CHAPTER IV.

THE CIRCLE.

40. We shall now consider loci whose equations are of the second degree, beginning with the circle, which is the simplest of these loci.

A circle is a plane figure bounded by a line every point of which is equally distant from a point within called the centre. In Analytic Geometry, the term **Circle** is applied generally not to the area of the figure but to the bounding line, while in Plane Geometry the term is confined to the area, the bounding line being called the circumference.

41. *To find the equation of the circle whose centre and radius are given.*

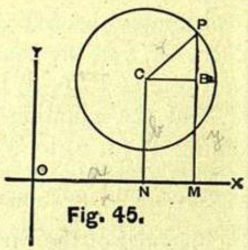

Fig. 45.

Let C be the centre of the circle, P any point on its circumference, and r the radius of the circle. Let a, b be the co-ordinates of C; x, y the co-ordinates of P. Draw CN, PM parallel to OY, and CB parallel to OX. Then we have

$$\overline{CB}^2 + \overline{BP}^2 = \overline{CP}^2;$$

or
$$(x-a)^2 + (y-b)^2 = r^2. \tag{1}$$

This equation is true for every position of P; hence it expresses the relation between the co-ordinates of every point of the circle, and is therefore the required equation.

If the axes are oblique, and inclined to each other at an angle $= \omega$, the equation is

$$(x-a)^2 + (y-b)^2 + 2(x-a)(y-b)\cos\omega = r^2. \tag{2}$$

COR. 1.—If the origin be transferred to the centre of the circle, then $a = 0$, $b = 0$, and equation (1) becomes

$$x^2 + y^2 = r^2. \tag{3}$$

This equation may be written in the symmetric form

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1. \tag{4}$$

NOTE.—We see from (1), (2), (3), and (4), that:

(1). *The equation of a circle is of the second degree.*

(2). *The coefficients of x^2 and y^2 are equal.*

(3). *There is no term involving the product xy in* (1) *or* (3).

COR. 2.—If the origin be transferred to the circumference, and the diameter which passes through the origin be taken for the axis of x, then $a = r$, $b = 0$, and equation (1) becomes

$$(x-r)^2 + y^2 = r^2,$$

or
$$y^2 = 2rx - x^2. \tag{5}$$

It may be observed here that, if the origin is on the curve, there will be no term which does not involve either x or y; for the equation is satisfied by the co-ordinates of the origin, $x = 0$, $y = 0$. The same argument proves that *if an equation of any degree wants the absolute term, the curve represented passes through the origin.*

In equation (5) we suppose the origin to be at the *left-hand* vertex of the diameter. This convention is adopted by custom.

COR. 3.—To find where (1) cuts the axis of x, we make $y = 0$, and have

$$x = a \pm \sqrt{r^2 - b^2}.$$

If $b^2 < r^2$, the two values of x are *real* and *unequal*, showing that the curve cuts the axis of x in two points.

If $b^2 = r^2$, the two values of x are *real* and *equal*, showing that the curve touches the axis of x; that is, is tangent to it.

If $b^2 > r^2$, the two values of x are *imaginary*, showing that the curve does not cut the axis of x.

Similarly, it may be shown that the curve cuts the axis of y in two points, is tangent to the axis of y, or does not cut the axis of y, according as $a^2 < r^2$, $= r^2$, or $> r^2$.

COR. 4.—To find where (3) cuts the axis of x, we make $y = 0$, and get $x = \pm r$, showing that the curve cuts the axis of x in two points on different sides of the origin, at the distance r from it.

To find where (3) cuts the axis of y, we make $x = 0$, and obtain $y = \pm r$, showing that the curve cuts the axis of y at r above and r below the origin.

Solving (3) with respect to y, we obtain,

$$y = \pm \sqrt{r^2 - x^2},$$

which shows that, for every value of x between $+r$ and $-r$, y has two real values, numerically equal, with contrary signs; hence the curve is symmetrically situated with respect to the axis of x. If $x = +r$ or $-r$, the two values of y are equal to 0, which shows that the ordinates at these two points are tangent to the curve. If $x > +r$ or $< -r$, y becomes imaginary, which shows that the curve does not extend beyond the two tangents just described.

Similarly it may be shown that the curve is symmetrical with respect to the axis of y, and that it does not extend beyond the two tangents drawn through the extremities of the vertical diameter.

COR. 5.—To find where (5) cuts the axis of x, we make $y = 0$, and obtain

$$x(2r - x) = 0.$$

This equation is satisfied by supposing $x = 0$, or

$$2r - x = 0,$$

from the last of which we get,

$$x = 2r;$$

hence the curve cuts the axis of x at the origin, and at the distance $2r$ to the right of it.

To find where the curve cuts the axis of y, we make $x = 0$, and obtain $y = \pm 0$, which shows that the curve touches the axis of y at the origin.

COR. 6.—If (x', y') and (x'', y'') be any two points on the curve, we shall have from (3),

$$y'^2 = r^2 - x'^2; \qquad y''^2 = r^2 - x''^2.$$

Hence, forming a proportion, we have,

$$y'^2 : y''^2 :: (r - x')(r + x') : (r - x'')(r + x'').$$

That is, the squares of any two ordinates to any diameter are to each other as the rectangles of the segments into which they divide the diameter.

EXAMPLES.

1. The equation of a circle is

$$x^2 + y^2 + 4x - 8y - 5 = 0;$$

find the co-ordinates of the centre, and the radius.

Writing the equation in the form of (1), it becomes

$$(x + 2)^2 + (y - 4)^2 = 25;$$

from which we see that the co-ordinates of the centre and the radius are $(-2, 4)$ and 5.

2. The equations of two circles are

$$x^2 + y^2 - 2x + 4y + 1 = 0;$$
$$3x^2 + 3y^2 - 5x - 7y + 1 = 0;$$

find the co-ordinates of the centre, and the radius in each circle.

Ans. $\begin{cases} (1, -2) \text{ and } 2 \text{ in the first case;} \\ (\frac{5}{6}, \frac{7}{6}) \text{ and } \frac{1}{6}\sqrt{62} \text{ in the second.} \end{cases}$

3. Form the equation of the circle whose centre is (3, 4), and whose radius = 2.

Ans. $x^2 + y^2 - 6x - 8y + 21 = 0.$

4. Form the equation of the circle whose centre is $(5, -3)$, and whose radius $= \sqrt{7}$, when $\omega = 60°$.

Ans. $x^2 + y^2 + xy - 7x + y + 12 = 0.$

5. Find the equation of the circle which passes through the points $(-6, -1)$, $(0, 0)$, $(0, -1)$; and also the co-ordinates of the centre, and the radius.

[These three sets of co-ordinates must each satisfy equation (1), giving three equations from which to obtain the values of a, b, and r.]

Ans. $x^2+y^2+6x+y=0$; and $(-3, -\frac{1}{2})$ and $\frac{1}{2}\sqrt{37}$.

6. Find the equation of a circle referred to its diameter and left-hand vertex that shall pass through the point (2, 3).

Ans. $y^2 = \frac{13}{2}x - x^2.$

42. *To find the equation of the tangent at any point of a circle.*

The **Tangent** *to any curve is the line joining two indefinitely near points on that curve.*

Hence, its equation will be found by first forming the equation of the secant drawn through any two points (x', y'), (x'', y'') on the curve, and then allowing the first point to remain fixed while the second moves on the curve up to the first; the secant in its limiting position will become the tangent to the curve at the first point, and the equation of the secant will become the equation of the tangent.

The equation of the circle, the origin at the centre, is

$$x^2 + y^2 = r^2. \tag{1}$$

The equation of the secant through (x', y') and (x'', y'') is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \tag{2}$$

Since (x', y') and (x'', y'') are both on the circle,* they will satisfy equation (1); therefore,

$$x'^2 + y'^2 = r^2, \tag{3}$$

and

$$x''^2 + y''^2 = r^2. \tag{4}$$

Subtract (4) from (3), transpose and factor, and we have $(y' - y'')(y' + y'') = -(x' - x'')(x' + x'')$; from which we obtain $\frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''}$. Hence, substituting in (2), it becomes

$$y - y' = -\frac{x' + x''}{y' + y''}(x - x'). \tag{5}$$

Now when the second point coincides with the first, we have $x'' = x'$, $y'' = y'$; therefore (5) becomes

$$y - y' = -\frac{x'}{y'}(x - x'), \tag{6}$$

which is the equation of the tangent at the point (x', y'), $-\frac{x'}{y'}$ being the tangent of the angle which the tangent to the curve at the point (x', y') makes with the axis of x.

Multiplying (6) by y', transposing, and remembering that $x'^2 + y'^2 = r^2$, we get

$$xx' + yy' = r^2, \tag{7}$$

a form very similar to the equation of the circle.

Equation (7) may be written in the symmetric form,

$$\frac{xx'}{r^2} + \frac{yy'}{r^2} = 1. \tag{8}$$

---

* The object of this transformation is to find the value of $\frac{y'-y''}{x'-x''}$ when the two points (x', y'), (x'', y'') are placed on the circle and then made to coincide.

note 9.

43. *To find the equation of the normal at any point of a circle.*

The **Normal** *at any point of a curve is the right line drawn through that point at right angles to the tangent to the curve at the same point.*

The equation of the tangent to a circle at the point (x', y'), Art. 42, is

$$xx' + yy' = r^2,$$

or

$$y = -\frac{x'}{y'}x + \frac{r^2}{y'};$$

therefore (Art. 28, Cor. 3), the equation of a right line through (x', y') perpendicular to the tangent at the same point, is

$$y - y' = \frac{y'}{x'}(x - x'), \qquad \text{or} \qquad y = \frac{y'}{x'}x.$$

Since this equation is satisfied by the values $x = 0$, $y = 0$, the normal at any point passes through the origin of co-ordinates, that is, through the centre of the circle.

SCH. 1.—The **Subtangent** is the distance from the point in which the tangent intersects the axis of x to the foot of the ordinate from the point of tangency; or it is the projection of the corresponding portion of the tangent upon the axis of x.

SCH.—The **Subnormal** is the distance from the foot of the ordinate of the point in the curve to which the normal is drawn to the point of intersection of the normal with the axis of x; or it is the projection of the corresponding portion of the normal upon the axis of x.

In Fig. 46, TP is the *tangent* to the curve at the point P; MP is the *ordinate of the point of tangency*; PN, the *normal*; MT, the *subtangent*; MN, the *subnormal*.

note 10

From the figure we have

$$TM = \frac{MP}{\tan MTP};$$

or, *Subtangent* (Art. 43) $= -\dfrac{y'^2}{x'}$

Also, $MN = MP \tan MTP$;

or, *Subnormal* $= -x'$,

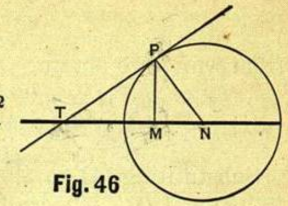

Fig. 46

which shows that the normal passes through the centre of the circle. (See Art. 43.)

EXAMPLES.

1. Find the equation of the tangent to the circle

$$x^2 + y^2 = 25,$$

at the point whose abscissa is $\sqrt{7}$.

Ans. $\sqrt{7}\,x \pm \sqrt{18}\,y = 25$.

2. Find the subtangent in the last example.

Ans. Subtangent $= -\dfrac{18}{\sqrt{7}}$.

3. Find the equations of two right lines which touch the circle $x^2 + y^2 = 10$, at points the common abscissa of which is one. *Ans.* $x \pm 3y = 10$.

note 11

44. If the equation of a circle be given in the form

$$(x - a)^2 + (y - b)^2 = r^2, \tag{1}$$

we may find the equation of the tangent at any point, in the same way as in Art. 42.

Let (x', y') be the point on the circle at which the tangent is drawn; (x'', y'') a second point on the circle; then these points will satisfy (1), giving

$$(x' - a)^2 + (y' - b)^2 = r^2, \tag{2}$$

$$(x'' - a)^2 + (y'' - b)^2 = r^2. \tag{3}$$

Subtract (3) from (2), transpose and factor, and we have

$$(x' - x'')(x'' + x' - 2a) + (y' - y'')(y'' + y' - 2b) = 0,$$

from which we obtain,

$$\frac{y' - y''}{x' - x''} = -\frac{x'' + x' - 2a}{y'' + y' - 2b}. \qquad (4)$$

Substituting (4) in the equation of the secant through (x', y') and (x'', y''), we have

$$y - y' = -\frac{x'' + x' - 2a}{y'' + y' - 2b}(x - x'). \qquad (5)$$

When the second point coincides with the first, we have $x'' = x'$, $y'' = y'$, and (5) becomes

$$y - y' = -\frac{x' - a}{y' - b}(x - x').$$

Clearing of fractions, transposing, and factoring, we have

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2, \qquad (6)$$

which is the equation of the tangent required, a form easily remembered, from its similarity to the corresponding equation of the circle.

EXAMPLES.

1. Find the equation of the tangent to the circle $(x - 2)^2 + (y - 3)^2 = 10$, at the point (5, 4).

Ans. $3x + y = 19$.

2. Find the equation of the tangent to the circle $x^2 + y^2 - 2y - 3x = 0$, at the origin.

Ans. $2y + 3x = 0$.

3. Find the equation of the right line passing through the origin, and tangent to the circle $x^2 + y^2 - 3x + 4y = 0$.

Ans. $4y - 3x = 0$.

4. Find the equations of the tangents to the circles $x^2 + y^2 - 6x - 2y = 0$, and $x^2 + y^2 - 5x + 6y = 0$, at the origin.

Ans. $3x + y = 0$, and $5x + 6y = 0$.

45. *To find the co-ordinates of the points in which a given right line* $\boldsymbol{y = ax + b}$ *intersects a given circle* $\boldsymbol{x^2 + y^2 = r^2}$.

Equating to each other the two values of y found from the two equations, we have, for determining the abscissas of the points of intersection, the equation

$$(1 + a^2)x^2 + 2abx = r^2 - b^2; \qquad (1)$$

hence,

$$x = \frac{-ab \pm \sqrt{(1 + a^2)r^2 - b^2}}{1 + a^2},$$

giving us two roots, *real* and *unequal*, *equal* or *imaginary*, according as $(1 + a^2)r^2$ is greater than, equal to, or less than b^2.

Hence, when the first of these conditions occurs, the right line will meet the circle in two *real* and *different* points; when the second, in two *consecutive* or *coincident* points, becoming a tangent (see Art. 42); when the third, in two *imaginary* points.

By **Consecutive Points** is meant *points whose distance apart is infinitely small;* that is, so small that we cannot assign a value too small for it. We may assign the value 0, and take the points as absolutely coincident, and hence they may be designated as **Coincident Points**, which is the language of pure Geometry; the term *consecutive* is peculiar to the Analytic method.

COR.—If the two values of x in equation (1) be equal, the two values of y in $y = ax \pm b$ must also be equal. Therefore the two points in which the line cuts the circle will be *coincident* if $b^2 = r^2(1 + a^2)$.

Hence the line $y = ax \pm r\sqrt{1 + a^2}$ will *touch* the circle $x^2 + y^2 = r^2$ for all values of a.

NOTE.—This enables us to write down at once the equation of the tangents to the given circle, which are inclined at a given angle ($\tan^{-1}a$) to the axis of x.

EXAMPLES.

1. Find the points of intersection of the circle $x^2+y^2=25$, and the line $y+x+1=0$.
Ans. $(-4, 3)$ and $(3, -4)$.

2. Find the points of intersection of the circle $x^2+y^2=25$, and the line $3y+4x+25=0$.
Ans. The line touches the circle at $(-4, -3)$.

3. Find the intersections of $x^2+y^2=65$ and $3x+y=25$.
Ans. $(7, 4)$ and $(8, 1)$.

4. Find the intersections of $x^2+y^2=25$ and $x+y=-5$.
Ans. $(0, -5)$ and $(-5, 0)$.

5. Find the points in which the circle $x^2+y^2=9$ intersects the lines $x+y+1=0$, $x+y-1=0$.
Ans. $\begin{cases}(1.55, -2.55) \text{ and } (-2.55, 1.55); \\ (2.55, -1.55) \text{ and } (-1.55, 2.55).\end{cases}$

6. Show that the circle $x^2+y^2+2x+2y+1=0$ touches the axes of co-ordinates, and find the points of contact.
Ans. $(-1, 0)$, $(0, -1)$.

7. Find the equations of the circles having their centres at the origin, and which touch the following three lines respectively:
(1) $y=2x+5$; (2) $3y=x+10$; (3) $3x+4y=10$.
Ans. (1) $x^2+y^2=5$; (2) $x^2+y^2=10$; (3) $x^2+y^2=4$.

8. Find the equations of the tangents to the circle $x^2+y^2=2$, which are inclined to the axis of x at the following angles:
(1) $45°$; (2) $120°$; (3) $-30°$; (4) $\tan^{-1}\frac{5}{12}$.
Ans. $\begin{cases}(1)\ y=x+2;\ (2)\ y+\sqrt{3}x=\pm 2\sqrt{2}; \\ (3)\ \sqrt{3}y+x=\pm 2\sqrt{2};\ (4)\ 12y=5x\pm 13\sqrt{2}.\end{cases}$

9. Show that the following lines and circles touch, and determine the points of contact in each case:
(1) $x^2+y^2+x+y=0$ and $x+y+2=0$;
(2) $y=x\sqrt{3}+9$ and $x^2+y^2=6y$.
Ans. (1) $(-1, -1)$; (2) $(-\frac{3}{2}\sqrt{3}, \frac{3}{2})$.

10. Find the equations of the tangents from the origin to the circle $x^2+y^2-6x-2y+8=0$.
Ans. $x-y=0$, and $x+7y=0$.

46. ***To find the length of the tangent drawn from any point to the circle.***

$$(x-a)^2+(y-b)^2-r^2=0. \qquad (1)$$

Let (x', y') be any point in the plane of the circle whose centre is (a, b); then (Art. 9), for the distance between (x', y') and (a, b), we have

$$\sqrt{(x'-a)^2+(y'-b)^2};$$

and since this distance is the hypothenuse of a right-angled triangle whose two sides are the radius of the circle and the corresponding tangent, we have, calling the tangent t,

$$t^2=(x'-a)^2+(y'-b)^2-r^2. \qquad (2)$$

Hence, *if the co-ordinates of any external point be substituted for x and y in the equation of a circle, in which the co-efficients of x^2 and y^2 are each unity, the result will be the square of the length of the tangent drawn from that point to the circle.*

46a. ***To find the locus of a point from which the tangents to two given circles are equal in length.***

Let $\qquad (x-a)^2+(y-b)^2-r^2=0,$
and $\qquad (x-a')^2+(y-b')^2-r'^2=0,$
be the equations of the two circles. Then by Art. 46 the squares of the tangents from any point (x, y) to the two circles are

$$(x-a)^2+(y-b)^2-r^2, \qquad (1)$$
$$(x-a')^2+(y-b')^2-r'^2. \qquad (2)$$

Since these two tangents are to be of equal length, (1) must equal (2), from which we find,

$$(x-a)^2+(y-b)^2-r^2=(x-a')^2+(y-b')^2-r'^2,$$
or $(a-a')x+(b-b')y+\frac{1}{2}(a'^2-a^2+b'^2-b^2+r^2-r'^2)=0, \qquad (3)$