CHAPTER II.

THE RIGHT LINE.

Ex I + N. 5

22. I. *To find the equation of a right line, in terms of its angle with the axis of* x*, and its intercept on the axis of* y.

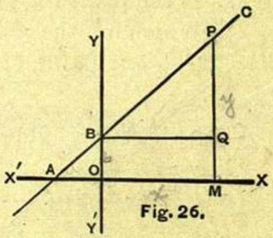

Fig. 26.

Let AC be any right line referred to the axes XX′ and YY′, and cutting the axis of y at B. Let P be any point in the given line, and draw PM perpendicular and BQ parallel to XX′; then will OM be the abscissa and MP the ordinate of the point P.

Let $\quad OM = x, \qquad MP = y, \qquad OB = b,$

$$\tan PAX = \tan PBQ = a;$$

then $y = PM = PQ + QM = BQ \tan PAX + BO = ax + b$,

that is, $$y = ax + b.$$

But the point P is, by hypothesis, *any point* of the line AC; therefore this equation, $y = ax + b$, expresses the relation between the co-ordinates of every point of AC, and hence it is the equation of that line, by the definition in Art. 15.

OB is called the **Intercept** on the axis of y; if the line cuts the axis of y below the origin, b will be negative. a denotes the tangent of the angle which the line AC makes with the axis of x, and is positive or negative, according as the angle is $<$ or $>$ 90°.

COR. 1.—If a is negative and b positive, the equation becomes $$y = -ax + b,$$

and the line cuts the axis of y *above* the origin, and makes with the axis of x an angle greater than 90°; it therefore cuts the latter at some point to the *right* of the origin, and so lies across the *first* angle.

If a and b are both positive, the equation becomes
$$y = ax + b;$$
the line lies across the *second* angle.

If a and b are both negative, the equation becomes
$$y = -ax - b;$$
the line lies across the *third* angle.

If a is positive and b negative, the equation becomes
$$y = ax - b;$$
the line lies across the *fourth* angle.

COR. 2.—If $b = 0$, the equation becomes
$$y = ax;$$
the line passes through the origin.

If $a = 0$, the equation becomes $y = b$;

the line is parallel to the axis of x.

If $a = \infty$, the line is parallel to the axis of y.

[The student may draw diagrams and verify these statements.]

SCH.—In the equation of a right line, so long as we consider the same line, a and b remain unchangeable; they are therefore called *constant quantities*, or *constants*. But x and y may have an *indefinite* number of values, since we may assign to one of them, as x, any value we please, and find the *corresponding value* of y from the equation
$$y = ax + b.$$
x and y are therefore called *variable quantities* or *variables*, as defined in Art. 15.

REM.—This form is often called the *tangent form* of the equation to a right line.

II. *To find the equation of a right line in terms of its intercepts on the two axes.*

Let A and B be the points where the right line cuts the axes of x and y respectively. Let $OA = a$, $OB = b$ be the intercepts on the axes of x and y, respectively; represent by x and y the co-ordinates OM and MP of any point P on the line. Draw PM parallel to YY′. Then, by similar triangles, we have

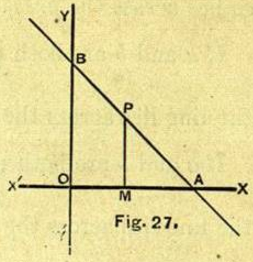

Fig. 27.

$$\frac{PM}{OB} = \frac{AM}{AO}, \quad \text{or} \quad \frac{y}{b} = \frac{a-x}{a};$$

therefore,
$$\frac{x}{a} + \frac{y}{b} = 1.$$

Cor.—By observing the signs of the arbitrary constants a and b in this equation, we can fix the position of the line with regard to the four angles, as in the preceding article.

When a and b are both positive, the line lies in the *first* angle.

When a is negative and b positive, the line lies in the *second* angle.

When a and b are both negative, the line lies in the *third* angle.

When a is positive and b negative, the line lies in the *fourth* angle.

Remark.—This form is known as the *symmetrical form* of the equation to a right line, and is frequently used. It has a close resemblance to the analogous equations of the conics; and it is applicable, as can be easily seen from the investigation, to rectangular and oblique axes alike.

III. *To find the equation of a right line in terms of the perpendicular on it from the origin, and the angle which the perpendicular makes with the axis of x.*

Let AB be the line, and OD the perpendicular on it from O; let $AOD = \alpha$, and $OD = p$. Let (x, y) be any point P on the line AB. Draw PM perpendicular to OA, MR perpendicular to OD, and PK perpendicular to MR.

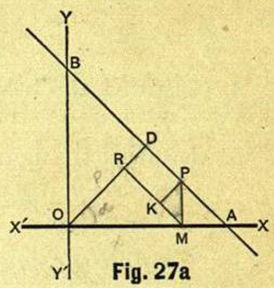

Fig. 27a

$$\begin{aligned} \text{Then } p = OD &= OR + RD \\ &= OR + KP \\ &= OM \cos MOR + MP \sin KMP \\ &= x \cos \alpha + y \sin \alpha; \end{aligned}$$

or
$$x \cos \alpha + y \sin \alpha = p, \qquad (1)$$

which is the equation required.

Sch. 1.—The coefficients of x and y in (1) are called the **Direction Cosines** of the line, since they are the cosines of the angles which the perpendicular makes with the axes of x and y respectively. In using this form, it must be carefully remembered that α is the angle which the perpendicular makes with the *positive direction* of the axis of x, and that α may have any value from 0 to 360°, while p is always *positive*, that is, measured from O so as to bound the angle α.

Sch. 2.—This form is known as the *normal form* of the equation to a right line.

EXAMPLES.

1. Across which of the four angles does the line $y = -7x + 5$ lie? The line $y = 3x + 4$? The line $y = -x - 3$? The line $y = 2x - 3$?

2. Trace the line $\frac{x}{3} + \frac{y}{2} = 1$; $-\frac{x}{2} - \frac{y}{3} = 1$.

3. In which of the angles lie the lines $\frac{x}{2} - \frac{y}{3} = 1$? $\frac{x}{3} - \frac{y}{2} = -1$? $\frac{x}{3} + \frac{y}{2} = 1$? $\frac{y}{7} + x = -1$?

4. Construct the triangle, the equations of whose sides are $y = \frac{3}{4}x + 3$, $y = \frac{1}{2}x - 1$, $y = -\frac{4}{5}x + 4$.

5. Construct the figure, the equations of whose sides are $y = x + 3$, $\frac{x+y}{2} = 2x - y - 4\frac{1}{2}$, $y + x = 3$, $x + y + 3 = 0$.

IV. *To find the equation of a right line referred to oblique axes.*

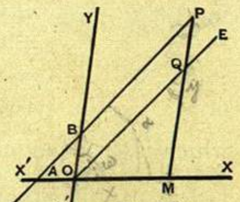

Fig. 28.

Let A and B be the points where the right line cuts the axes of x and y respectively. Draw PM parallel to YY′, and OE through the origin parallel to AB. Let x and y represent the co-ordinates OM and MP of any point P on the line. Denote the inclination of the axes by ω; and let OB $= b$, and the angle BAX $= \alpha$. Then we have,

$$y = \text{PM} = \text{PQ} + \text{QM} = \text{OB} + \text{QM}. \qquad (1)$$

But
$$\frac{\text{QM}}{\text{OM}} = \frac{\sin \text{BAO}}{\sin \text{ABO}} = \frac{\sin \alpha}{\sin (\omega - \alpha)}.$$

Therefore
$$\text{QM} = \frac{\sin \alpha}{\sin (\omega - \alpha)} \text{OM},$$

which in equation (1) gives

$$y = \frac{\sin \alpha}{\sin (\omega - \alpha)} x + b,$$

the required equation.

If we put a for $\frac{\sin \alpha}{\sin (\omega - \alpha)}$, the equation becomes

$$y = ax + b,$$

which is of the same form as the equation in Art. 22, I. The meaning of b is the same as before; a is the *ratio of the sines of the angles* which the line makes with the two axes respectively. If the axes become rectangular, $\omega = 90°$, and therefore

$$a = \frac{\sin \alpha}{\sin (90° - \alpha)} = \tan \alpha,$$

which agrees with Art. 22, I.

EXAMPLES.

1. Find the equation of a right line which makes an angle of 135° with the axis of x, and cuts off an intercept $= -3$ on the axis of y, (1) if the axes are rectangular, and (2) if they are inclined at an angle of 45°.

(1) Putting $b = -3$ and $a = \tan 135° = -1$ in the equation of Art 22, I,

$$y = ax + b,$$

we have, for the required equation,

$$y = -x - 3.$$

(2) Putting $\alpha = 135°$, $\omega - \alpha = 45° - 135° = -90°$ in the equation of Art. 22, IV,

$$y = \frac{\sin \alpha}{\sin (\omega - \alpha)} x + b,$$

we have for the required equation,

$$y = \frac{\frac{1}{2}\sqrt{2}}{-1} x - 3,$$

or
$$y = -\frac{\sqrt{2}}{2} x - 3.$$

2. Find the equation of a right line which makes an angle of 30° with the axis of x, and cuts off an intercept of 4 on the axis of y, if the axes are inclined at an angle of 60°.

Ans. $y = x + 4$.

23. *Every equation of the first degree between two variables is the equation of a right line.*

The general equation of the first degree with two variables is of the form

$$Ax + By + C = 0, \tag{1}$$

in which A and B are the collected coefficients of x and y, and C is the sum of the absolute terms.

Solving this equation for y, we obtain,

A = -3
B = 2

$$y = -\frac{A}{B}x - \frac{C}{B}, \tag{2}$$

which is the same as $y = ax + b$, if we take $a = -\frac{A}{B}$ and $b = -\frac{C}{B}$.

Hence (2), and therefore also (1), is the equation of a right line making with the axis of x an angle whose tangent is $-\frac{A}{B}$, and cutting the axis of y at a distance $-\frac{C}{B}$ from the origin.

If $A = 0$, then (1) becomes

$$By + C = 0,$$

or

$$y = -\frac{C}{B},$$

and, from Art. 22, I, this equation represents a right line parallel to the axis of x.

If $B = 0$, then (1) becomes,

$$Ax + C = 0,$$

or

$$x = -\frac{C}{A},$$

and, by Art. 22, I, this equation represents a right line parallel to the axis of y.

If A and B have like signs, the line makes an obtuse angle with the axis of x; and if they have unlike signs, it makes an acute angle. If B and C have like signs, the line cuts the axis of y below the origin; and if they have unlike signs, it cuts the axis of y above the origin.

If $C = 0$, then (1) becomes

$$Ax + By = 0,$$

or

$$y = -\frac{A}{B}x,$$

and the line passes through the origin. Hence the equation

$$Ax + By + C = 0$$

always represents a right line.

COR.—*To reduce the equation* $Ax + By + C = 0$ (1)
to the normal form $x \cos \alpha + y \sin \alpha = p$. (2)

Let HK denote the given line.

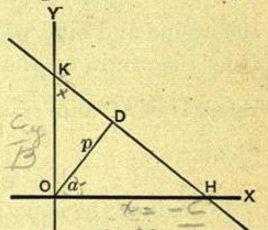

Fig. 28a

The intercepts made by the line, $Ax + By + C = 0$, on the axes are (Art. 21),

$$OH = -\frac{C}{A}; \quad OK = -\frac{C}{B}.$$

$$\therefore HK = \pm \frac{C}{AB}\sqrt{A^2 + B^2}.$$

HK : OH :: OK : p

But $HK \cdot p = OH \cdot OK$ (where p = the perpendicular OD).

$$\therefore p = \frac{OH \cdot OK}{HK} = \frac{C}{\pm\sqrt{A^2 + B^2}}.$$

Now p is always positive (Art. 22, III, Sch. 1); therefore we must take the radical with the same sign as C. Thus, if C be itself a positive quantity,

$$p = \frac{C}{\sqrt{A^2 + B^2}}.$$

$$\therefore \cos \alpha = \frac{p}{OH} = -\frac{A}{\sqrt{A^2 + B^2}}; \ \sin \alpha = -\frac{B}{\sqrt{A^2 + B^2}}.$$

Substituting these values in (2), we have

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}, \tag{3}$$

which is identical in form with $x \cos \alpha + y \sin \alpha = p$.

If C be negative, we must take the negative sign of the radical throughout, and (2) becomes

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=\frac{-C}{\sqrt{A^2+B^2}}. \qquad (4)$$

Hence to reduce any equation of the form

$$Ax+By+C=0,$$

to the form $\qquad x\cos\alpha+y\sin\alpha=p,$

transpose the absolute term to the second member, make it positive by changing the signs of all the terms if necessary, and divide each term by $\sqrt{A^2+B^2}$.

Rem.—This reduction is important in finding the length of the perpendicular from any point to any line.

Sch. $-\dfrac{A}{\sqrt{A^2+B^2}}$ and $-\dfrac{B}{\sqrt{A^2+B^2}}$ are respectively the cosine and sine of the angle which the perpendicular from the origin on the line $Ax+By+C=0$ makes with the axis of x, and $\dfrac{C}{\sqrt{A^2+B^2}}$ is the length of this perpendicular.

EXAMPLES.

1. Reduce the equation $3x-4y+12=0$ to the form $x\cos\alpha+y\sin\alpha=p$.

Transposing the constant term 12, and making it positive, we have $\qquad -3x+4y=12.$

Dividing by $\sqrt{(-3)^2+4^2}=5$, we obtain

$$-\tfrac{3}{5}x+\tfrac{4}{5}y=\tfrac{12}{5},$$

which is identical with $x\cos\alpha+y\sin\alpha=p$, where $\cos\alpha=-\frac{3}{5}$, $\sin\alpha=\frac{4}{5}$, and $p=\frac{12}{5}$.

2. Show that the equation $x+y+5=0$ is equivalent to

$$x\cos\frac{5\pi}{4}+y\sin\frac{5\pi}{4}=\frac{5}{\sqrt{2}}.$$

Reduce the following equations to the normal form:

3. $4x+3y-10=0$. $\qquad$ *Ans.* $\frac{4}{5}x+\frac{3}{5}y=2$.
4. $3x+4y-15=0$. $\qquad$ *Ans.* $\frac{3}{5}x+\frac{4}{5}y=3$.
5. $12x-5y+10=0$. $\qquad$ *Ans.* $-\frac{12}{13}x+\frac{5}{13}y=\frac{10}{13}$.
6. $3x+\sqrt{3}y-3\sqrt{3}=0$. $\qquad$ *Ans.* $\frac{1}{2}\sqrt{3}x+\frac{1}{2}y=\frac{3}{2}$.

24. *To find the length of the perpendicular from any point* (x', y') *to the line* $x\cos\alpha+y\sin\alpha=p$.

Let (x', y') be the given point P, and AB the given line.

From the given point P draw PR parallel, and PN perpendicular to the given line AB. PN will be the perpendicular required.

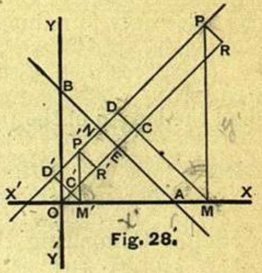

Fig. 28.

From the figure we have

$$\begin{aligned} PN &= PD + DN \\ &= PD + CO - EO \\ &= PM\sin DMP + OM\cos COM - EO \\ &= x'\cos\alpha + y'\sin\alpha - p. \end{aligned}$$

We have taken P on the side of the line *opposite* the origin. If the point were taken on the *same* side as the origin, as at P′, we would have,

$$\begin{aligned} P'N &= OE - OR' = OE - (OC' + D'P') \\ &= p - x'\cos\alpha - y'\sin\alpha. \end{aligned}$$

Hence, if the equation of a line is

$$x\cos\alpha+y\sin\alpha-p=0,$$

where p is a positive quantity, the length of the perpendicular on it from (x', y') is

$$\pm(x'\cos\alpha+y'\sin\alpha-p),$$

according as the point and the origin lie on *opposite* sides, or the *same* side of the line; that is, is equal to the result obtained by substituting in the left-hand member of the equation of the given line the co-ordinates of the given point, with the above restriction as to sign.

If the point (x', y') is *on* the line, its perpendicular becomes

$$x'\cos\alpha+y'\sin\alpha-p=0 \text{ (Art. 22, III).}$$

If the equation of the line were given in the form

$$Ax + By + C = 0,$$

we have only to reduce it to the form

$$x \cos \alpha + y \sin \alpha - p = 0 \text{ (Art. 23, Cor.)},$$

and the length of the perpendicular from any point (x', y') is

$$\pm \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}.$$

SCH.—Comparing this expression for the perpendicular from (x', y') with that for the perpendicular from the origin (Art. 23, Sch.), we see that (x', y') lies on the *same* side of the line as the origin, or on the *opposite* side, according as $Ax' + By' + C$ has the *same* sign as C, or the opposite sign.

EXAMPLE.

Find the length of the perpendicular from the origin to

$$a(x - a) + b(y - b) = 0.$$

This equation, reduced to the form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

becomes (Art. 23, Cor.),

$$-\frac{ax - a^2 + by - b^2}{\sqrt{a^2 + b^2}} = 0.$$

Ans. $\sqrt{a^2 + b^2}$.

25. *To find the equation of a right line passing through a given point.*

Let (x', y') be the given point, and the equation of the line be

$$y = ax + b. \qquad (1)$$

Since the given point (x', y') is on the right line, its co-ordinates must satisfy the equation of the line; that is,

the equation being true for *every* point on the line, must be true for the point (x', y'). Hence (1) becomes

$$y' = ax' + b. \qquad (2)$$

Eliminating b by subtracting (2) from (1), we obtain

$$y - y' = a(x - x'), \qquad (3)$$

which is the required equation. For it is the equation, by Art. 23, of *some* right line, since it is of the first degree between two variables; and it is the equation of a right line passing through the *given point*, because it is evidently satisfied when x' and y' are substituted in it for x and y. The constant a is the *tangent of the angle* which the line makes with the axis of x, or the *ratio of the sines of the angles* which the line makes with the two axes respectively, according as the line is referred to rectangular or oblique axes. By giving a suitable value to a, we may make equation (3) represent *any* right line which passes through the given point.

This equation (3) can easily be obtained geometrically. For let AB be any right line passing through the given point P′, the co-ordinates of which are x' and y'. Let P be *any* point on the line, x and y its co-ordinates. Draw the ordinates PM, P′N, and P′C parallel to the axis of x; then we have

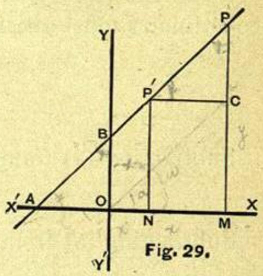

Fig. 29.

$$\frac{PC}{P'C} = \tan BAX \quad \text{or} \quad = \frac{\sin BAX}{\sin ABO},$$

according as the axes are rectangular or oblique; that is,

$$\frac{y - y'}{x - x'} = \tan BAX \quad \text{or} \quad = \frac{\sin \alpha}{\sin(\omega - \alpha)},$$

according as the axes are rectangular or oblique.

Hence, $$y - y' = \tan \text{BAX}\,(x - x')$$

or $$= \frac{\sin \alpha}{\sin(\omega - \alpha)}(x - x'),$$

or $$y - y' = a(x - x'),$$

in which a is the *tangent of the angle* which the line makes with the axis of x, or the *ratio of the sines of the angles* which the line makes with the two axes respectively, according as the line is referred to rectangular or oblique axes. This is the same as equation (3).

Ex. I

26. *To find the equation of the right line which passes through two given points.*

Let the two given points be (x', y') and (x'', y''), and the equation of the line be

$$y = ax + b. \qquad (1)$$

Since the two given points are on the right line, their co-ordinates must satisfy the equation of the line, giving

$$y' = ax' + b, \qquad (2)$$

$$y'' = ax'' + b. \qquad (3)$$

Subtracting (2) from (1), we obtain

$$y - y' = a(x - x'). \qquad (4)$$

Subtracting (3) from (2), we obtain

$$y' - y'' = a(x' - x''). \therefore a = \frac{y' - y''}{x' - x''},$$

which in (4) gives $y - y' = \dfrac{y' - y''}{x' - x''}(x - x'),$ (5)

which is the required equation.

NOTE.—Equations (2) and (3) are the *conditions* that the two points (x', y') and (x'', y'') may lie on the line $y = ax + b$. (Art. 15.)

Observe that the only variables in (5) are x and y, and that x', y', x'', y'' are constants.

To obtain equation (5) geometrically, let P be any point (x, y) on the line AB, and P′ and P″ the two given points (x', y') and (x'', y''); then we have, from the figure,

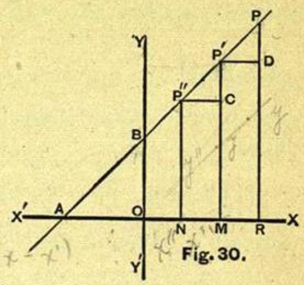

Fig. 30.

$$\frac{\text{PD}}{\text{P'D}} = \frac{\text{P'C}}{\text{P''C}},$$

or $$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''}.$$

Hence, $$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$

which is the same as equation (5).

$$\frac{y' - y''}{x' - x''} = \tan \text{BAX} = \tan \alpha,$$

if the axes are rectangular.

$$\frac{y' - y''}{x' - x''} = \frac{\sin \text{BAX}}{\sin \text{ABO}} = \frac{\sin \alpha}{\sin(\omega - \alpha)},$$

if the axes are oblique; which agrees with the results of Art. 25.

COR. 1.—Suppose $x'' = x'$; then

$$\frac{y' - y''}{x' - x''} = \frac{y' - y''}{0} = \infty,$$

which, being the tangent of 90°, shows that the line is parallel to the axis of y, which is as it clearly should be, since, if $x'' = x'$, the points P′ and P″ are equally distant from the axis of y.

If $y'' = y'$, $\dfrac{y' - y''}{x' - x''} = \dfrac{0}{x' - x''} = 0$, which, being the tangent of 0°, shows that the line is parallel to the axis of x, which is as it clearly should be, since if $y'' = y'$, the points P′ and P″ are equally distant from the axis of x.

In the case of oblique axes, if $x'' = x'$,

$$\frac{y' - y''}{x' - x''} = \frac{y' - y''}{0} = \frac{\sin \alpha}{\sin (\omega - \alpha)},$$

therefore, $\qquad \sin (\omega - \alpha) = 0,$

and hence $\omega = \alpha$; that is, the line is parallel to the axis of y.

If $y'' = y'$, $\dfrac{y' - y''}{x' - x''} = \dfrac{0}{x' - x''} = \dfrac{\sin \alpha}{\sin (\omega - \alpha)}$; therefore $\sin \alpha = 0$, and hence the line is parallel to the axis of x.

Cor. 2.—If P$''$ coincides with P$'$, we shall have,

$$x'' = x' \quad \text{and} \quad y'' = y',$$

and equation (5) becomes,

$$y - y' = \frac{0}{0}(x - x'), \tag{6}$$

which is the equation of a right line passing through a given point; and by representing the indeterminate expression $\frac{0}{0}$ by a, this equation becomes

$$y - y' = a(x - x'),$$

which agrees with equation (3), Art. 25.

Cor. 3.—If we make $x' = 0$ and $y' = b$, equation (6) becomes

$$y - b = ax,$$

or

$$y = ax + b,$$

which is the equation of a line passing through a point on the axis of y, at the distance of b from the origin. This equation agrees with the one found in Art. 22, I, as it clearly should.

Cor. 4.—If one of the points (x', y') be the origin, equation (5) becomes $y = \frac{y''}{x''}x$, which is therefore the equation of a line passing through the origin and (x'', y'').

EXAMPLES.

1. Find the equation of the right line passing through the points $(-2, 3)$ and $(3, -2)$.

Here $x' = -2$, $x'' = 3$, $y' = 3$, $y'' = -2$. Now, substituting these values in equation (5), we get

$$y - 3 = \frac{3 + 2}{-2 - 3}(x + 2),$$

and, reducing to the form $y = ax + b$, we get

$$y = -x + 1, \textit{ Ans.}$$

2. Find the equation of the line passing through the points $(4, -2)$, $(-3, -5)$. *Ans.* $7y - 3x + 26 = 0$.

3. Find the equations of the sides of the triangle, the co-ordinates of whose vertices are $(2, 1)$, $(3, -2)$, and $(-4, -1)$.

$$\textit{Ans.} \begin{cases} x + 7y + 11 = 0, \\ 3y - x - 1 = 0, \\ 3x + y - 7 = 0. \end{cases}$$

4. Find the equations of the sides of the triangle, the co-ordinates of whose vertices are $(2, 3)$, $(4, -5)$, and $(-3, -6)$. *Ans.* $x - 7y = 39$, $9x - 5y = 3$, $4x + y = 11$.

5. Find the equation of the line passing through the origin and the point $(3, -2)$. *Ans.* $3y + 2x = 0$.

27. *To find the angle between two right lines whose equations are given.*

Let AC and BC be the two right lines whose equations are respectively

$$y = ax + b,$$

and

$$y = a'x + b',$$

and call ϕ the angle between them.

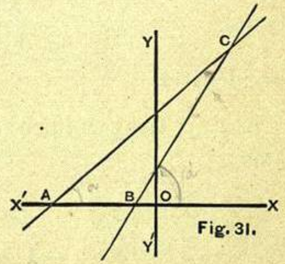

Fig. 31.