CHAPTER V.

THE PARABOLA.

51. In the previous chapters we investigated various properties of right lines and circles. We shall now proceed to consider three curves, commonly called *conic sections*, which rank next in importance and interest to the right line and circle.

A **Conic Section** is the locus of a point moving in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line. If this ratio is *unity*, the locus is a **Parabola**; if *less* than unity an **Ellipse**; if *greater* than unity, an **Hyperbola**. The fixed point is called the **Focus**, and the fixed right line is called the **Directrix.***

We might begin by producing the general equation of a conic section, and afterwards applying it to the parabola, ellipse, and hyperbola, in succession; † but we prefer to find the equation of each conic section separately from its definition, beginning with the *parabola*, because it is the simplest of the three.

REMARK.—It will be shown hereafter, that if a right cone with a circular base be cut by a plane, the curve of intersection will be one of the following: a parabola, an ellipse, an hyperbola, a circle, one right line, two right lines, or a point. Hence, the parabola, ellipse, and hyperbola are called conic sections, which term may also be extended to include the circle, one right line, two right lines, and the point. It was from this point of view that these curves were first examined by geometers. It will be shown hereafter that every equation of the second degree between two variables is the equation of a conic section.

* Todhunter's Conic Sections, p. 116.
† See O'Brien's Co-ordinate Geometry, p. 62.

52. A **Parabola** is the locus of a point moving in a plane so that its distance from a fixed point is equal to its distance from a fixed right line. The fixed point is called the **Focus**; the fixed right line is called the **Directrix**; the right line through the focus perpendicular to the directrix is called the **Axis** of the curve; the point in which the axis cuts the curve is called the **Principal Vertex.**

From the definition, the parabola may readily be constructed by points, thus: Let F be the focus, CD the directrix, and OX through F perpendicular to CD the axis. The point A, midway between O and F, is a point of the curve, and is the *vertex*. Take any point on the axis, as M, and erect MP perpendicular to it. With F as a centre and OM as a radius, describe an arc cutting MP at P. This will be a point of the curve, for we have FP = DP. In the same way, any number of points may be constructed; drawing a line through them, it will be the required curve.

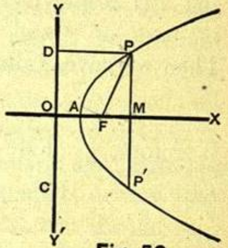

Fig. 50.

The curve may also be described by a continuous movement. Let CD be the directrix and F the focus. Take a triangular ruler, RDE, right-angled at D, and place one side DE on the directrix; take a string, equal in length to RD, and attach one end at R, and the other at F; then press a pencil against the string, keeping it continually tight, with the point P against the ruler, and slide the ruler along the directrix; the path of the pencil will be a parabola, for in every position of P we shall have

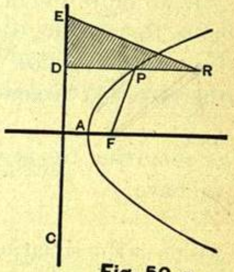

Fig. 50.a

$$PD = FP.$$

53. *To find the equation of the parabola.*

Let F be the focus, YY′ the directrix, OX the axis of the parabola. Take OX and OY for the co-ordinate axes. Let x, y be the co-ordinates of any point P in the locus, and put $p =$ the constant distance OF. Draw PM and PD respectively perpendicular to the axes of x and y, and join FP. Then we have, from the definition,

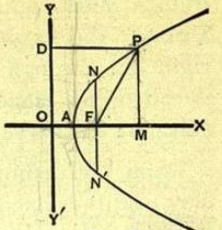

Fig. 51.

$$FP = PD;$$

therefore, $\quad FM^2 + MP^2 = PD^2;$

that is, $\quad (x-p)^2 + y^2 = x^2,$

or
$$y^2 = 2p\,(x - \tfrac{1}{2}p), \tag{1}$$

which is true for every position of P; hence it is the equation required.

When $y = 0$, $x = \frac{1}{2}p$, which shows that the curve cuts the axis of x at the distance $\frac{1}{2}p$ to the right of the origin, or midway between O and F.

If we move the origin to A, and keep the new axes parallel to the old, the equation will be simplified. The formulæ for transformation (Art. 33) are $x = \frac{1}{2}p + x'$, $y = y'$; therefore (1) becomes

$$y'^2 = 2px';$$

or removing the accents, since x and y are general variables, we have

$$y^2 = 2px, \tag{2}$$

which is the equation of the parabola referred to its axis and the tangent at the principal vertex.

Cor. 1.—When $y = 0$ in (2), we have $x = 0$, which shows that the curve cuts the axis of x at the origin. When $x = 0$, $y = \pm 0$, which shows that the axis of y is tangent to the curve at the origin.

Cor. 2.—Solving (2), for y, we get

$$y = \pm\sqrt{2px},$$

which shows that for positive values of x there are *two real* values of y, numerically equal, but with contrary signs. Hence, for every point P on one side of the axis of x, there is a point P′ on the other side, at the same distance from it; and therefore the curve is symmetrical with respect to the axis of x. If we suppose p to be positive, which is the case when the focus is to the *right* of the origin, we see that negative values of x do not give *real* values of y; hence, no point of the curve lies to the left of the axis of y. As x may have any positive value whatever, the curve extends to an infinite distance in the direction of positive abscissas. In the same way, if we suppose p to be negative, or the focus to be to the *left* of the origin, it may be shown that no part of the curve lies to the right of the origin, while it extends without limit to the left of it.

Cor. 3.—To find the value of the ordinate passing through the focus, make $x = \frac{1}{2}p$, and get, from (2),

$$y^2 = p^2, \qquad \text{or} \qquad 2y = 2p.$$

Hence, the double ordinate passing through the focus is equal to the constant quantity $2p$. This double ordinate through the focus of a conic section is called the **Principal Parameter**, or **Latus Rectum**.

Cor. 4.—From (2) we have the proportion,

$$x : y :: y : 2p;$$

that is, $2p$, *the latus rectum, is a third proportional to any abscissa and its corresponding ordinate.*

Cor. 5.—If (x', y') and (x'', y'') be any two points on the curve, we shall have, from equation (2),

$$y'^2 = 2px'; \qquad y''^2 = 2px''.$$

Hence, forming a proportion, we have

$$y'^2 : y''^2 :: x' : x''.$$

That is, *the squares of any two ordinates are to each othe1 as their corresponding abscissas.*

Cor. 6.—A point is *outside, on*, or *inside* the parabola, according as

$$y^2 - 2px >, =, \text{ or } < 0.$$

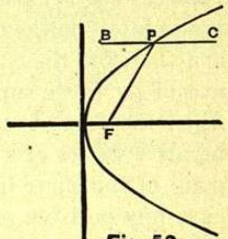

Fig. 52.

Thus, if the point is *on* the curve, as at P, its co-ordinates satisfy the equation of the curve, giving

$$y^2 - 2px = 0.$$

If the point is *outside* of the curve, as at B, its abscissa will be less than at P, while its ordinate will be the same, giving

$$y^2 - 2px > 0.$$

If the point is *inside* of the curve, as at C, its abscissa will be greater than at P, while its ordinate will be the same, giving

$$y^2 - 2px < 0.$$

54. *To find the equation of the tangent at any point of a parabola* (see Def., Art. 42).

Let (x', y') and (x'', y'') be any two points on the curve. The equation of the secant through these points is (Art. 26)

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \qquad (1)$$

Since (x', y') and (x'', y'') are on the parabola, they will satisfy its equation, giving us

$$y'^2 = 2px', \qquad (2)$$

$$y''^2 = 2px''. \qquad (3)$$

Subtracting (3) from (2), factoring, and dividing, we have

$$\frac{y' - y''}{x' - x''} = \frac{2p}{y' + y''},$$

which, in (1), gives $y - y' = \dfrac{2p}{y' + y''}(x - x').$ (4)

When the points become consecutive, $y'' = y'$; hence (4) becomes

$$y - y' = \frac{p}{y'}(x - x'). \qquad (5)$$

Clearing of fractions, and substituting for y'^2 its value in (2), we have $yy' = p(x + x')$, (6)
which is the required equation of the tangent at (x', y').

Cor. 1.—To find the point in which the tangent cuts the axis of x, make $y = 0$, in (6), and we have

$$0 = p(x + x'); \qquad \therefore x = -x';$$

that is, *the subtangent is bisected at the vertex.*

Sch.—This result enables us to draw a tangent to the curve at a given point. Let P be the given point, and MP its ordinate. Lay off AT to the left of the origin equal to AM. Draw a line through T and P, and it will be the tangent required. Ex 7.

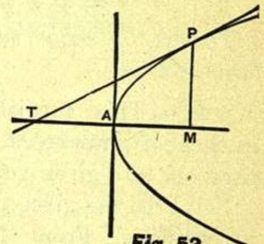

Fig. 53.

Cor. 2. From equation (6) we have

$$y = \frac{p}{y'}x + \frac{p}{y'}x' = \frac{p}{y'}x + \frac{\frac{1}{2}y'^2}{y'} = \frac{p}{y'}x + \frac{y'}{2}. \qquad (7)$$

Put $\dfrac{p}{y'} = a$, and $\therefore \dfrac{y'}{2} = \dfrac{p}{2a}$; and (7) becomes

$$y = ax + \frac{p}{2a}. \qquad (8)$$

We might have found (8) by the method of Art. 45, Cor.

Note.—In the solution of examples we should take whichever form of the equation of a tangent appears the more suitable for the particular case.

Note 15

55. *To find the equation of the normal at any point of a parabola.*

Let (x', y') be the point; the equation of the tangent at (x', y'), (Art. 54), is

$$y = \frac{p}{y'}(x + x'). \qquad (1)$$

The equation of a right line through (x', y') perpendicular to (1) is, by Art. 27, Cor. 2,

$$y - y' = -\frac{y'}{p}(x - x'), \qquad (2)$$

which is the required equation of the normal at the point (x', y').

Cor.—To find the point in which the normal at (x', y') cuts the axis of x, we make $y = 0$ in (2), and get, after reduction,

$$x = x' + p; \qquad \text{or} \qquad x - x' = p.$$

That is, *the subnormal is constant, and equal to half the latus rectum.*

Sch.—This furnishes a second method of drawing a tangent to a parabola, at a given point.

Let P be the given point, and PM its ordinate. From the foot of the ordinate lay off a distance MG on the axis, to the right, equal to half the latus rectum, and draw GP; through P draw PT perpendicular to GP. PT will be the tangent required, and GP will be the normal.

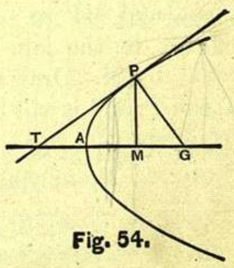

Fig. 54.

56. *To prove that a tangent to the parabola at any point makes equal angles with the axis of the curve and the focal line to the point of contact.*

A **Focal Line** is a line drawn from the focus to a point of the curve.

Let PT be the tangent at P, FP the focal line to the point of contact, MP the ordinate, GP the normal, and OD the directrix. Then (Art. 54, Cor.),

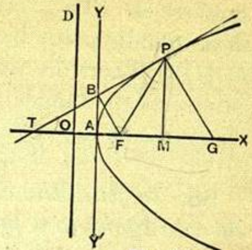

Fig. 55.

$$AT = AM;$$

also,

$$\begin{aligned} FT &= AT + AF \\ &= AM + AF \\ &= AM + AO = OM; \end{aligned}$$

that is,

$$FT = FP \text{ (Art. 52).}$$

Hence the angle FTP = angle FPT.

57. *To find the locus of the intersection of the tangent to a parabola with the perpendicular on it from the focus.*

The equation of any tangent to the parabola is (Art. 54, Cor. 2),

$$y = ax + \frac{p}{2a}. \qquad (1)$$

The equation of the line through the focus perpendicular to (1) is (Art. 27, Cor. 2),

$$y = -\frac{1}{a}(x - \tfrac{1}{2}p),$$

or

$$y = -\frac{x}{a} + \frac{p}{2a}. \qquad (2)$$

Since the lines (1) and (2) have the same intercept on the axis of y they meet in that axis. Hence the axis of y, or the tangent to the curve at the vertex, is the required locus.

Cor.—The result of this Art. can be easily obtained from geometric considerations. Thus, let FB, Fig. 55, be a perpendicular from the focus to the tangent PT. It will intersect PT at its middle point B, because the triangle TFB is isosceles. The vertical tangent at A also intersects TP at its middle point B, because it bisects MT and is parallel to MP. Therefore the point B, at which the perpendicular intersects the tangent, is on the axis of y, or the tangent to the curve at the vertex.

Ex T N.17

58. *To find the co-ordinates of the point of contact of a tangent to a parabola from a fixed point.*

Let (x', y') be the required point of contact, and (x'', y'') be the fixed point through which the tangent passes.

Since (x', y') is on the parabola, we have

$$y'^2 = 2px'. \qquad (1)$$

The equation of the tangent at (x', y') is

$$yy' = p(x + x').$$

Since this tangent passes through (x'', y''), we have

$$y''y' = p(x'' + x'). \qquad (2)$$

Solving (1) and (2) for x' and y', we have

$$px' = y''^2 - px'' \pm y'' \sqrt{y''^2 - 2px''},$$

$$y' = y'' \pm \sqrt{y''^2 - 2px''}.$$

These values indicate that from any fixed point *two* tangents can be drawn to a parabola, *real*, *coincident*, or *imaginary*, according as $y''^2 - 2px'' > 0, = 0$, or < 0; that is, according as the point (x'', y'') is *without*, *on*, or *within* the curve (see Art. 53, Cor. 6).

Cor.—The ordinate of the middle point of the chord joining the *two real* points of contact is equal to the half-sum of the ordinates of the two points; that is, it is equal to y''. Hence, a line through the fixed point, parallel to the axis of the curve, bisects the chord joining the two points of contact. This chord is called the **Chord of Contact.**

EXAMPLES

1. Are the points (6, 6), (4, 6), (4, 3), (4, 4), (4, —5) outside, on, or inside the parabola $y^2 = 6x$?

2. Are the points (0, 0), (0, 1), $(\frac{4}{5}, -2)$, $(b^2, b\sqrt{5})$ on the parabola $y^2 = 5x$?

3. The distance from the focus of a parabola to the directrix is 4: find its equation when the origin is (1) at the vertex, (2) at the focus, and (3) at the intersection of the axis and directrix.

   Ans. (1) $y^2 = 8x$; (2) $y^2 = 8x + 16$; (3) $y^2 = 8x - 16$.

4. At what point of the parabola $y^2 = 16x$ is the ordinate equal to twice the abscissa? *Ans.* (4, 8.)

5. If the distance of a point in the parabola $y^2 = 2px$ from the focus is equal to $2\frac{1}{2}p$, what is the abscissa of the point? *Ans.* $2p$.

6. The equation of a parabola is $y^2 = 9x$: find the equation of the chord through the points whose ordinates are 3 and 6. *Ans.* $y = x + 2$.

7. Find the equations of the tangent and the normal to the parabola $y^2 = 4x$ at the point whose abscissa is 9 and ordinate positive.

   Ans. Tangent, $x - 3y + 9 = 0$; normal, $y + 3x = 33$.

8. Find the equations of the tangents and the normals to the parabola $y^2 = 8x$ at the ends of its latus rectum.

   Ans. $\left\{ \text{Tangents, } \begin{matrix} x - y + 2 = 0 \\ x + y + 2 = 0 \end{matrix}; \text{ normals, } \begin{matrix} y + x - 6 = 0, \\ y - x + 6 = 0. \end{matrix} \right.$

9. Find the equations of the tangents and the normals to the parabola $y^2 = 4ax$ at the ends of its latus rectum.

   Ans. $x \mp y + a = 0$; $y \pm x \mp 3a = 0$.

10. Find the points where the line $y = 3x - a$ cuts the parabola $y^2 = 4ax$. *Ans.* $(a, 2a)$, $(\frac{1}{9}a, -\frac{2}{3}a)$.