which is the equation of the required locus; this locus is a right line, and is called *the* **Radical Axis** *of the two given circles.*

Hence, *the* **Radical Axis of two Circles** *is a right line from any point of which the tangents drawn to the two circles are of equal lengths.*

Cor. 1.— When the given circles intersect, the locus (3) passes through their points of intersection.

Hence, *when two circles intersect or are tangent their radical axis is their common chord or tangent.*

Cor. 2.— The equation of the line through the centres of the given circles is

$$y - b = \frac{b - b'}{a - a'}(x - a), \qquad \text{(Art. 26)}$$

which is perpendicular to the line (3). (Art. 27, Cor. 1)

Hence, *the radical axis of two circles is perpendicular to the line which joins their centres.*

Cor. 3.— Let the equations of three circles be

$$(x - a)^2 + (y - b)^2 - r^2 = 0, \qquad (4)$$

$$(x - a_1)^2 + (y - b_1)^2 - r_1^2 = 0, \qquad (5)$$

$$(x - a_2)^2 + (y - b_2)^2 - r_2^2 = 0. \qquad (6)$$

Let the radical axis of (4) and (5) meet the radical axis of (4) and (6) in P.

Then the tangents from P to (4) and (5) are equal, also the tangents from P to (4) and (6) are equal.

Therefore the tangents from P to (5) and (6) are equal; that is, P is also on the radical axis of (5) and (6).

Hence, *the three radical axes of three circles taken in pairs pass through one common point.*

The point in which the three radical axes meet is called the **Radical Centre**.

EXAMPLES.

1. Find the radical axis of the circles
$x^2 + y^2 - 4x + 4y = 1$, $x^2 + y^2 + 6x - 3y = 1$.
Ans. $10x - 7y = 0$.

2. Find the radical axis of

$$(x - 5)^2 + (y - 4)^2 = 4,$$
$$(x - 2)^2 + (y - 1)^2 = 1.$$

Ans. $x + y = \frac{11}{2}$.

3. Find the radical axis of

$$(x - 1)^2 + (y - 2)^2 = 6,$$
$$(x - 2)^2 + (y - 3)^2 = 8.$$

Ans. $x + y = 3$.

4. Find the radical centre of the three circles,

$$(x - 1)^2 + (y - 2)^2 = 7, \qquad (1)$$
$$(x - 3)^2 + y^2 = 5, \qquad (2)$$
$$(x + 4)^2 + (y + 1)^2 = 9. \qquad (3)$$

Ans. $(-\frac{1}{16}, -\frac{25}{16})$.

5. Find the radical centre of $(x - 5)^2 + (y - 6)^2 = 4$, $(x - 3)^2 + (y - 1)^2 = 1$, $(x + 1)^2 + (y + 2)^2 = 9$.

Ans. $(-\frac{79}{28}, \frac{166}{28})$.

47. *Tangents are drawn to a circle from a given external point; to find the equation of the chord joining the points of contact.*

Let x', y' be the co-ordinates of the external point P′; x_1, y_1, the co-ordinates of the point P_1, where one of the tangents from P′ meets the circle, x_2, y_2 the co-ordinates of the point P_2, where the other tangent from P′ meets the circle. Then P_1 P_2 will be the line whose equation is required.

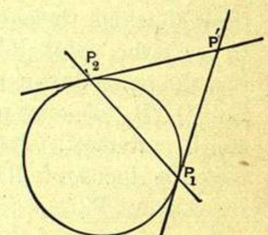

Fig. 47.

The equations of the tangents at P_1 and P_2 (Art. 42) are

$$xx_1 + yy_1 = r^2, \qquad (1)$$
$$xx_2 + yy_2 = r^2. \qquad (2)$$

Since these tangents pass through P′ (x', y'), the co-ordinates of P′ must satisfy both equations.

$$\therefore x'x_1 + y'y_1 = r^2, \qquad (3)$$
$$x'x_2 + y'y_2 = r^2. \qquad (4)$$

But we see that (3) and (4) are the conditions (Art. 15) that the two points (x_1, y_1) and (x_2, y_2) may lie on the line whose equation is

$$xx' + yy' = r^2. \qquad (5)$$

Hence (5) is the equation of the right line through the two points (x_1, y_1) and (x_2, y_2). Therefore it is the required equation of the chord joining the two points of contact.

The chord $P_1\ P_2$ is called the **Chord of Contact.**

Note that the co-ordinates of P_1 and P_2 do not appear in the final result.

48. *Through any fixed point a chord is drawn to a circle, and tangents to the circle are drawn at the extremities of the chord; to find the equation of the locus of the intersection of these tangents when the chord is turned about the fixed point.*

Let (x', y') be the fixed point P′ through which the chord passes; and (x'', y'') the point P″ in which the two tangents drawn at the extremities Q, R, of one position of the chord, intersect. It is required to find the locus of P″ as the chord turns about P′.

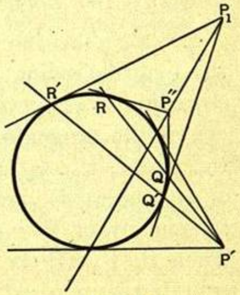

Fig. 48

The equation of the chord of contact (Art. 47) is

$$xx'' + yy'' = r^2. \qquad (1)$$

But since this chord passes through (x', y'), we have

$$x'x'' + y'y'' = r^2. \qquad (2)$$

Now (2) is the condition that the point P″ (x'', y'') lies on the right line whose equation is

$$xx' + yy' = r^2, \qquad (3)$$

and this is true for any position of the chord P′ Q R passing through P′; thus, if P′ Q R be turned about P′ into any other position as P′ Q′ R′, the point P″ will move along the fixed line $P''P_1$, whose equation is (3).

Therefore (3) is the equation required, and the locus is a right line.

49. The line $xx' + yy' = r^2$ is called the **Polar** of the point (x', y') with regard to the circle $x^2 + y^2 = r^2$; and the point (x', y') is called the **Pole** of the line.

It will be seen (Art. 48) that if (x', y') be *any point whatever*, the equation $xx' + yy' = r^2$ represents the locus of the intersection of the tangents at the extremities of the chord through (x', y').

If (x', y') be an *external point*, the equation $xx' + yy' = r^2$ represents the chord of contact (Art. 47).

If (x', y') be *on the circle*, the equation $xx' + yy' = r^2$ represents the tangent at that point (Art. 42).

That is, the three equations are identical; the position of the point (x', y') in Art. 48 is not subject to any limitation; hence, wherever the point (x', y') may be, the equation $xx' + yy' = r^2$ represents the locus of the intersection of tangents drawn at the extremities of chords which all pass through (x', y'). If the point be *without* the circle, this locus is identical (Art. 47) with the chord joining the points of contact of tangents drawn from (x', y'). If the point be *on* the circle, the locus is (Art. 42) the tangent at the point (x', y').

NOTE.—The limits of this treatise forbid us from pursuing this subject further. The student who wishes to go on with it, is referred to more extended works on Conic Sections, such as Salmon's, Todhunter's, Puckle's, etc.

49a. *If the polar of a point P′ pass through P″, then the polar of P″ will pass through P′.*

Let (x', y') be the point P′, and (x'', y'') the point P'', and let the equation of the circle, as before, be $x^2 + y^2 = r^2$.

The equations of the polars of P′ and P″ are

$$xx' + yy' = r^2, \qquad (1)$$
$$xx'' + yy'' = r^2. \qquad (2)$$

If P″ be on the polar of P′, its co-ordinates must satisfy (1); $\quad \therefore \quad x''x' + y''y' = r^2. \qquad (3)$

But (3) is also the condition that P′ may be on the line (2): that is, on the polar of P″. Therefore the polar of P″ passes through P′.

EXAMPLES.

1. Let tangents be drawn from the point (3, 4) to the circle $\quad x^2 + y^2 = 9.$

To find the equation of their chord of contact.

Let (x', y') and (x'', y'') be the points Q and R respectively.*

The equation of Q P, the tangent at (x', y') is $xx' + yy' = 9$, (1) and that of R P, the tangent at (x'', y''), is $xx'' + yy'' = 9$. (2)

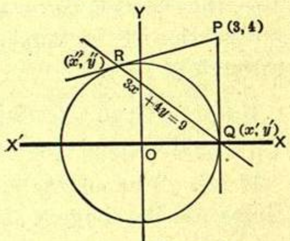

Fig. 48a

Since the point (3, 4) is on both these tangents,

$$\therefore \quad 3x' + 4y' = 9, \qquad (3)$$
$$3x'' + 4y'' = 9. \qquad (4)$$

But (3) and (4) are the conditions that the two points (x', y') and (x'', y'') are on the line whose equation is

$$3x + 4y = 9,$$

which is therefore the equation of the chord Q R.

2. Given the circle $x^2 + y^2 = 9$, and the points P (3, 4) and P′ (− 5, 6); to show that the polar Q′ R′ of P passes through P′, and that the polar Q R of P′ passes through P.

* The values of these co-ordinates we do not require.

The equation of Q′ R′, the polar of (3, 4), is $3x + 4y = 9$. (1)

This polar will pass through the point P′ if (− 5, 6) satisfy (1). But (− 5, 6) *does* satisfy (1), since $3 \times -5 + 4 \times 6 = -15 + 24 = 9.$

Therefore the polar of (3, 4) passes through the point

P′ (− 5, 6).

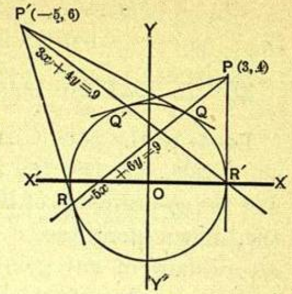

Fig. 48b

Also the equation of Q R, the polar of (− 5, 6), is $-5x + 6y = 9.$ (2)

This polar will pass through the point P if (3, 4) satisfy (2).

But (3, 4) *does* satisfy (2), since

$$-5 \times 3 + 6 \times 4 = -15 + 24 = 9.$$

Therefore the polar of (− 5, 6) passes through P (3, 4).

3. Find the pole of the line $3x - 5y = 4$, (1) with respect to the circle $\quad x^2 + y^2 = 16.$ (2)

Let (x', y') be the pole. Then the polar of (x', y') is the line $\quad xx' + yy' = 16.$ (3)

Comparing (3) with (1) in the form $12x - 20y = 16$, we have clearly

$$x' = 12,\ y' = -20.$$

Therefore the required pole is the point (12, − 20).

4. Find the poles, with respect to $x^2 + y^2 = 14$ of the lines (1) $2x + 3y = 7$; (2) $3x - y = 2$; (3) $x - y = 14$; (4) $3x = 7$.

Ans. (1) (4, 6); (2) (21, − 7); (3) (1, − 1); (4) (6, 0).

5. Find the polars with respect to $x^2 + y^2 = 14$, of the points (1) (6, 8); (2) (21, − 35); (3) (− 3, 1); (4) (0, 1).

Ans. (1) $3x + 4y = 7$; (2) $3x - 5y = 2$; (3) $y - 3x = 14$; (4) $y = 14$.

POLAR EQUATION.

50. *To find the polar equation of the circle.*

Let O be the pole, C the centre of the circle, and OX the initial line. Let the co-ordinates of the centre be the known quantities, r', θ', and the co-ordinates of any point P be r, θ, and R the radius of the circle. Then we have (Art. 14),

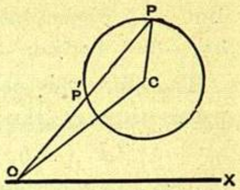

Fig. 49.

$$R = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta')};$$

or $$r^2 - 2rr'\cos(\theta - \theta') + r'^2 - R^2 = 0, \qquad (1)$$

which is the equation required.

Cor. 1.—Solving (1) for r, we obtain

$$r = r'\cos(\theta - \theta') \pm \sqrt{R^2 - r'^2\sin^2(\theta - \theta')}. \qquad (2)$$

These two values of r in (2) are the two distances from the pole O to P and P', and are *real* and *unequal*, or *real* and *equal*, according as $r'^2\sin^2(\theta - \theta') <$ or $= R^2$; or when $\sin^2(\theta - \theta') <$ or $= \frac{R^2}{r'^2}$. But when

$$\sin^2(\theta - \theta') = \frac{R^2}{r'^2},$$

we have $$\sin(\theta - \theta') = \pm\frac{R}{r'},$$

showing that there are *two* positions in which r is tangent to the circle. The condition

$$\sin(\theta - \theta') = +\frac{R}{r'}$$

gives the upper point of tangency, for which $\theta > \theta'$. The condition

$$\sin(\theta - \theta') = -\frac{R}{r'}$$

gives the lower point of tangency, for which $\theta < \theta'$, or $\theta - \theta'$ is $-$, and hence $\sin(\theta - \theta')$ is $-$. From equation (2) we see that the two values of r have the same, or different signs, according as

$$\sqrt{R^2 - r'^2\sin^2(\theta - \theta')} < \text{or} > r'\cos(\theta - \theta').$$

In the former case, the pole is *without* the circle; in the latter it is *within*.

Cor. 2.—If $\theta' = 0$, the diameter is the initial line, and (1) becomes

$$r^2 - 2rr'\cos\theta + r'^2 - R^2 = 0. \qquad (3)$$

If, in addition, the pole be on the circumference, $r' = R$, and (3) becomes

$$r = 2R\cos\theta, \qquad (4)$$

a result which we might have obtained at once geometrically from the property that the inscribed angle in a semicircle is a right angle.

These polar equations may be deduced from the equations referred to rectangular axes (Art. 41) by putting $r\cos\theta$ and $r\sin\theta$ for x and y respectively. The student should deduce these equations by this method.

EXAMPLES.

1. Find the points where the axes are cut by

$$x^2 + y^2 - 5x - 7y + 6 = 0.$$

By making alternately $y = 0$, $x = 0$, in the given equation, we find that the points are determined by the quadratics

$$x^2 - 5x + 6 = 0, \qquad y^2 - 7y + 6 = 0,$$

giving us the points,

$$x = 3, \quad x = 2; \qquad y = 6, \quad y = 1.$$

2. Find the centre and radius of the circle whose equation is $x^2 + y^2 - x - y = 0$;

Ans. Centre $(\frac{1}{2}, \frac{1}{2})$, radius $\frac{1}{2}\sqrt{2}$.

3. Find the centres and radii of the circles whose equations are (1) $x^2 + y^2 + 6x + 8y + 2 = 0$;
(2) $x^2 + y^2 - x - y + \frac{1}{4} = 0$.

Ans. $\begin{cases} (1) \text{ centre } (-3, -4), \text{ radius } \sqrt{23}; \\ (2) \text{ centre } (\frac{1}{2}, \frac{1}{2}), \text{ radius } \frac{1}{2}. \end{cases}$

4. Find the centre and radius of each of the circles
(1) $x^2 + y^2 - 4x - 2y - 31 = 0$;
(2) $x^2 + y^2 - 4x + 2y + 1 = 0$.

Ans. (1) $(2, 1)$, 6; (2) $(2, -1)$, 2.

5. Find the equation of the circle whose centre is $(1, 2)$, and whose radius is 3.

Ans. $x^2 + y^2 - 2x - 4y - 4 = 0$.

6. Find the equation of the circle whose centre is $(3, 0)$, and whose radius is 5. *Ans.* $x^2 + y^2 - 6x - 16 = 0$.

7. Find the equation of the circle passing through the origin and the point (x', y'), and having its centre on the axis of x. *Ans.* $(x^2 + y^2)\,x' - (x'^2 + y'^2)\,x = 0$.

8. Find the equation of the circle which passes through the points $(0, 0)$, $(a, 0)$, $(0, b)$.

Ans. $x^2 + y^2 - ax - by = 0$.

9. Find the equation of the circle which passes through the points $(a, 0)$, $(-a, 0)$, $(0, b)$.

Ans. $(x^2 + y^2)\,b + (a^2 - b^2)\,y = a^2 b$.

10. Find the equation of the circle passing through the points $(0, 1)$, $(1, 0)$, $(2, 1)$.

Ans. $x^2 + y^2 - 2x - 2y + 1 = 0$.

11. Find the equation of the circle passing through the points $(2, 0)$, $(-2, 0)$, $(0, 3)$.

Ans. $x^2 + y^2 - \frac{5}{3}y - 4 = 0$.

note 13

12. If the equation $x^2 + y^2 + xy + 2x + 2y = 0$ represent a circle, show that the axes are inclined at an angle of 60°, and find the centre and radius of the circle.

Ans. Centre $(-\frac{2}{3}, -\frac{2}{3})$, radius $\frac{2}{3}\sqrt{3}$.

$\cos w = \frac{1}{2}$ $w = 60°$

$x^2 + y^2 - 2xa - 2yb + (2xy - 2xb - 2ay + 2ab)\cos w = r^2$

13. Find the equation of the circle which touches the axes at the distance of 5 from the origin.

Ans. $x^2 + y^2 - 10x - 10y + 25 = 0$.

14. Find the equation of the circle whose centre is at the origin, and whose radius $= 3$, the axes being inclined at an angle of 45°. *Ans.* $x^2 + y^2 + xy\sqrt{2} - 9 = 0$.

15. Find the equation of the circle whose centre is at $(-\frac{1}{3}, -\frac{1}{3})$, and whose radius $= \frac{2}{\sqrt{3}}$, the axes being inclined at an angle of 60°.

Ans. $x^2 + y^2 + xy + x + y - 1 = 0$.

16. Find the relation between a, b, r, in order that the line

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (1)$$

may touch the circle $x^2 + y^2 = r^2$. (2)

Comparing (1) with (8) in Art. 42, we have

$$\frac{1}{a} = \frac{x'}{r^2} \text{ or } \frac{x'}{r} = \frac{r}{a}; \qquad \text{and} \qquad \frac{1}{b} = \frac{y'}{r^2} \text{ or } \frac{y'}{r} = \frac{r}{b}.$$

Substituting these values of $\frac{x'}{r}$ and $\frac{y'}{r}$ in (4) of Art. 41, we have

$$\frac{r^2}{a^2} + \frac{r^2}{b^2} = 1; \qquad \text{or} \qquad \frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

17. Find the equation of the circle whose centre is at the origin of co-ordinates, and which is touched by the line $y = 2x + 3$. *Ans.* $x^2 + y^2 = \frac{9}{5}$.

18. On a circle whose radius $= 6$, a tangent is drawn at the point whose ordinate is 4. Find where the tangent cuts the two axes, and also determine the angle which it makes with the axis of x.

Ans. It cuts the axes at $\frac{18}{\sqrt{5}}$ and 9; angle $= \tan^{-1} - \frac{1}{2}\sqrt{5}$.

19. Show that the point (2, 3) lies on the circle

$$x^2 + y^2 - 6x - 8y + 23 = 0,$$

and find the equation of the tangent to the circle at this point. *Ans.* Equation of tangent is $x + y = 5$.

20. Find (1) where the circle $x^2 + y^2 - 8x - 12y = 48$ cuts the axis of x; and (2) find the equations of the tangents at these points, and show that they are equally inclined to the axis of x.

$$\textit{Ans.} \begin{cases} (1)\ (12, 0), (-4, 0); \\ (2)\ y = \tfrac{4}{3}x - 16, y = -\tfrac{4}{3}x - \tfrac{16}{3}. \end{cases}$$

21. The circle $x^2 + y^2 - ax - by = 0$ passes through the origin. Find the equations of the tangent at the origin, and of the tangents at the points in which it cuts the axes.
Ans. $ax + by = 0$, $ax - by = a^2$, $ax - by + b^2 = 0$.

22. Find the points where the line $y = 2x + 1$ cuts the circle $x^2 + y^2 = 2$. *Ans.* $(-1, -1)$, $(\frac{1}{5}, \frac{7}{5})$.

23. Show that the line $3x - 2y = 0$ touches the circle

$$x^2 + y^2 - 3x + 2y = 0.$$

24. Show that the circles $x^2 + y^2 - 6x - 6y + 10 = 0$ and $x^2 + y^2 = 2$ touch each other at the point (1, 1).

25. Show that the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches the axes of x and y.

26. Find the equation of the circle which touches the lines $x = 0$, $y = 0$, $x = c$.
Ans. $4x^2 + 4y^2 - 4cx + 4cy + c^2 = 0$.

27. Find the length of the tangent (1) from the point (2, 5) to $x^2 + y^2 - 2x - 3y - 1 = 0$, and (2) from the point (4, 1) to $4x^2 + 4y^2 - 3x - y = 7$. *Ans.* (1), 3; (2), $2\sqrt{3}$.

28. Find the radical axis of $x^2 + y^2 + 2x + 3y = 7$ and $x^2 + y^2 - 2x - y + 1 = 0$. *Ans.* $x + y = 2$.

29. Find the radical axis of $x^2 + y^2 + bx + by = c$ and $ax^2 + ay^2 + a^2x + b^2y = 0$. *Ans.* $ax - by + \dfrac{ca}{a - b} = 0$.

30. Find the radical centre of the three circles
$x^2 + y^2 + 4x + 7 = 0$, $2x^2 + 2y^2 + 3x + 5y + 9 = 0$,
$x^2 + y^2 + y = 0$. *Ans.* $(-2, -1)$.

31. Find the pole of $3x + 4y = 7$ with regard to

$x^2 + y^2 = 14$. *Ans.* (6, 8).

32. Find the poles, with respect to $x^2 + y^2 = 35$, of

(1) $4x + 6y = 7$; (2) $3x - 2y = 5$; (3) $ax + by = 1$.

Ans. (1) (20, 30); (2) (21, −14); (3) $(35a, 35b)$.

33. Show that the polar of the point (x', y') with regard to the circle $(x - a)^2 + (y - b)^2 = r^2$ is

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2.$$

34. Find the polar of (4, 4) with regard to

$$(x - 1)^2 + (y - 2)^2 = 13.$$

Ans. $3x + 2y = 20$.

35. Find the polar of (4, 5) with regard to

$$x^2 + y^2 - 3x - 4y = 8.$$

Ans. $5x + 6y = 48$.

36. Find the pole of $2x + 3y = 6$ with regard to

$$(x - 1)^2 + (y - 2)^2 = 12.$$

Ans. (−11, −16).

37. Find the polar equation of the circle whose centre is at $\left(8, \dfrac{\pi}{4}\right)$, and whose radius is 10; and determine where the circle cuts the initial line.

Ans. Equation is $r^2 - 8\sqrt{2}(\sin\theta + \cos\theta)\, r = 36$; cuts the initial line at $r = \left(4 \pm \sqrt{34}\right)\sqrt{2}$.

38. Find the polar equation of the circle whose centre is $\left(15, \dfrac{\pi}{2}\right)$, and whose radius is 10; and determine the values of θ when the radius-vector is tangent to the circle.

$$\textit{Ans.} \begin{cases} \text{Equation is } r^2 - 30r\sin\theta = -125; \\ \theta = \cos^{-1}(\pm \tfrac{2}{3}). \end{cases}$$

39. Determine what is represented by the equation

$$r^2 - ra\cos 2\theta \sec\theta - 2a^2 = 0.$$

Ans. $\begin{cases} \text{A circle whose equation is} & r = 2a\cos\theta, \\ \text{and a right line whose equation is} & r = -a\sec\theta. \end{cases}$

40. Determine the radius and the centre of the circle

$$r^2 - 2r(\cos\theta + \sqrt{3}\sin\theta) = 5.$$

[Compare with (1) in Art. 50.]

Ans. Radius $= 3$; $r' = 2$, $\theta' = \frac{\pi}{3}$.

41. A limited right line moves so that its extremities are always on the co-ordinate axes; show that the locus of its middle point is a circle.

42. Show what the equation of the circle becomes when the origin is on the circumference, and the axes are inclined at an angle of 120°, the parts of them intercepted by the circle being h and k.

Since the origin is on the curve, the absolute term is zero (Art. 41, Cor. 2); therefore the equation of the circle referred to oblique axes (Art. 41), when expanded, becomes

$$x^2 + y^2 + 2xy\cos\omega - 2(a + b\cos\omega)x - 2(b + a\cos\omega)y = 0. \quad (1)$$

Making alternately $y = 0$, $x = 0$, we have, for determining the intercepts on the two axes,

$$x^2 - 2(a + b\cos\omega)x = 0,$$
$$y^2 - 2(b + a\cos\omega)y = 0.$$

$$\therefore \quad x = 2(a + b\cos\omega) = h,$$
$$y = 2(b + a\cos\omega) = k.$$

When $\omega = 120°$, $\cos\omega = -\frac{1}{2}$; $\therefore$ (1) becomes,

$$x^2 + y^2 - xy - hx - ky = 0, \quad \textit{Ans.}$$

43. Find the inclination of the axes in order that each of the equations (1) $x^2 + y^2 + xy - hx - hy = 0$,
(2) $x^2 + y^2 - xy - hx - hy = 0$,
may represent a circle; and find the centres and radii.
[Compare with (2), Art. 41.]

Ans. (1) 60°, $\left(\frac{h}{3}, \frac{h}{3}\right)$, $\frac{h}{\sqrt{3}}$; (2) 120°, (h, h), h.

44. Two lines are drawn through the points $(a, 0)$, $(-a, 0)$ respectively, and make an angle θ with each other: find the locus of their intersection.

Ans. $x^2 + y^2 - a^2 = \pm 2ay\cot\theta$.

45. A circle touches one given straight line and cuts off a constant length $(2l)$ from another straight line perpendicular to the former: find the locus of its centre.

Ans. $y^2 - x^2 = l^2$.

46. Given the base of a triangle $= 2m$, and the sum of the squares on its sides $= 2s^2$, to find the locus of its vertex.
[Take the base and a perpendicular through its centre for axes.] *Ans.* $x^2 + y^2 = s^2 - m^2$.

47. A point moves so that the sum of the squares of its distances from the four sides of a square is constant; show that the locus of the point is a circle.

48. Find the locus of the vertex of a triangle, given the base $= 2m$ and the vertical angle $= \alpha$. [Take axes as in Ex. 46.] *Ans.* $x^2 + y^2 - m^2 - 2my\cot\alpha = 0$.

49. Find the locus of the vertex of a triangle, given the base $= 2m$ and the ratio of the two sides $= a : b$. [Take axes as before.] *Ans.* $x^2 + y^2 - 2m\frac{a^2 + b^2}{a^2 - b^2}x + m^2 = 0$.

50. Given the base $= 2m$ and vertical angle $= \alpha$, to find the locus of the intersection of the perpendiculars from the extremities of the base to the opposite sides. [Take axes as before.] *Ans.* $x^2 + y^2 + 2m\cot\alpha y - m^2 = 0$.