Then, Art. 22, I,

$$a = \tan \text{CAX} \quad \text{and} \quad a' = \tan \text{CBX},$$

and $$\tan \text{ACB} = \tan (\text{CBX} - \text{CAX}),$$

by Trigonometry, $$= \frac{\tan \text{CBX} - \tan \text{CAX}}{1 + \tan \text{CBX} \cdot \tan \text{CAX}},$$

or $$\tan \phi = \frac{a' - a}{1 + aa'}.$$

Sch.—In applying this formula to examples, we may obtain two results numerically equal, with contrary signs. Thus, if the two lines are

$$y = 3x + 2 \quad \text{and} \quad y = 4x - 7,$$

and we let $a' = 3$ and $a = 4$, we have

$$\tan \phi = \frac{3 - 4}{1 + 12} = -\frac{1}{13}.$$

But if we let $a' = 4$, and $a = 3$, we have

$$\tan \phi = \frac{4 - 3}{1 + 12} = \frac{1}{13}.$$

This ambiguity is as it should be, since the two lines form with each other two equal acute and two equal obtuse angles; and as these angles are supplements of each other, their tangents are numerically equal, with contrary signs.

Cor. 1.—If the two lines are parallel, we have

$$\phi = 0, \quad \text{and} \quad \therefore \quad \tan \phi = 0;$$

hence, $$\frac{a' - a}{1 + aa'} = 0,$$

which gives $$a' = a.$$

Also, if the two lines are perpendicular to each other, we have

$$\phi = 90° \quad \text{and} \quad \therefore \quad \tan \phi = \infty;$$

hence, $$\frac{a' - a}{1 + aa'} = \infty,$$

which gives, $$1 + aa' = 0,$$

or $$a' = -\frac{1}{a}.$$

Hence, $$y = -\frac{1}{a}x + b'$$

negative reciprocability is the condition for perpendicularity

represents a right line perpendicular to the right line

$$y = ax + b.$$

Cor. 2.—We found, Art. 25, that the equation of a right line passing through a given point (x', y') is

$$y - y' = a(x - x');$$

hence, by Cor. 1, $$y - y' = -\frac{1}{a}(x - x')$$

is the equation of a line passing through a given point (x', y') and perpendicular to the line $y = ax + b$.

EXAMPLES.

1. Find the angle between the lines

$$2y + x + 1 = 0,$$
$$3y - x - 1 = 0.$$

Solving both equations with respect to y, we have

$$y = -\tfrac{1}{2}x - \tfrac{1}{2}.$$
$$y = \tfrac{1}{3}x + \tfrac{1}{3}.$$

Here $a' = -\frac{1}{2}$, $a = \frac{1}{3}$; hence,

$$\tan \phi = \frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{6}} = -1.$$

$$\therefore \quad \phi = 135°.$$

2. Find the angle between the lines

$$3x + 2y - 12 = 0,$$
$$4x + y - 6 = 0.$$

Ans. $\tan \phi = \frac{5}{14}$, or $\phi = 19° 3$[illegible]

3. Find the angle between the lines

$$y = -x + 2,$$
$$y = 3x - 6.$$

Ans. $\tan\phi = -2. \quad \therefore \ \phi = 116^\circ\ 34$

4. Find the equation of the line passing through the point $(3, -4)$ and perpendicular to the line

$$5x - 4y - 52 = 0.$$

Ans. $5y = -4x - 8.$

5. Find the equation of the line passing through the point $(4, 1)$ and perpendicular to the line

$$4y = 5x - 31.$$

Ans. $5y = -4x + 21.$

28. *To find the equation of a right line which makes any given angle with a given line.*

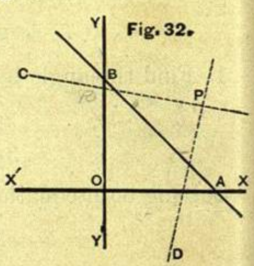

Fig. 32.

Let ϕ be the given angle, and let $\tan\phi = m$; let the equation of the given line AB be

$$y = ax + b, \qquad (1)$$

and the equation of the required line be

$$y = a'x + b', \qquad (2)$$

where a' is to be determined from the conditions of the problem.

Now, it is evident that the required line may be either PC or PD, since each makes the same angle with the given line AB. Hence we have, by Art. 27,

$$m = \frac{a' - a}{1 + aa'}, \quad \text{or} \quad = \frac{a - a'}{1 + aa'}.$$

Therefore, $$a' = \frac{a \pm m}{1 \mp am},$$

which in (2) gives $$y = \frac{a \pm m}{1 \mp am} x + b', \qquad (3)$$

where b' is undetermined, as it should be, since there may be an infinite number of lines drawn fulfilling this condition, all having the same inclination to the axis of x.

Cor. 1.—If the required line is to pass through a given point (x', y'), the equation will be, Art. 25,

$$y - y' = a'(x - x'),$$

or $$y - y' = \frac{a \pm m}{1 \mp am}(x - x'). \qquad (4)$$

Cor. 2.—If the required line is to pass through a given point, and parallel to a given line, $m = 0$, and (4) becomes

$$y - y' = a(x - x').$$

Cor. 3.—If the required line is to pass through a given point and perpendicular to a given line, $m = \infty$, and (4) becomes

$$y - y' = \frac{a \pm m}{1 \mp am}(x - x') = \frac{\frac{a}{m} \pm 1}{\frac{1}{m} \mp a}(x - x')$$

$$= \frac{\frac{a}{\infty} \pm 1}{\frac{1}{\infty} \mp a}(x - x') = \frac{\pm 1}{\mp a}(x - x'),$$

or $$y - y' = -\frac{1}{a}(x - x'),$$

which agrees with Art. 27, Cor. 2.

EXAMPLES.

1. Find the equations of the lines which pass through the point $(1, 2)$, and make an angle of 45° with the line

$$3x + 4y + 7 = 0.$$

Here $a = -\frac{3}{4}$, $m = 1$, and taking the upper sign in equation (4), we have

$$y - 2 = \frac{-\frac{3}{4} + 1}{1 + \frac{3}{4}}(x - 1) = \frac{1}{7}(x - 1).$$

or $$7y - x - 13 = 0, \quad \textit{Ans.}$$

And taking the lower sign in (4), we have,

$$y - 2 = \frac{-\frac{3}{4} - 1}{1 - \frac{3}{4}}(x - 1) = -7(x - 1),$$

or $$y + 7x - 9 = 0, \quad \textit{Ans.}$$

2. Find the equations of the lines which pass through the point (4, 4), and make an angle of 45° with the line

$$y = 2x.$$

$$\textit{Ans.}\ y - 4 = -3(x - 4), \text{ and}$$
$$y - 4 = \tfrac{1}{3}(x - 4).$$

3. Find the equations of the lines which pass through the origin, and make an angle of 60° with the line

$$x + y\sqrt{3} = 1.$$

$$\textit{Ans.}\quad y = \frac{1}{\sqrt{3}}x, \quad \text{and} \quad x = 0.$$

4. Find the equations of the lines which pass through the point (0, 1), and make an angle of 30° with the line

$$y + x = 2.$$

$$\textit{Ans.}\ \begin{cases} y - 1 = (\sqrt{3} - 2)x, & \text{and} \\ y - 1 = -(\sqrt{3} + 2)x. \end{cases}$$

5. Find the equation of the line which cuts the axis of y at a distance of 8 from the origin, and is perpendicular to the line

$$8y + 5x - 3 = 0.$$

$$\textit{Ans.}\quad 5y - 8x - 40 = 0.$$

$y = \frac{8}{5}x + \frac{40}{5}$

29. *To find the co-ordinates of the point of intersection of two right lines whose equations are given.*

Let the equations of the lines be,

$$y = ax + b, \tag{1}$$
$$y = a'x + b'. \tag{2}$$

Each equation expresses a relation which must be satisfied by the co-ordinates of every point on that line; therefore, the co-ordinates of the point where the lines *intersect* must satisfy *both* equations; hence, we must make (1) and (2) simultaneous, and find the values of x and y from them. Thus

$$x = \frac{b - b'}{a' - a}; \qquad y = \frac{ba' - b'a}{a' - a}. \tag{3}$$

EXAMPLES.*

1. Find the co-ordinates of the intersection of the two lines

$$3x + 7y = 47,$$
$$8x - y = 27. \qquad \textit{Ans.}\ (4, 5).$$

2. Find the intersection of the lines

$$\tfrac{1}{2}x - \tfrac{1}{3}y = 1,$$
$$y = -2x + 4. \qquad \textit{Ans.}\ (2, 0).$$

3. Find the intersection of the two lines

$$3y + 4x - 11 = 0,$$
$$4y + 3x - 10 = 0. \qquad \textit{Ans.}\ (2, 1).$$

4. Find the vertices of the triangle, the equations of whose sides are

$$x + y = 2, \qquad x - 3y = 4, \qquad 3x + 5y = -7.$$

$$\textit{Ans.}\ (-\tfrac{1}{14}, -\tfrac{19}{14}), (\tfrac{17}{2}, -\tfrac{13}{2}), (\tfrac{5}{2}, -\tfrac{1}{2}),$$

* These examples may be solved either by substituting the values of a, b, a', b' in (3), or by solving the equations directly for x and y.

29a. *To find the equations of the two bisectors of the angle between the lines*

$$x \cos \alpha + y \sin \alpha = p, \; x \cos \alpha' + y \sin \alpha' = p'.$$

It is clear that every point in either bisector is equally distant from the sides of the angle; hence if (x', y') be any point in either bisector, then

$$x' \cos \alpha + y' \sin \alpha - p = \pm (x' \cos \alpha' + y' \sin \alpha' - p');$$

for this merely expresses that the perpendicular from (x', y') to the one line is equal to the perpendicular from the same point to the other line. Hence the point (x', y') is on one or other of the lines

$$x \cos \alpha + y \sin \alpha - p = \pm (x \cos \alpha' + y \sin \alpha' - p'),$$

which are therefore the equations of the bisectors, the upper or lower signs being taken according as the angle bisected is *toward* the origin or the *supplement* of this angle (Art. 24).

EXAMPLES.

1. Find the bisectors of the angles between the lines $3x + y - 7 = 0$ (1), and $x - 3y + 5 = 0$ (2).

Let (x, y) be a point on one of the bisectors; then the lengths of the perpendiculars from (x, y) to the lines (1) and (2) are (Art. 24), respectively,

$$\frac{3x + y - 7}{\sqrt{10}} \quad (3), \quad \text{and} \quad \frac{x - 3y + 5}{\sqrt{10}} \qquad (4)$$

without regard to signs.

Since the perpendiculars are to be equal, (3) and (4) must be equal, or equal and of opposite signs.

$$\therefore \frac{3x + y - 7}{\sqrt{10}} = \pm \frac{x - 3y + 5}{\sqrt{10}};$$

and the two bisectors are $x + 2y - 6 = 0$, $2x - y - 1 = 0$.

2. Find the bisectors of the angles between the lines $2x + y = 1$, and $3x + y = 2$.

Ans. $\sqrt{2}\,(2x + y - 1) = \pm (3x + y - 2)$.

3. Find the bisectors of the angles between the lines $3x + 4y = 12$, and $4x + 3y = 24$.

Ans. $x - y = 12$, $7x + 7y = 36$.

30. *Given the equations of two right lines, to find the equation of a third line passing through their point of intersection.*

The method of solving this question, which would naturally occur to the student, would be to obtain the co-ordinates of the point of intersection, by Art. 29, and then to substitute the values of these co-ordinates for x' and y' in equation (3) of Art. 25, viz., $y - y' = a(x - x')$.

The question, however, admits of an easier solution.

Let the equations of two right lines be

$$y - ax - b = 0, \qquad (1)$$

$$y - a'x - b' = 0. \qquad (2)$$

Multiply either equation, (2) for instance, by an arbitrary constant, k, and add the result to (1). We have,

$$(y - ax - b) + k(y - a'x - b') = 0, \qquad (3)$$

which is the required equation.

For, equation (3) denotes *some* right line, since it is of the first degree (Art. 23); and it is clear that any co-ordinates which satisfy (1) and (2) must also satisfy (3), for the left member of this equation must vanish whenever $y - ax - b$ and $y - a'x - b'$ are each equal to zero. That is, equation (3) represents a line passing through a point whose co-ordinates satisfy equations (1) and (2); but this point is the intersection of the two lines, by Art. 29. Hence equation (3) denotes a line passing through the intersection of the given lines.

Since k is an arbitrary quantity, equation (3) will represent an infinite number of lines fulfilling one condition only, viz., all passing through the intersection of (1) and (2). We can therefore impose a second condition by giving the proper value to k; for example, we can make equation (3) represent a line passing through the point (x', y') by substituting x' and y' for x and y in (3), finding the value of k, and substituting this value for k in (3).

EXAMPLES.

1. Find the equation of the line passing through the intersection of

$$2x + 3y + 1 = 0, \tag{1}$$
$$3x - 4y - 5 = 0, \tag{2}$$

and the point (2, 3).

The equation of a line through the intersection of (1) and (2), by Art. 30, is

$$(2x + 3y + 1) + k(3x - 4y - 5) = 0. \tag{3}$$

As (3) is to pass through (2, 3), these co-ordinates, when substituted for x and y in (3), must satisfy it, giving us

$$(4 + 9 + 1) + k(6 - 12 - 5) = 0.$$
$$\therefore \quad k = \tfrac{14}{11},$$

which in (3) gives

$$(2x + 3y + 1) + \tfrac{14}{11}(3x - 4y - 5) = 0,$$

or, $\quad 64x - 23y - 59 = 0,$ *Ans.*

2. Given
$$\begin{cases} 2y - x + 6 = 0, & (1) \\ y + 4x + 8 = 0, & (2) \\ 3y + 2x - 30 = 0, & (3) \end{cases}$$

to find the equation of the perpendicular from the intersection of (1) and (2) to (3).

The line passing through the intersection of (1) and (2) is

$$(2y - x + 6) + k(y + 4x + 8) = 0. \tag{4}$$

Solving for y, we get

$$y = \frac{1 - 4k}{2 + k}x - \frac{6 + 8k}{2 + k}. \tag{5}$$

As (5) is to be perpendicular to (3), we must have, Art. 27, Cor. 1,

$$\frac{1 - 4k}{2 + k} = \tfrac{3}{2}, \quad \text{or} \quad k = -\tfrac{4}{11}.$$

which in (4) gives

$$(2y - x + 6) - \tfrac{4}{11}(y + 4x + 8) = 0,$$

or, $\quad 18y - 27x + 34 = 0,$ *Ans.*

3. Find the equation of the right line passing through the point (a, b), and the intersection of the right lines,

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \text{and} \quad \frac{x}{b} + \frac{y}{a} = 1.$$

Ans. $\dfrac{x}{a^2} - \dfrac{y}{b^2} = \dfrac{1}{a} - \dfrac{1}{b}.$

4. Find the equation of the line passing through the origin and the intersection of

$$7x + 3y + 2 = 0, \quad \text{and} \quad 4x - 5y - 7 = 0,$$

Ans. $11y + 57x = 0.$

31. *To find the polar equation of a right line.*

Let AB be a right line, OQ the perpendicular on it from the pole O, OX the initial line, P any point in the line. Let OQ $= p$, and the angle QOX $= \alpha$. Let (r, θ) be the polar co-ordinates of P; then

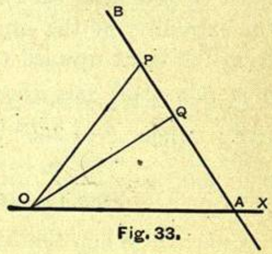

Fig. 33.

$$\text{OQ} = \text{OP} \cos \text{POQ};$$

that is, $\quad p = r\cos(\theta - \alpha).$

$$\therefore \quad r = \frac{p}{\cos(\theta - \alpha)}, \tag{1}$$

which is the required equation.

COR. 1.—If the right line AB were perpendicular to the initial line, we would have $\alpha = 0$, and the equation would become

$$r = \frac{p}{\cos\theta}, \qquad (2)$$

which is the equation of a right line perpendicular to the initial line.

COR. 2.—When $\theta = 0$, (1) becomes,

$$r = \frac{p}{\cos(-\alpha)} = \frac{\text{OQ}}{\cos \text{XOQ}} = \text{OA},$$

which is as it should be.

COR. 3.—When $\theta = \alpha$, (1) becomes

$$r = \frac{p}{\cos 0} = p, \text{ as it should.}$$

COR. 4.—When $\theta = 90° + \alpha$, (1) becomes

$$r = \frac{p}{0} = \infty,$$

as it should, since in this case the radius-vector becomes parallel to the line, and hence ∞.

COR. 5.—When $\theta > 90° + \alpha$ and $< 270° + \alpha$, r is negative, as it should be, since, in order to reach the line AB, it must be produced backward from the pole, directly opposite the extremity of the arc θ measured from the initial line from the right upward to the left.

COR. 6.—When $\theta > 270° + \alpha$ and $< 360° + \alpha$, r is positive; when $\theta = 360° + \alpha$, $r = p$, as it should; when $\theta = 360°$, $r = \dfrac{p}{\cos(-\alpha)} = \text{OA}$.

COR. 7.—When the line AB passes through the pole,

$$r = \frac{0}{\cos(\theta - \alpha)},$$

which is 0 for every value of θ except $90° + \alpha$, for which value $r = \frac{0}{0}$, or indeterminate, as it should.

EXAMPLES.

1. Find the perpendicular distance from the point (10, 2.9) to the line

$$5y - 4x + 5 = 0. \qquad (1)$$
$$4x - 5y - 5 = 0.$$

Reducing to the normal form, we have,

$$\frac{4x}{\sqrt{4^2 + (-5)^2}} - \frac{5y}{\sqrt{4^2 + (-5)^2}} - \frac{5}{\sqrt{4^2 + (-5)^2}} = 0. \qquad (2)$$

By Art. 24, the perpendicular is

$$\frac{4(10) - 5(2.9) - 5}{\sqrt{41}} = \frac{20.5}{\sqrt{41}} = \frac{(20.5)\sqrt{41}}{41} = .5\sqrt{41}$$
$$= .5(6.4) = 3.2, \textit{ Ans.}$$

2. Find the intersection of the perpendicular from $(-3, 8)$ to the line $y = \frac{1}{3}x - 5$. *Ans.* $(1\frac{1}{5}, -4\frac{3}{5})$.

3. Find the angle between the lines $x + y = 1$ and $y = x + 2$; also find the co-ordinates of the point of intersection. *Ans.* $90°$; $(-\frac{1}{2}, \frac{3}{2})$.

4. Find the angle between the lines $x + y\sqrt{3} = 0$ and $x - y\sqrt{3} = 2$. *Ans.* $60°$.

5. Find the length of the perpendicular from the point (2, 3) to the line $2x + y - 4 = 0$. *Ans.* $\dfrac{3}{\sqrt{5}}$.

6. Find the lengths of the perpendiculars from each vertex to the opposite sides of the triangle (2, 1), (3, − 2), and (− 4, − 1). *Ans.* $2\sqrt{2}$, $\sqrt{10}$, $2\sqrt{10}$.

7. Find the lengths of the perpendiculars from each vertex to the opposite side of the triangle (0, 0), (1, − 1), (3, 2). *Ans.* $\dfrac{5}{\sqrt{2}}$, $\dfrac{5}{\sqrt{13}}$, $\dfrac{5}{\sqrt{13}}$.

8. Find the length of the perpendicular from the point $(-1, 2)$ to the line $5x-2y=4$. *Ans.* $\frac{13}{29}\sqrt{29}$.

9. Find the perpendicular distances of the point $(2, 3)$ from the lines $4x+3y=7$, $5x+12y=20$. *Ans.* 2.

10. Find the angle between the lines $y=2x+5$ and $3x+y=7$. *Ans.* $45°$.

11. Find the equation of the line through $(4, 5)$ parallel to $2x-3y=5$. *Ans.* $2x-3y+7=0$.

12. Find the equation of the line through $(2, 1)$ parallel to the line joining $(2, 3)$ and $(3, -1)$. *Ans.* $4x+y=9$.

13. Find the equations of the sides of the triangle whose vertices are $(1, 2)$, $(2, 3)$, $(-3, -5)$.

Ans. $8x-5y=1$, $4y-7x=1$, $y-x=1$.

14. Find the equations of the lines from the vertices to the middle points of the opposite sides of the triangle in Ex. 13. *Ans.* $y=2x$, $2y=3x$, $3y=5x$.

15. In what ratio is the line joining the points $(1, 2)$ and $(4, 3)$ divided by the line joining $(2, 3)$ and $(4, 1)$?

Ans. The line is bisected.

16. Write the equations of the lines through the origin perpendicular to the lines $3x+2y=5$ and $4x+3y=7$. Find the co-ordinates of the points where these perpendiculars meet the lines, and show that the equation of the line joining these points is $23x+11y=35$.

17. Find the area of the triangle whose vertices are

(x_1, y_1), (x_2, y_2), (x_3, y_3).

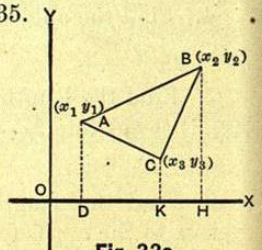

Fig. 33a

Draw AD, BH, CK parallel to the axis of y. Then area $ABC = ABHD - BCKH - ACKD = \frac{1}{2}[(y_1+y_2)(x_2-x_1)+(y_2+y_3)(x_3-x_2)+(y_3+y_1)(x_1-x_3)]$.

$\therefore$ Area $= \frac{1}{2}[y_1(x_2-x_3)+y_2(x_3-x_1)+y_3(x_1-x_2)]$.

18. Find the area of the triangle whose vertices are $(0, 0)$, $(3, 5)$, $(4, 3)$. *Ans.* $5\frac{1}{2}$.

19. Find the area of the triangle formed by the lines $x+2y=5$, $2x+y=7$, $y-x=1$. *Ans.* $1\frac{1}{2}$.

20. Find the area of the triangle formed by the lines $y=x$, $y=-x$, $x=c$. *Ans.* c^2.

21. Find the area of the triangle formed by the lines $y=2x$, $y=3x$, $y=5x+4$. *Ans.* $1\frac{1}{3}$.

22. Find the area of the triangle formed by the lines $y=2x+4$, $2y+3x=5$, $y+x+1=0$. *Ans.* $\frac{338}{21}$.

23. Find the bisectors of the angles between the lines $3x+4y=7$, and $8x+6y=13$.

Ans. $2x-2y+1=0$, $14x+14y=27$.

24. Find the bisectors of the angles between the lines $4x+3y=3$, and $5x-12y=8$.

Ans. $77x-21y=79$, $27x+99y+1=0$.

25. Find the equation of the line joining the point of intersection of the lines $4x-5y+6=0$, and $3x-5y+12=0$ to the point $(3, 4)$. *Ans.* $2x-3y+6=0$.

26. Find the equation of the line joining the point of intersection of the lines $3x+2y=5$, and $4x+3y+7=0$ to the point $(3, 1)$. *Ans.* $21x+13y=76$.

27. Find the equation of the line joining the point of intersection of the lines $y=x+1$, and $y=2x+2$ to the point $(0, 3)$. *Ans.* $y=3x+3$.

28. Find the equation of the line joining the origin to the point of intersection of the lines $x-4y=7$ and $y+2x=1$. *Ans.* $13x+11y=0$.

29. Find the equation of the line joining $(1, 1)$ to the point of intersection of the lines $3x+4y=2$ and $x-2y+5=0$. *Ans.* $7x+26y=33$.

30. Find the equation of the line through the point of intersection of $y-4x=1$ and $2x+5y=6$, perpendicular to $3y+4x=0$. *Ans.* $88y-66x=101$.

31. Find the equations of the two lines through the point $(2, 3)$ which make an angle of $45°$ with $x+2y=0$.

Ans. $x-3y+7=0$, $3x+y=9$.

32. Find the equation of the line through the intersection of $x-7y+13=0$, and $7x+y=9$ and parallel to the line $3x+4y+2=0$. *Ans.* $3x+4y=11$.

33. Find the equation of the line in Ex. 32, which is perpendicular to $3x+4y+2=0$. *Ans.* $4x+3y=2$.

34. Find the equation of the line that joins the points of intersection of the two pairs of lines,

$$\begin{cases} 2x + 3y - 4a = 0, \\ 2x + \ \ y - \ \ a = 0, \end{cases} \quad \text{and} \quad \begin{cases} x + 6y - 7a = 0, \\ 3x - 2y + 2a = 0. \end{cases}$$

Ans. $4(x + y) - 5a = 0.$

35. The co-ordinates of two points are (3, 5) and (4, 4), respectively; find the equation of the line which bisects the distance between them and makes an angle of 45° with the axis of x. *Ans.* $y - x - 1 = 0.$

36. Find the perpendicular distance from the origin to the line $\frac{x}{2} + \frac{y}{3} = 1.$ *Ans.* $\frac{6}{\sqrt{13}}.$

37. An equilateral triangle whose sides $= a$, has its vertex at the origin and its sides equally inclined to the positive directions of the axes; find the co-ordinates of the other two vertices and of the point bisecting the base.

$$\textit{Ans.} \begin{cases} x = \frac{a}{4}(\sqrt{6} + \sqrt{2}), & y = \frac{a}{4}(\sqrt{6} - \sqrt{2}); \\ x = \frac{a}{4}(\sqrt{6} - \sqrt{2}), & y = \frac{a}{4}(\sqrt{6} + \sqrt{2}); \\ x = \frac{a}{4}\sqrt{6}, & y = \frac{a}{4}\sqrt{6}. \end{cases}$$

38. Find the equation of the lines which pass through the point (1, 3) and make an angle of 30° with the line $2y - x + 1 = 0.$

Ans. $11y - (8 \pm 5\sqrt{3})\,x - 5\,(5 \mp \sqrt{3}) = 0.$

39. Find the cosine of the angle between the lines $y - 4x + 8 = 0$ and $y - 6x + 9 = 0.$ *Ans.* $\frac{25}{\sqrt{629}}.$

40. Find the equations of the diagonals of the four-sided figure, the equations of whose sides are

$$x = 4, \quad y = 5, \quad y = x, \quad y = 2x.$$

Ans. $4y = 5x$ and $3y + 2x - 20 = 0.$

41. Find the points of intersection of the lines

$$x + 2y - 5 = 0, \quad 2x + y - 7 = 0, \quad \text{and} \quad y - x - 1 = 0,$$

and find the area of the triangle which the lines form.

Ans. Area $= 1\frac{1}{2}.$

42. The axes of co-ordinates being inclined to each other at an angle of 45°, a right line passes through the points (2, 3) and (3, 2). Find its equation and the value of α.

Ans. $y = -x + 5,\ \alpha = \tan^{-1} - (1 + \sqrt{2}).$

43. The axes of co-ordinates being inclined to each other at an angle of 60°, find the equation of a line parallel to the line $(x + y = 3a)$, and at a distance from it equal to $\frac{1}{2}a\sqrt{3}.$

Ans. $x + y = 2a$ or $x + y = 4a$ (according to the side on which the line is drawn).

44. Find the polar equation of a line the nearest point in which is 8 from the pole, and the perpendicular to which makes an angle of 30° with the initial line. Where does the line cut the initial line? What values of θ make r infinite?

Ans. $r = \frac{8}{\cos(\theta - 30°)};\ r = \frac{16}{\sqrt{3}};\ \theta = 120°$ and $300°.$

45. Find the polar equation of the line perpendicular to the initial line, and which cuts it at 3 to the left of the pole. What is the value of r when $\theta = 60°$? What is the value of r when θ is 120°?

Ans. $r = -\frac{3}{\cos\theta};\ r = -6;\ r = +6.$

46. Find the polar co-ordinates of the intersection of the lines

$$r = \frac{2a}{\cos\left(\theta - \frac{\pi}{2}\right)} \quad \text{and} \quad r = \frac{a}{\cos\left(\theta - \frac{\pi}{6}\right)},$$

and also the angle between them.

Ans. $r = 2a,\ \theta = \frac{\pi}{2};$ angle $= \frac{\pi}{3}.$