}{6}\pi$.

14. *To find the distance between two points in terms of their polar co-ordinates.*

Let P and Q be the two points; represent the co-ordinates of P by ρ', θ', and of Q by ρ'', θ'', and the distance PQ by d. Then in the triangle OPQ, OP = ρ', OQ = ρ'', and

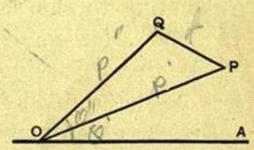

Fig. 14.

the angle POQ $= \theta'' - \theta'$. Hence, from Trigonometry,

$$d = \sqrt{\rho''^2 + \rho'^2 - 2\rho''\rho' \cos(\theta'' - \theta')}. \qquad (1)$$

COR.—If $\rho'' = 0$, and $\theta'' = 0$, we have, for the distance of any point (ρ', θ') from the origin, $d = \rho'$.

EXAMPLES.

1. Find the distance between $\rho = 3$, $\theta = \frac{1}{3}\pi$, and $\rho = 4$, $\theta = \frac{5}{6}\pi$. *Ans.* 5.

2. Find the distance (1) between $\rho = 5, \theta = 75°$, and $\rho = 4$, $\theta = 15°$; and (2) between (5, 30°) and (6, 225°).
Ans. (1) $\sqrt{21}$; (2) 10.9.

DEFINITIONS.

15. The **Equation of a Line** is the equation which expresses the relation between the co-ordinates of every point of the line.

The term **Locus** is nearly synonymous with **Geometric Figure;** it is the series of positions to which a point or line is restricted by given conditions.

The **Locus of a Point** is the line generated by the point when moving according to some given law.

The **Locus of a Line** is the surface generated by that line when moving according to a given law.

The **Locus of an Equation** is the line or surface, the co-ordinates of all of whose points are determined by the equation, while the equation is the *analytic representative* of the line or surface. In the equation, $y = x + 4$, we may assign to x any value we please, as 1, and from the equation determine the corresponding value of y equal to 5. A point $x = 1$ and $y = 5$ is thus determined. In like manner corresponding to the values 2, 3, 4, etc., for x, we have 6, 7, 8, etc., for y, determining the points (2, 6), (3, 7), (4, 8), etc. The line passing through all the points that may be determined in this way is called the *locus of the equation,* which may therefore be regarded as the *geometric equivalent* of the equation.

cos 195 = cos 180 + 15 = cos 15

Every equation between variables which denote the co-ordinates of a point represents a locus, and every locus has an equation.

When a point is on a locus its co-ordinates must *satisfy* the equation of the locus, that is, they must reduce the equation to an identity when they are substituted in it for x and y. Thus, 1 and 5 substituted for x and y respectively in $y = x + 4$ give the identity $5 = 1 + 4$. The resulting equation is called the *condition* that the point may lie on the locus. Thus, the equation $y' = x' + 4$ is the condition that the point (x', y') may lie on the locus $y = x + 4$.

There are two kinds of quantities used in Analytic Geometry: 1st, **Constants,** whose values do not change in the same discussion, and are represented by the leading letters of the alphabet; and 2d, **Variables,** which are susceptible of an infinite number of values within certain limits that are determined by the nature of the problem, and are represented by the final letters of the alphabet.

CONSTRUCTING EQUATIONS.

16. To construct an equation, or find its locus, is to trace, by means of determined points, the geometric figure which the equation represents.

To construct any curve from its equation, we solve the equation for either of its variables, usually for y, whose value or values we find in terms of x and constants. Then substitute for x a series of arbitrary values, and find the corresponding values of y. Now draw the axes, and lay down the points corresponding to the co-ordinates thus found. A curve traced through these points will approximately represent the locus of the equation. The closer the points are to each other, the more exact is the locus, unless it be a right line, which needs but two points to determine it.

SCH.—Although it is customary to solve the equation for y, yet if it is above the second degree with respect to either