CHAPTER III.

TRANSFORMATION OF CO-ORDINATES.

32. We saw in Art. 22 that the *general* equation of a right line is of the form $y = ax + b$, but that the equation takes simpler forms in particular cases. If the origin is *on the line*, the equation becomes $y = ax$; if the axis of x *coincides with the line*, the equation becomes $y = 0$. In a similar manner, we shall see that the equation of a *curve* often assumes simpler forms, according to the position of the origin and of the axes. For example, the circle, Fig. 33′, when referred to the axes XX′, YY′, has for its equation

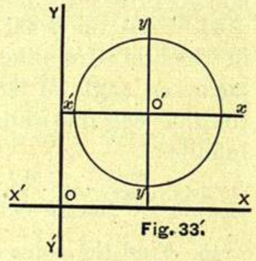

Fig. 33′.

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0,$$

where (a, b) is the centre O', and c is the radius; but when referred to the axes xx', yy', its equation is

$$x^2 + y^2 = c^2.$$

It becomes therefore desirable to be able to change the reference of any locus from one set of axes to another, or from one system of co-ordinates to another. The operation is called the **Transformation of Co-ordinates,** and may consist either in changing the origin without disturbing the directions of the axes, or changing the directions of the axes without disturbing the origin, or changing both the position of the origin and the directions of the axes.

The axes or system from which we pass is called the **Old,** or **Primitive Axes** or **System**; the axes or system *to* which we pass is called the **New Axes** or **System.** The transformation is effected by substituting for the old co-

ordinates of any point their values in terms of the new co-ordinates of the same point and certain constants.

33. *To find the formulæ for passing from one system of co-ordinates to another, the new axes being parallel to the old.*

Let OX, OY be the old axes; $O'x$, $O'y$ the new axes respectively parallel to the old. Let m and n be the co-ordinates of the new origin referred to the old axes. Let P be any point; x, y its co-ordinates referred to the old axes, and x', y' its co-ordinates referred to the new axes. Then

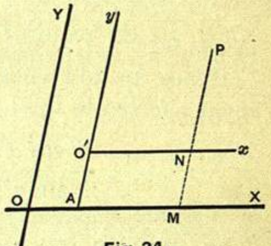

Fig. 34.

$$OA = m; \qquad AO' = n;$$

$$x = OM = OA + AM = OA + O'N = m + x';$$

$$y = MP = MN + NP = AO' + NP = n + y';$$

that is, $\qquad x = m + x', \quad$ and $\quad y = n + y',$

which are the required formulæ.

These formulæ are equally true whether the axes be rectangular or oblique.

34. *To find the formulæ for changing the direction of the axes without changing the origin, both systems being rectangular.*

Let OX, OY be the old axes; Ox, Oy the new axes. Let the angle $XOx = \alpha$. Let P be any point; x, y its co-ordinates referred to the old axes; x', y' its co-ordinates referred to the new axes. Draw PM and PR parallel to OY and Oy respectively;

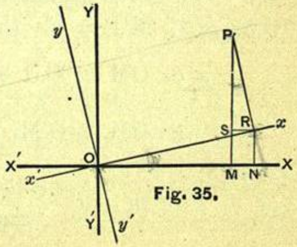

Fig. 35.

and RN and RS parallel to OY and OX respectively. Then

$$x = \text{OM} = \text{ON} - \text{SR} = \text{OR} \cos \text{XO}x - \text{PR} \sin \text{SPR}$$
$$= x' \cos \alpha - y' \sin \alpha,$$
$$y = \text{PM} = \text{RN} + \text{SP} = \text{OR} \sin \text{RON} + \text{RP} \cos \text{SPR}$$
$$= x' \sin \alpha + y' \cos \alpha,$$

which are the required formulæ.

Hence, to find what the equation of any locus becomes when referred to the new axes, we must substitute in it

$x \cos \alpha - y \sin \alpha$ for x,

and $x \sin \alpha + y \cos \alpha$ for y, and reduce.

35. *To find the general formulæ for passing from one rectilinear system to another.*

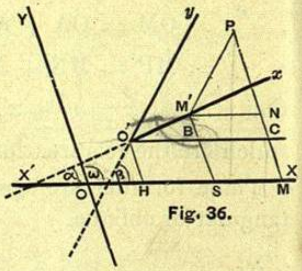
Fig. 36.

Let OX, OY be the old axes; O$'x$, O$'y$ the new axes. Let the angle which the new axis of x makes with the old axis of x be α; the angle which the new axis of y makes with the old axis of x be β; the angle included between the old axes be ω; and let the co-ordinates of the new origin be $\text{OH} = m$, $\text{HO}' = n$. Let P be any point, its co-ordinates referred to the old axes being $\text{OM} = x$, $\text{MP} = y$; its co-ordinates referred to the new axes being $\text{O}'\text{M}' = x'$, $\text{M}'\text{P} = y'$; then we have,

$$x = \text{OM} = \text{OH} + \text{O}'\text{B} + \text{M}'\text{N}$$
$$= \text{OH} + \text{O}'\text{M}' \frac{\sin \text{O}'\text{M}'\text{B}}{\sin \text{O}'\text{BM}'} + \text{M}'\text{P} \frac{\sin \text{M}'\text{PN}}{\sin \text{M}'\text{NP}},$$

or $$x = m + x' \frac{\sin(\omega - \alpha)}{\sin \omega} + y' \frac{\sin(\omega - \beta)}{\sin \omega}. \qquad (1)$$

$$y = \text{PM} = \text{HO}' + \text{BM}' + \text{NP}$$
$$= \text{HO}' + \text{O}'\text{M}' \frac{\sin \text{BO}'\text{M}'}{\sin \text{O}'\text{BM}'} + \text{M}'\text{P} \frac{\sin \text{NM}'\text{P}}{\sin \text{M}'\text{NP}},$$

or $$y = n + x' \frac{\sin \alpha}{\sin \omega} + y' \frac{\sin \beta}{\sin \omega}, \qquad (2)$$

which are the required formulæ.

Cor. 1.—If the old axes are rectangular, $\omega = \frac{\pi}{2}$, and (1) and (2) become

$$x = m + x' \cos \alpha + y' \cos \beta,$$
$$y = n + x' \sin \alpha + y' \sin \beta,$$

which are the *formulæ to pass from rectangular axes to oblique.*

Cor. 2.—If the new axes are rectangular, $\beta = \frac{\pi}{2} + \alpha$, and (1) and (2) become,

$$x = m + x' \frac{\sin(\omega - \alpha)}{\sin \omega} - y' \frac{\cos(\omega - \alpha)}{\sin \omega},$$

$$y = n + x' \frac{\sin \alpha}{\sin \omega} + y' \frac{\cos \alpha}{\sin \omega},$$

which are the *formulæ to pass from oblique axes to rectangular.*

Cor. 3.—If both axes are rectangular,

$$\omega = \frac{\pi}{2} \quad \text{and} \quad \beta = \frac{\pi}{2} + \alpha,$$

and (1) and (2) become

$$x = m + x' \cos \alpha - y' \sin \alpha,$$
$$y = n + x' \sin \alpha + y' \cos \alpha,$$

which are the *formulæ to pass from one set of rectangular axes to another set of rectangular axes, not parallel to the old.*

36. *To find the formulæ for passing from a rectangular to a polar system.*

Let OX, OY be the rectangular axes; O′ the *pole*, and O′A or O′A′ the *initial line*. Let m, n, be the co-ordinates of O′ referred to the rectangular axes. Let P be any point in a locus, its co-ordinates being $\mathrm{OM} = x$, $\mathrm{PM} = y$, when referred to the rectangular axes; r, θ its polar co-ordinates. Let the angle $x\mathrm{O'A}$ or $x\mathrm{O'A'} = \alpha$. Then

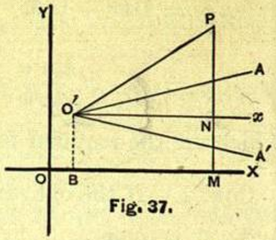

Fig. 37.

$$x = \mathrm{OM} = \mathrm{OB} + \mathrm{O'N} = m + \mathrm{O'P} \cos \mathrm{PO'N}$$
$$= m + r \cos (\theta \pm \alpha), \qquad (1)$$
$$y = \mathrm{MP} = \mathrm{BO'} + \mathrm{NP} = n + \mathrm{O'P} \sin \mathrm{PO'N}$$
$$= n + r \sin (\theta \pm \alpha), \qquad (2)$$

which are the required formulæ.

COR.—If the initial line is parallel to the old axis of x, $\alpha = 0$, and (1) and (2) become,

$$x = m + r \cos \theta. \qquad (3)$$
$$y = n + r \sin \theta. \qquad (4)$$

If the pole is at the origin, (3) and (4) become

$$x = r \cos \theta, \quad (5), \qquad y = r \sin \theta. \quad (6)$$

NOTE.—Formulæ (5) and (6) are the ones most generally used, and should be carefully remembered.

37. *To find the formulæ for passing from a polar to a rectangular system of co-ordinates.*

It is easily seen from Fig. 37 that

$$r = \sqrt{(y-n)^2 + (x-m)^2}.$$

$$\cos (\theta \pm \alpha) = \frac{x - m}{\sqrt{(y-n)^2 + (x-m)^2}}.$$

$$\sin (\theta \pm \alpha) = \frac{y - n}{\sqrt{(y-n)^2 + (x-m)^2}}.$$

38. The student will bear in mind that no change is made in the locus by any of these transformations. The assemblage of points which the new equation represents is exactly the same as that represented by the old; but the new axes to which the locus is referred occupy a different position from that occupied by the old axes; and therefore the equation which expresses this *relative* position is not the same as before.

EXAMPLES.

1. The equation of a right line is

$$3x + 5y - 15 = 0;$$

find the equation of the same line referred to parallel axes whose origin is at (1, 2). *Ans.* $3x + 5y = 2$.

2. The equation of a locus is

$$x^2 + y^2 - 4x - 6y = 18;$$

what will this equation become if the origin be moved to the point (2, 3)? *Ans.* $x^2 + y^2 = 31$.

3. What does the equation $x^2 - y^2 + 2x + 4y = 0$ become when the origin is transformed to the point $(-1, 2)$, the new axes being parallel to the old? *Ans.* $x^2 - y^2 + 3 = 0$.

4. What does $x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2$ become when the new origin is at the point (a, b), the new axes being parallel to the old? *Ans.* $x^2 + y^2 = r^2$.

5. What does the equation $ax + by + c = 0$ become when the new origin is at the point $(-\frac{c}{a}, 0)$, the new axes being parallel to the old? *Ans.* $ax + by = 0$.

6. Show that $6x^2 + 5xy - 6y^2 - 17x + 7y + 5 = 0$, when referred to axes through a certain point parallel to the old axes will become $6x^2 + 5xy - 6y^2 = 0$.

Find what the following eight equations become when the origin is transformed to the point (1, 1), the new axes being parallel to the old:

7. $x^2 + xy - 3x - y + 2 = 0$. *Ans.* $x^2 + xy = 0$.

8. $xy - x - y + 1 = 0$. *Ans.* $xy = 0$.

9. $xy - y^2 - x + y = 0$. *Ans.* $xy - y^2 = 0$.

10. $x^2 - y^2 - 2x + 2y = 0$. *Ans.* $x^2 - y^2 = 0$.

11. $x^2 + y^2 = 2$. *Ans.* $x^2 + y^2 + 2x + 2y = 0$.

12. $x^2 + y^2 - 2x - 2y = 0$. *Ans.* $x^2 + y^2 = 2$.

13. $x^2 + y^2 - 2x = 0$. *Ans.* $x^2 + y^2 + 2y = 0$.

14. $x^2 + y^2 - 2y = 0$. *Ans.* $x^2 + y^2 + 2x = 0$.

15. Transform the equations,

$$x^2 + y^2 - 2hx - 2ky = a^2 - h^2 - k^2,$$

and

$$x^2 + y^2 = c^2 + h^2 + k^2$$

to parallel axes through the point (h, k).

Ans. $x^2 + y^2 = a^2,\ x^2 + y^2 + 2hx + 2ky = c^2$.

16. Transform the equations,

$$x^2 + y^2 = a^2,\ x^2 + y^2 + 2gx + 2fy + c^2 = 0,$$

to parallel axes through the point $(-g, -f)$.

$$Ans. \begin{cases} x^2 + y^2 - 2gx - 2fy + g^2 + f^2 - a^2 = 0, \\ x^2 + y^2 - g^2 - f^2 + c^2 = 0. \end{cases}$$

17. Find what the following four equations become when the axes are turned through an angle of 45°:

$$(1)\ xy = 0;\ (2)\ x + y = \sqrt{2};\ (3)\ x - y = \tfrac{1}{2}\sqrt{2};$$
$$(4)\ x^2 + y^2 = 1.$$

$$Ans. \begin{cases} (1)\ x^2 - y^2 = 0;\ (2)\ x = 1; \\ (3)\ y = -\tfrac{1}{2};\ (4)\ x^2 + y^2 = 1. \end{cases}$$

18. What does the equation $y^2 - x^2 = 4$ become when the axes are turned through an angle of 45°? *Ans.* $xy = 2$.

19. What does the equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 1$ become when the axes are turned through an angle of 30°?

Ans. $5x^2 + y^2 = 1$.

20. Show that the equation $4xy - 3x^2 = a^2$ will become $x^2 - 4y^2 = a^2$ when the axes are turned through the angle whose tangent is 2.

21. Transform the equation $x^2 - 2xy + y^2 + x - 3y = 0$ to axes through the point $(-1, 0)$ parallel to the lines bisecting the angles between the old axes.

Ans. $\sqrt{2}\,y^2 - x = 0$.

22. What does the equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ become when the origin is transformed to the point (2, 3), the new axes being turned through an angle of 45°? *Ans.* $4x^2 + 2y^2 = 1$.

23. Transform the equation $2x^2 - 3xy + 2y^2 = 1$ from a rectangular to an oblique system, the origin remaining the same, the new axis of x coinciding with the old, and the new axis of y bisecting the angle between the old axes.

Ans. $4x^2 + \sqrt{2}xy + y^2 = 2$.

24. The equation of a locus is $y^2 - x^2 = 16$; what will this equation become if transformed to axes bisecting the angles between the given axes? *Ans.* $xy = 8$.

25. Transform the equation $2x^2 - 5xy + 2y^2 = 4$ from axes inclined to each other at an angle of 60°, to the axes which bisect the angles between the given axes.

Ans. $x^2 - 27y^2 + 12 = 0$.

26. Transform the equation $y^2 + 4ay \cot \alpha - 4ax = 0$ from a rectangular system to an oblique system inclined at an angle α, the origin remaining the same, and the new axis of x coinciding with the old.

Ans. $y^2 \sin^2 \alpha = 4ax$.

27. The equation of a locus is $x^4 + y^4 + 6x^2y^2 = 2$; what will be the equation if the axes are turned through an angle of 45°? *Ans.* $x^4 + y^4 = 1$.

28. Transform $xy = 0$ and $x^2 - y^2 = 0$ to the point (2, 3), the new axes making an angle of 30° with the old, both sets of axes being rectangular.

$$Ans. \begin{cases} (x^2 - y^2)\sqrt{3} + 2xy + (4 + 6\sqrt{3})\,x + (4\sqrt{3} - 6)\,y \\ +24 = 0, \text{ and } x^2 - y^2 - 2\sqrt{3}xy + 2\sqrt{3}\,(2 - \sqrt{3})\,x \\ -2\,(2 + 3\sqrt{3})\,y = 10. \end{cases}$$

29. Transform $y^2 - 4ax = 0$ to the point $(am^2, 2am)$ as origin, the new axes making an angle $\cot^{-1} m$ with the old, both sets being rectangular.

Ans. $(x + my)^2 + 4a\,(1 + m^2)^{\frac{3}{2}}\,y = 0$.

30. Transform $x^2 + y^2 = 7ax$ to polar co-ordinates, the pole being at the origin, and the initial line coincident with the axis of x.

Ans. $r = 7a \cos \theta$.

31. Change the equations $r^2 = a^2 \cos 2\theta$ and $r^2 \cos 2\theta = a^2$ into equations between x and y.

Ans. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ and $x^2 - y^2 = a^2$.

3rd day

39. The following exercises are designed to give the student an opportunity for making an effort to produce the equations himself. The fundamental idea of Analytic Geometry is that every geometric condition to be fulfilled by a point leads to an equation which must be satisfied by its co-ordinates. It is important that the student should become able to express by an equation any given geometric condition; he should understand that ability to investigate, to reason for himself, is the chief object for the attainment of which he should strive. For this purpose he should diligently apply himself in working out examples, until he has acquired readiness and accuracy in so doing. In attempting to solve these examples, the student will find that very much depends upon a proper selection of the origin and axes of co-ordinates, and the application of the proper equations and formulæ. He should, in every case, consider the problem well, and form a definite plan before he attempts the solution. He will often be unable to carry out his original plan, and will have to abandon it, although it may have seemed at first the most suitable. Such failures, however, are not to be considered as waste of time; for it is only by thorough application that the student is enabled, gradually, to become expert in obtaining solutions; and a *failure* will often suggest some method by which a problem may be solved.

[The student need not necessarily tarry till he has mastered all the examples in any one article.]

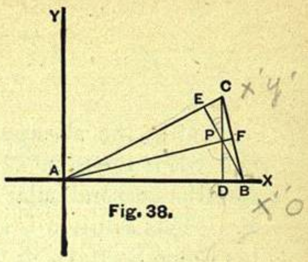

Fig. 38.

1. Prove that the perpendiculars drawn from the vertices of a triangle to the opposite sides meet in a point.

Let ABC be the triangle; AF, BE, CD the perpendiculars. Assume AX and AY as the rectangular axes; and let the co-ordinates of B and C be x'', 0, and x', y', respectively. Now, if it can be shown that x' is the abscissa of the point of intersection of the perpendiculars AF and BE, the proposition will be proved. We therefore have to find the equations of AF and BE, and then their intersection.

Since AC passes through the origin and the point C, (x', y'), its equation (Art. 26, Cor. 4) is

$$y = \frac{y'}{x'}x. \tag{1}$$

Since BC passes through B $(x'', 0)$ and C (x', y'), its equation (Art. 26) is

$$y = \frac{y'}{x' - x''}(x - x''). \tag{2}$$

Since BE passes through B $(x'', 0)$ and is perpendicular to (1), its equation (Art. 28, Cor. 3) is

$$y = -\frac{x'}{y'}(x - x''). \tag{3}$$

Since AF passes through the origin $(0, 0)$ and is perpendicular to (2), its equation is

$$y = -\frac{x' - x''}{y'}x. \tag{4}$$

At the point P, where (3) and (4) intersect, their ordinates must be identical; hence, equating their values, we have

$$\frac{x'}{y'}(x - x'') = \frac{x' - x''}{y'} x.$$

$$\therefore \quad x = x'.$$

That is, the abscissa of the point of intersection of AF and BE is the same as the abscissa of the point C; therefore the perpendicular CD passes through the intersection P. [This solution is similar to the one given by Puckle in his Conic Sections, p. 77.]

2. Given the base ($= 2m$) of a triangle, and the difference between the squares of its sides ($= n^2$), to find the locus of its vertex.

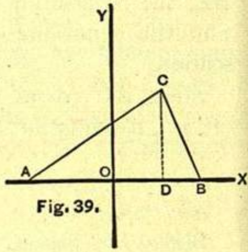

Fig. 39.

Take for axes the base and a perpendicular through its middle point, and let the co-ordinates of the vertex C be x, y. Then

$$\overline{AC}^2 = (m + x)^2 + y^2;$$

$$\overline{BC}^2 = (m - x)^2 + y^2.$$

$$\therefore \qquad \overline{AC}^2 - \overline{BC}^2 = 4mx = n^2,$$

or

$$x = \frac{n^2}{4m},$$

the equation required. The locus is therefore a line perpendicular to the base, at the distance of $\frac{n^2}{4m}$ from the middle point.

3. A line is drawn parallel to the base of a triangle, and its extremities are joined transversely to those of the base; to find the locus of the point of intersection of the joining lines.*

* This solution is from Salmon's Conic Sections, p. 44.

Take for axes the sides of the triangle, AB and AC. Let AB $= a$, AC $= b$, and let the lengths of the proportional intercepts made by the parallel be ka, kb. Then the equations of the transversals will be as follows:

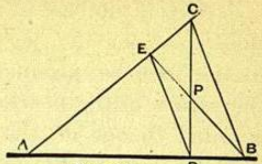

Fig. 40.

Equation of BE (Art. 22) is $\dfrac{x}{a} + \dfrac{y}{kb} = 1.$

Equation of CD (Art. 22) is $\dfrac{x}{ka} + \dfrac{y}{b} = 1.$

Subtract one from the other; divide by the constant, $\left(1 - \frac{1}{k}\right)$, and we get for the equation of the locus,

$$\frac{x}{a} - \frac{y}{b} = 0, \qquad \text{or} \qquad y = \frac{b}{a}x,$$

a right line passing through the origin and the middle of BC.

4. Given the base of a triangle $= 2m$, and the sum of the cotangents of the base angles $= n$, to find the locus of its vertex.

From Fig. 39 we have,

$$\cot A = \frac{AD}{DC} = \frac{m + x}{y},$$

$$\cot B = \frac{m - x}{y}.$$

Hence the required equation is

$$\frac{2m}{y} = n, \qquad \text{or} \qquad y = \frac{2m}{n},$$

a right line parallel to the base, at the distance $\frac{2m}{n}$ from it.

5. Given the base of a triangle $= 2m$, and the sum of the sides $= s$; let the perpendicular to the base be produced beyond the vertex until its whole length is equal to one of the sides; to find the locus of the extremity of the perpendicular.

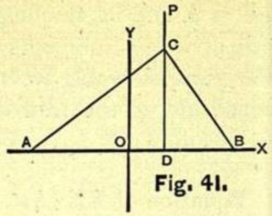

Fig. 41.

Take the origin at the middle of the base, axes rectangular, as in Fig. 41. The abscissa of P is $OD = x$, and the ordinate is $DP = AC = y$.

$$BC = s - AC = s - y;$$

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2AB \times AD,$$

or $$(s - y)^2 = y^2 + 4m^2 - 4(m + x)\,m,$$

or $$s^2 - 2sy = -4mx;$$

therefore, $$y = \frac{2m}{s}x + \frac{s}{2},$$

which is the equation of the required locus, the equation of a right line.

6. Prove that the three perpendiculars through the middle points of the sides of a triangle meet in a point.

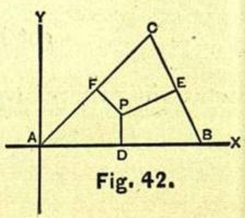

Fig. 42.

Suggestions.—1st, find equation of AC; 2d, find equation of BC; 3d, find equation of FP perpendicular to AC; 4th, find equation of EP perpendicular to BC; 5th, find abscissa of point of intersection of FP and EP; $\therefore$ etc.

7. A point moves so that its distances from two points (3, 4) and (5, -2) are equal to each other: find the equation of its locus. *Ans.* $x - 3y = 1$.

8. A point moves so that the sum of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is constant $(2c^2)$: find the equation of its locus.
Ans. $x^2 + y^2 = c^2 - a^2$.

9. A point moves so that the difference of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is constant (c^2): find the equation of its locus.
Ans. $4ax = \pm c^2$.

10. A point moves so that its distance from the origin is twice its distance from the axis of x: find the equation of its locus. *Ans.* $3y^2 - x^2 = 0$.

11. A point moves so that it is always equally distant from the axis of x and from the point (1, 1): find the equation of its locus. *Ans.* $x^2 - 2x - 2y + 2 = 0$.

12. A point moves so that the difference of its distances from two fixed lines perpendicular to each other is constant and $= k$: find the equation of its locus. *Ans.* $x - y = k$.

13. A point moves so that the sum of its distances from two fixed lines inclined to each other at an angle of 30° is constant $= k$: find the equation of its locus.
Ans. $x + y = 2k$.

14. A point moves so that the ratio of its distances from two fixed lines is constant and $= k$: find the locus.
Ans. $y = kx$.

15. A point moves so that the square of its distance from the origin is twice the square of its distance from the axis of x: find the equation of its locus. *Ans.* $y^2 - x^2 = 0$.

16. A point moves so that its distance from the axis of x is three times its distance from the axis of y: find the equation of its locus. *Ans.* $y = 3x$.

17. A point moves so that the squares of its distances from the origin and the point (2, 0) are equal: find the equation of its locus. *Ans.* $x = 1$.

18. Find the locus of a point equidistant from the points (1, 1) and (− 1, − 1). *Ans.* $x + y = 0$.

19. Find the locus of a point which moves so that the sum of the squares of its distances from the axes is equal to 2. *Ans.* $x^2 + y^2 = 2$.

20. Find the locus of a point the square of whose distance from the point (0, 1) is equal to unity.

Ans. $x^2 + y^2 - 2y = 0$.

21. Find the locus of a point such that the square of its distance from the point (4, 0) is four times the square of its distance from the point (1, 0). *Ans.* $x^2 + y^2 = 4$.

22. Find the locus of a point which moves so that the difference of the squares of its distances from two given fixed points is always a constant $= k$.

Let $2a$ be the distance between the given points; take this line as axis of x and the perpendicular at its mid point as axis of y. *Ans.* $x = \dfrac{k}{4a}$.

23. Find the equation of the line which is equidistant from the lines $x + 1 = 0$ and $x = 3$. *Ans.* $x = 1$.

24. Find the equation of the line which is equidistant from the lines $y = b$ and $y = b'$. *Ans.* $y = \frac{1}{2}(b + b')$.

25. Find (1) the equations of the lines through the point (0, 2) making angles $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$ with the axis of x; and (2) the lines parallel to them cutting the axis of y at a distance 2 below the origin.

$$\textit{Ans.} \begin{cases} (1)\ y = x\sqrt{3} + 2, \text{ and } y = -x\sqrt{3} + 2; \\ (2)\ y = x\sqrt{3} - 2, \text{ and } y = -x\sqrt{3} - 2. \end{cases}$$

26. From a point P perpendiculars PM, PN are dropped on two fixed lines OX and OY: find the locus of P when OM + ON = a constant k.

Ans. Taking the fixed lines for axes, and θ for the included angle, the equation is $(x + y)(1 + \cos\theta) = k$.

27. Prove that the lines drawn from the vertices of a triangle to the middle points of the opposite sides pass through the same point.

[Take for axes EB and EC in Fig. 43.]

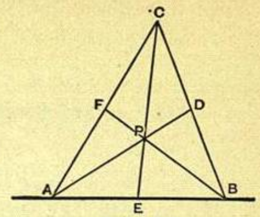

Fig. 43.

28. Given two fixed points A and B, one on each of the axes of co-ordinates, at the respective distances a and b from the origin; if A′ and B′ be taken on the axes so that OA′ + OB′ = OA + OB, find the locus of the intersection of AB′ and A′B. *Ans.* $x + y = a + b$.

29. PP′ $= a$, and QQ′ $= b$ are any two parallels to the sides of a given parallelogram, to find the locus of the intersection of the lines PQ and P′Q′.

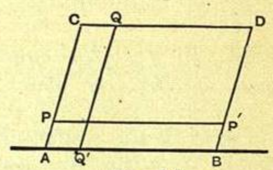

Fig. 44.

Take AB, AC for the axes of co-ordinates; let AQ′$=m$, AP$=n$. Then, 1st, find the equation of the line joining P $(0, n)$ to Q (m, b); 2d, find the equation of the line joining P′ (a, n) to Q′ $(m, 0)$; 3d, add these two equations together, and get for the locus, $y = \dfrac{b}{a}x$, the equation of the diagonal of the parallelogram.

30. On the two sides of a right-angled triangle, squares are constructed; from the acute angles, diagonals are drawn, crossing the triangle to the vertices of these squares; and from the right angle a perpendicular is let fall upon the hypothenuse; prove that the diagonals and the perpendicular meet in one point. [Take the two sides for axes, and call their lengths a and b.]