variable, it is expedient to solve it with respect to the variable which is least involved. Thus, to construct

$$2x + y^2 = 3y^3 + 2y - 8,$$

we solve it with respect to x, and find

$$x = \frac{3y^3 + 2y - 8 - y^2}{2}.$$

Then substitute arbitrary values for y, and find the corresponding values of x.

17. The **Independent Variable** is the one to which arbitrary values are assigned, usually x. The other is called the **Dependent Variable.** This distinction is made for convenience ; either variable may be treated as the *independent variable,* and the other as the *dependent variable ;* the latter is said to be a **Function** of the former.

One quantity is a function of another when so connected with it that no change can take place in the latter, without producing a corresponding change in the former. Thus,

$$y = ax + b,$$

y is a function of x ; the ordinate of a curve is a function of the abscissa.

Functions are divided into two classes, *algebraic* and *transcendental.*

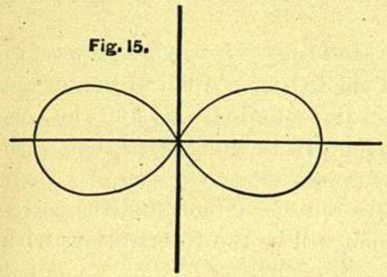
An **Algebraic Function** is one in which the relation between the function and its variable can be expressed by the ordinary operations of algebra, that is, *addition, subtraction, multiplication, division, involution,* and *evolution,* or the algebraic sum of many such functions. Thus, in each of the following expressions,

$$y = 2x^2 + x^3, \quad y = 4x - \sqrt{x}, \quad y = (ax^3 - 2x^4)^{\frac{1}{2}},$$

y is an algebraic function of x.

A **Transcendental*** **Function** is one in which the relation between the function and its variable cannot be expressed by the ordinary operations of algebra.

Fig. 15.



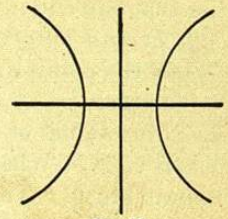


Fig. 16.

18. A curve is **Continuous** when it has no interruption in its extent, and no abrupt change in its curvature. A circle, an ellipse, and the curve in Fig. 15 are examples of *continuous* curves. The curve, Fig. 16, is *discontinuous,* having an interruption in *extent.* Fig. 17 is an example of a *discontinuous* curve, having an abrupt change in its curvature.

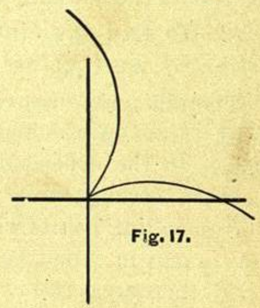


Fig. 17.

19. A **Branch** is the continuous part of a curve. In Figs. 16 and 17 the curves have two branches.

20. A curve is **Symmetrical** with respect to any line when it has the same form on both sides of the line; that is, when every point on one side of the line has a corresponding point on the other side of the line. The curves in Figs. 15 and 16 are symmetrical with respect to the axis of x. The curve in Fig. 17 is not symmetrical with respect to any line.

* Transcendental functions are further subdivided, but this division is not necessary in the present work.

DISCUSSION OF EQUATIONS.

21. The **Discussion** of an equation consists in observing the peculiarities of the loci which appear from the form of the equation, by making different hypotheses on the quantities that enter it.

1st. *To find where the locus cuts the axes of x and y.*

When the points are on the axis of x their ordinates are 0. Therefore put $y = 0$ in the equation, and find the corresponding values of x, which will be the intersections with the axis of x. When the points are on the axis of y their abscissas are 0. Therefore put $x = 0$, and find the corresponding values of y, which will be the intersections with the axis of y.

Thus, in the locus, $y = x + 2$, if $x = 0$, $y = 2$, and if $y = 0$, $x = -2$. Therefore the locus cuts the axis of x at the distance 2 to the left of the origin, and the axis of y at the distance 2 above the origin.

The distances from the origin to the points where the locus cuts the axes are called the *intercepts* on the axes.

2d. *To find the limits between which the locus is situated, and to test for continuity in extent.*

The limits are discovered by determining the greatest and least values of the independent variable which give *real* values to the dependent one. If all values assigned to x between certain limits give rise to *real* values for y, the corresponding points will be *real;* that is, the curve will be *continuous* in extent between these limits. If, on the contrary, there are certain values of x which render y *imaginary*, the corresponding points will be *imaginary;* that is, the locus is *interrupted* at such points, and therefore is *discontinuous.* If between any *two* values of either variable, the corresponding values of the other variable are *all* imaginary, the locus does not exist between the corresponding limits. And the limits of discontinuity are the limits between which the values of the dependent variable are imaginary.

Thus, in the locus,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

by solving for y we obtain,

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2};$$

so that y is *real* for every value of x that lies *beyond* the limits $x = a$ and $x = -a$, but is *imaginary* for every value of x lying *between* them; and the locus is interrupted in the latter region.

3d. *To test for symmetry with respect to an axis.*

Note whether, for each *real* value of one variable, the other has *two* values numerically equal, but with contrary signs; if so, there are points similarly situated on opposite sides of the axis from which the variable, having two values, is reckoned; and hence the locus is symmetrical with respect to that axis.

Thus, in the locus last considered, we see that for any value of x beyond the limits $+a$ and $-a$, y has two values numerically equal, with contrary signs. Hence this locus is symmetrical with respect to the axis of x.

EXAMPLES.

1. Construct and discuss the equation

$$2y - 6x - 12 = 0.$$

Solving the equation for y, we have,

$$y = 3x + 6.$$

Making successively $x = 0$ and $y = 0$, we obtain,

$$y = 6, \qquad \text{and} \qquad x = -2.$$

The locus, therefore, cuts the axis of x at a distance 2 to the left of the origin, and the axis of y at a distance 6 above the origin. Draw the axes XX′ and YY′, and lay down the corresponding points.

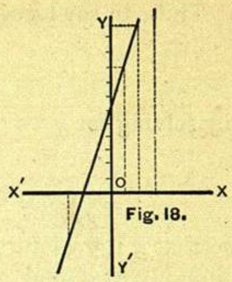


Fig. 18.

Now, give x the following arbitrary values, and find the corresponding values of y:

When $x = 1$, $y = 9$, giving the point $(1, 9)$.
" $x = 2$, $y = 12$, " " $(2, 12)$.
" $x = 3$, $y = 15$, " " $(3, 15)$.

All positive values of x give *positive, real,* and *single* values to y. The equation being of the first degree, the locus has but one branch, which extends to the right of the axis of y indefinitely, and above the axis of x.

Giving negative values to x, we have:

When $x = -1$, $y = 3$, giving the point $(-1, 3)$.
" $x = -2$, $y = 0$, " " $(-2, 0)$.
" $x = -3$, $y = -3$, " " $(-3, -3)$.

For all subsequent negative values of x, y has *real, negative,* and *single* values. Hence, the locus has a single branch extending indefinitely in the third angle. Laying down the points $(-3, -3)$, $(-2, 0)$, $(-1, 3)$, $(1, 9)$, $(2, 12)$, $(3, 15)$, we find that they all come upon the right line drawn through $(0, 6)$ and $(-2, 0)$, which is therefore the locus represented by the given equation.

If any other values be assigned to x, either positive or negative, integral or fractional, and the corresponding values of y be deduced, the points so determined will all fall upon the same line.

2. Construct and discuss the equation $\frac{1}{3}y + x = 2$.

Result.—A straight line cutting the axis of x at $(2, 0)$, and the axis of y at $(0, 6)$.

3. Construct and discuss the equation

$$8y - 6x - 5 = 0.$$

Result.—A right line passing through the points $(0, \frac{5}{8})$ and $(-\frac{5}{6}, 0)$.

4. Construct and discuss the equation

$$x^2 + y^2 = 16.$$

Solving the equation for y, we get

$$y = \pm \sqrt{16 - x^2}.$$

When $x = 0$, $y = +4$ and -4.

Hence the locus cuts the axis of y at $(0, 4)$ and $(0, -4)$.

When $y = 0$, $x = +4$ and -4.

Hence the locus cuts the axis of x at $(4, 0)$ and $(-4, 0)$.

As every value of x between $+4$ and -4 gives two *real* values for y, numerically equal, with contrary signs, the locus is symmetrical with respect to the axis of x, and continuous between these limits. But when $x > 4$ or < -4, y becomes imaginary, and therefore the locus has no point beyond its intersection with the axis of x.

Similarly, $x = \pm \sqrt{16 - y^2}$ shows that the locus is symmetrical with respect to the axis of y, and continuous between $y = 4$ and -4. When y is > 4 or < -4, the values of x become *imaginary;* and hence the locus has no point beyond its intersection with the axis of y.

Now giving to x arbitrary values between $+4$ and -4, we find the following:

When $x = 1$, $y = \pm 3.9$ nearly, giving us the points $(1, 3.9)$ and $(1, -3.9)$.

When $x = 2$, $y = \pm 3.5$ nearly, giving us the points $(2, 3.5)$ and $(2, -3.5)$.

When $x = 3$, $y = \pm 2.6$ nearly, giving us the points $(3, 2.6)$ and $(3, -2.6)$.

The negative values of x give us the following points:
$(-1, 3.9)$ and $(-1, -3.9)$,
$(-2, 3.5)$ and $(-2, -3.5)$,
$(-3, 2.6)$ and $(-3, -2.6)$.

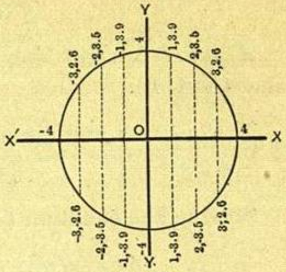


Fig. 19.

Constructing the points thus found, we find the figure to be the circumference of a circle whose radius is 4, and which is symmetrical to both axes. If any fractional values be given to x between the limits $+4$ and -4, and the corresponding values of y be found, the points so determined will all fall upon the same circumference.

The same result might have been reached by considering that $x^2 + y^2 = 16$ shows that the distance of any point (x, y) from the origin is constantly equal to 4. (Art. 9.)

5. Construct and discuss the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Solving the equation for y, we get,

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}.$$

Making $x = 0$, we get $y = \pm 2$; hence the curve cuts the axis of y in two points, one at the distance 2 above the origin, and the other at the same distance below it.

Making $y = 0$, we get $x = \pm 3$; hence the curve cuts the axis of x in two points equally distant from the origin and on opposite sides of it. For each value of x between $+3$ and -3, y is *real*, and has two values numerically equal, with contrary signs; hence the curve is symmetrical with respect to the axis of x, and continuous between these limits. When $x > 3$ or < -3, y becomes imaginary, and hence the curve has no point beyond its intersection with the axis of x.

Similarly, from $x = \pm \frac{3}{2}\sqrt{4 - y^2}$, we learn that the curve is continuous between $y = 2$ and $y = -2$, and symmetrical with respect to the axis of y, but has no point beyond its intersection with the axis of y. Since the curve is symmetrical with respect to the axis of y, we need consider only positive values of x.

Giving now to x the following values, we have the following corresponding values for y:

When $x = 0$, $y = \pm 2$, giving the points $(0, 2)$ and $(0, -2)$.

When $x = .5$, $y = \pm 1.97$, giving the points $(.5, 1.97)$ and $(.5 - 1.97)$.

When $x = 1$, $y = \pm 1.89$, giving the points $(1, 1.89)$ and $(1, -1.89)$.

When $x = 1.5$, $y = \pm 1.73$, giving the points $(1.5, 1.73)$ and $(1.5, -1.73)$.

When $x = 2$, $y = \pm 1.49$, giving the points $(2, 1.49)$ and $(2, -1.49)$.

When $x = 2.5$, $y = \pm 1.1$, giving the points $(2.5, 1.1)$ and $(2.5, -1.1)$.

When $x = 2.75$, $y = \pm 0.8$, giving the points $(2.75, 0.8)$ and $(2.75, -0.8)$.

When $x = 3$, $y = 0$, giving the points $(3, +0)$ and $(3, -0)$.

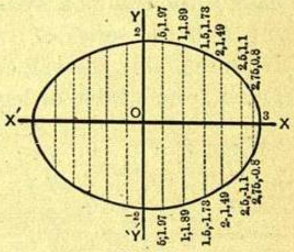


Fig. 20.

Laying down the points thus found, and a similar set on the left of the axis of y, we determine the figure to be an ellipse, whose axes are 6 and 4.

6. Construct and discuss the equation $y^2 = 3x - 9$.

Ans. It cuts the axis of x at the distance 3 to the right of the origin, and lies entirely to the right of this point, extending indefinitely in two branches that are symmetrical with respect to the axis of x. The curve is a parabola.

7. Construct and discuss the equation

$$x^2 - y^2 = 16.$$

Ans. The curve cuts the axis of x at two points; one at the distance 4 to the right of the origin, and the other at the same distance to the left of it. It has no point between these intersections, but extends to the right and left of these points indefinitely, and is symmetrical with respect to both axes, and is known as the hyperbola.

8. Construct and discuss the following equations:

$$y^2 + x^2 - 6x + 10y + 9 = 0,$$
$$16(y+3)^2 = 200 - 25(x-5)^2;$$
$$y^2 = x^4 - x^3.$$

9. Construct and discuss the equation

$$x = \log y \qquad \text{or} \qquad y = a^x.$$

Assuming $a = 10$, which is the base of the common system, and giving to x the series of values in the following table, the values of y can be found from a table of logarithms.

When	$x =$ 0,	$y =$ 1.	
"	$x =$ .2,	$y =$ 1.58	nearly.
"	$x =$ .4,	$y =$ 2.51	"
"	$x =$ .6,	$y =$ 3.98	"
"	$x =$ .8,	$y =$ 6.31	"
"	$x =$ 1,	$y =$ 10.00.	
"	$x = -$ .1,	$y =$ .8	nearly.
"	$x = -$ .2,	$y =$ .6	"
"	$x = -$ .4,	$y =$ .4	"
"	$x = -$ .7,	$y =$ .2	"
"	$x = -1$,	$y =$ .1	"
"	$x = -2$,	$y =$ .01.	

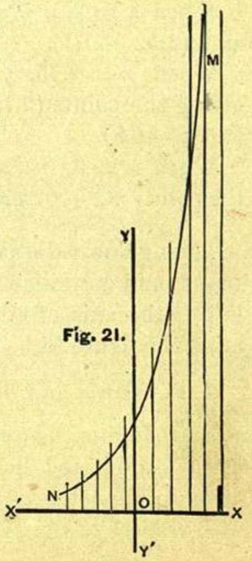


Fig. 21.

Laying down the points thus found, we have the curve MN (Fig. 21), which is called the *logarithmic curve.* It lies entirely above the axis of x, since negative numbers have no logarithms, but extends on both sides of the axis of y indefinitely and cuts it at (0, 1), through which point all logarithmic curves pass, since log. $1 = 0$ in any system. The curve is not symmetrical with respect to the axis of y; as it continues to the left of the origin, the ordinates diminish more and more, but can never reduce to 0 while x is finite. When the ordinate becomes infinitely small, the abscissa becomes infinitely great, and negative.

10. Construct and discuss $x = \log y$ or $y = a^x$.

Assume $a = 2.718$, which is the base of the Naperian system, and we get $y = (2.718)^x$.

When	$x =$ 1,	$y =$ 2.718.
"	$x =$ 2,	$y =$ 7.389.
"	$x =$ 3,	$y =$ 20.085.
"	$x = -$ 1,	$y =$ 0.368.
"	$x = -$ 2,	$y =$ 0.135.

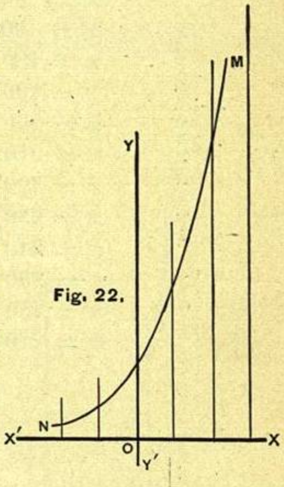


Fig. 22.

Laying down the points thus found, we have the curve MN (Fig. 22), which is called the logarithmic curve for the Naperian base.

11. Construct and discuss the equation of the sinusoid

$$y = \sin x.$$

The unit of angular measure is the angle at the centre, measured by an arc equal in length to the radius, as this angle is of an invariable magnitude, whatever be the length of the radius. The semi-circumference being 3.1416, when the radius is unity, the number of degrees in an arc equal to the length of the radius is equal to $\frac{180°}{3.1416} = 57°.3$ nearly. Hence the following series of values:

When	$x = 0° = 0,$	$y =$	$0.$
"	$x = 10° = .17,$	$y =$	$.17.$
"	$x = 20° = .35,$	$y =$	$.34.$
"	$x = 30° = .52,$	$y =$	$.50.$
"	$x = 40° = .70,$	$y =$	$.64.$
"	$x = 50° = .87,$	$y =$	$.77.$
"	$x = 60° = 1.05,$	$y =$	$.87.$
"	$x = 70° = 1.22.$	$y =$	$.94.$
"	$x = 80° = 1.40,$	$y =$	$.98.$
"	$x = 90° = 1.57,$	$y =$	$1.00.$
"	$x = 180° = 3.14,$	$y =$	$0.$
"	$x = 190° = 3.31,$	$y =$	$-.17.$
"	$x = 200° = 3.49,$	$y =$	$-.34.$
"	$x = 210° = 3.66,$	$y =$	$-.50.$
"	$x = 220° = 3.84,$	$y =$	$-.64.$
"	$x = 230° = 4.01,$	$y =$	$-.77.$
"	$x = 240° = 4.19,$	$y =$	$-.87.$
"	$x = 250° = 4.36,$	$y =$	$-.94.$
"	$x = 260° = 4.54,$	$y =$	$-.98.$
"	$x = 270° = 4.71,$	$y =$	$-1.00.$

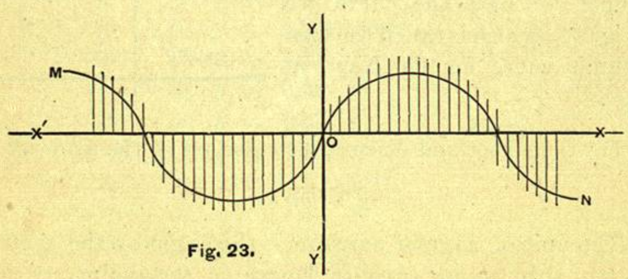


Fig. 23.

Laying down the points thus found, we have the curve MN, which is called the *Sinusoid*, or the *curve of sines*.

12. Construct and discuss

$$y = \tan x;$$
$$y = \cot x;$$
$$y = \cos x;$$
$$y = \text{vers } x;$$
$$y = \text{covers } x;$$
$$y = \sec x;$$
$$y = \text{cosec } x.$$

These loci may be constructed with sufficient accuracy without computing their numerical values. Thus, in the example,

$$y = \sec x.$$

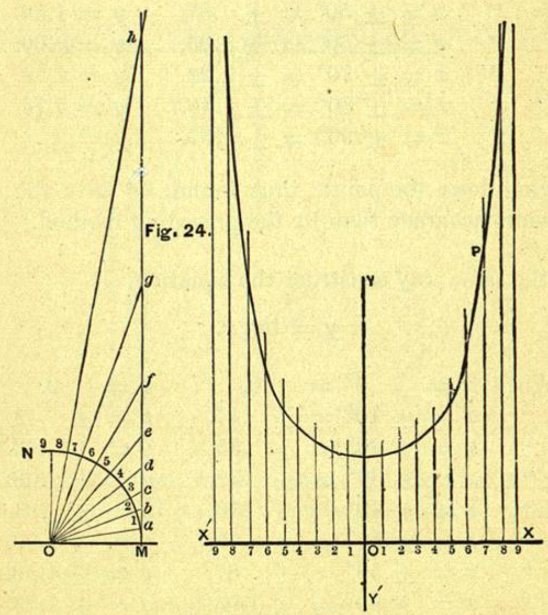


Fig. 24.

Divide a quadrant MN into any number of equal parts, say nine, so small that for practical purposes the chord and arc may be considered equal. Now measure the arcs from

M, and the secants of those arcs; then lay off the arcs on the axis of x for abscissas, and draw perpendiculars equal to the corresponding secants for the ordinates.

For example, measure O7 = the arc M7, and at 7 draw the perpendicular 7P = Og. P will be a point of the curve.

This example may be solved in the same way as Ex. 11; thus,

When	$x = \pm\ 0° = \pm\ 0,$	$y = 1.00.$
"	$x = \pm\ 10° = \pm\ .17,$	$y = 1.02.$
"	$x = \pm\ 20° = \pm\ .35,$	$y = 1.06.$
"	$x = \pm\ 30° = \pm\ .52,$	$y = 1.16.$
"	$x = \pm\ 40° = \pm\ .70,$	$y = 1.31.$
"	$x = \pm\ 50° = \pm\ .87,$	$y = 1.56.$
"	$x = \pm\ 60° = \pm\ 1.05,$	$y = 2.00.$
"	$x = \pm\ 70° = \pm\ 1.22,$	$y = 2.92.$
"	$x = \pm\ 80° = \pm\ 1.40,$	$y = 5.76.$
"	$x = \pm\ 90° = \pm\ 1.57,$	$y = \infty.$

Laying down the points thus found, we have the curve with more accuracy than by the preceding method.

In the same way construct the equation,

$$y = \tan x.$$

When	$x = 0° = 0,$	$y = 0.$
"	$x = \pm\ 10° = \pm\ .17.$	$y = \pm\ .18.$
"	$x = \pm\ 20° = \pm\ .35,$	$y = \pm\ .36.$
"	$x = \pm\ 30° = \pm\ .52,$	$y = \pm\ .58.$
"	$x = \pm\ 40° = \pm\ .70,$	$y = \pm\ .84.$
"	$x = \pm\ 45° = \pm\ .79,$	$y = \pm\ 1.00.$
"	$x = \pm\ 50° = \pm\ .87,$	$y = \pm\ 1.19.$
"	$x = \pm\ 60° = \pm\ 1.05,$	$y = \pm\ 1.73.$
"	$x = \pm\ 70° = \pm\ 1.22,$	$y = \pm\ 2.75.$
"	$x = \pm\ 80° = \pm\ 1.40,$	$y = \pm\ 5.67.$
"	$x = \pm\ 90° = \pm\ 1.57,$	$y = \infty.$

Laying down the points thus found, we have the curve MN.

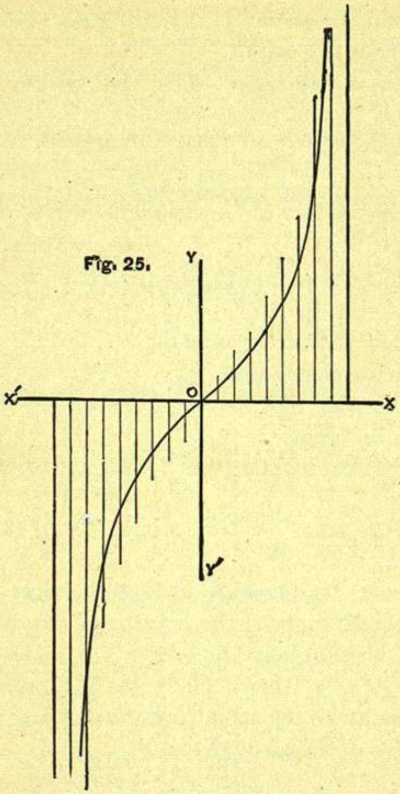


Fig. 25.

It is clear that when we take the + values of x we must take the + values of y; and when we take the — values of x we must take the — values of y, confining ourselves to the first quadrant.

EXAMPLES.

1. Find the points (1, 2) and (— 3, — 1); and show that the distance between them is 5.

2. Find the distance from the origin to each of the points $(2, 3)$, $(-2, 3)$, $(-3, -2)$. *Ans.* $\sqrt{13}$, $\sqrt{13}$, $\sqrt{13}$.

3. Show that the points $(1, 3)$, $(2, \sqrt{6})$ $(2, -\sqrt{6})$ are equidistant from the origin.

4. Find the distances between the following pairs of points: (1) $(1, 0)$ and $(-1, 0)$; (2) $(-3, 4)$ and $(5, -6)$; (3) $(-3, 7)$ and $(6, -5)$; (4) $(2, 0)$ and $(0, -2)$.

Ans. (1) 2; (2) $2\sqrt{41}$; (3) 15; (4) $2\sqrt{2}$.

5. Find the sides of a triangle whose vertices are $(-1, -2)$, $(1, 2)$, $(2, -3)$. *Ans.* $\sqrt{20}$, $\sqrt{10}$, $\sqrt{26}$.

6. Show that the points $(3, 0)$, $(0, 3\sqrt{3})$, $(6, 3\sqrt{3})$ form an equilateral triangle.

7. Show that the points $(1, 1)$, $(-1, -1)$, $(-\sqrt{3}, \sqrt{3})$ form an equilateral triangle.

8. Show that the four points $(0, -1)$, $(-2, 3)$, $(6, 7)$, $(8, 3)$ form a rectangle.

9. Show that the points $(0, -1)$, $(2, 1)$, $(0, 3)$, $(-2, 1)$ form a square.

10. Show the same of the points $(2, 1)$, $(4, 3)$, $(2, 5)$, $(0, 3)$.

11. Construct the triangle whose vertices are $(0, 0)$, $(2, 3)$, $(3, 2)$, and find (1) the lengths of the sides, and (2) the cosine of the angle at the origin.

Ans. (1) $\sqrt{13}$, $\sqrt{13}$, $\sqrt{2}$; (2) $\frac{12}{13}$.

12. Express by an equation that the distance of the point (x, y) from $(-1, 2)$ is equal to 3.

Ans. $\sqrt{(x+1)^2 + (y-2)^2} = 3$.

13. Express by an equation that the point (x, y) is equidistant from the points $(-1, 1)$ and $(2, 3)$.

Ans. $6x + 4y = 11$.

14. Express by an equation that the point (x, y) is equidistant from the points $(3, 4)$ and $(1, -2)$.

Ans. $x + 3y = 5$.

15. Show that the point equidistant from the points $(-1, 1)$, $(1, 2)$, $(1, -2)$ is the point $(\frac{3}{4}, 0)$.

16. Find the lengths of the sides of a triangle whose vertices are $(0, 0)$, $(3, 4)$, $(-3, 4)$. *Ans.* 5, 5, 6.

17. Find the co-ordinates of the point midway between the points $(-6, 2)$ and $(4, -2)$. *Ans.* $(-1, 0)$.

18. The co-ordinates of P are $(3, -1)$, and of Q $(10, 6)$; find the point R so that PR : RQ = 3 : 4. *Ans.* $(6, 2)$.

19. Find the distance between the points whose polar co-ordinates are $(2, 40°)$ and $(4, 100°)$. *Ans.* $\sqrt{12}$.

20. Find the distance between $(4, 50°)$ and $(3, 110°)$.

Ans. $\sqrt{13}$.

21. Is the point $(3, 9)$ on the line $y = 2x + 3$?

22. Which of the following points are on the curve $y = 3x^2 + 5x$: $(2, 3)$, $(1, 8)$, $(-2, 2)$, $(-3, 10)$, $(-3, 12)$, $(3, 3)$?

Find where the following loci cut the axes of x and y:

23. $y = x + 2$.
24. $y = (x-2)(x-3)$.
25. $y^2 - 2y = x^2 - 3x$.
26. $y = x^2 - 4$.
27. $2x + 3y = 6$.
28. $x^2 + y^2 = 4$.
29. $16x^2 + 9y^2 = 144$.
30. $y^2 = x^2 - x^3$.
31. $9x^2 + 6xy + 9y^2 = 4$.
32. $x^2 + 6x + y^2 - 4y = 3$.

Construct the following equations:

33. $x + y = 4$.
34. $3x + 2y = 6$.
35. $2x - 5y = 10$.
36. $3x - 4y = -12$.
37. $4x + 3y = -10$.
38. $6x - 4y = 12$.
39. $x^2 + y^2 = 81$.
40. $y^2 = 4x^2$.
41. $y^2 = 4$.
42. $y = x^2$.
43. $x^2 = xy$.
44. $x^2 + 2x + 10y - 8 = 0$.
45. $x^2y = 4(2 - y)$.
46. $(x^2 + y^2)^2 = (x^2 - y^2)$.
47. $14x^2 - 4xy + 11y^2 - 60 = 0$.
48. $3x^2 + 4xy + 5y^2 - 2x - 7y - 4 = 0$.
49. $3x^2 + 8xy - 3y^2 + 6x - 10y + 5 = 0$.
50. $2x^2 + xy - 15y^2 - x + 19y - 6 = 0$.
51. $x^2 - 2xy + y^2 - 6x - 6y + 9 = 0$.
52. $\theta = 0$; $\theta = 1$; $\theta = \frac{1}{2}\pi$;
53. $r = 0$; $r = 4$; $r = 4 \sin^2 \theta$.