59. *Tangents are drawn to a parabola from a given external point; to find the equation of the chord of contact.*

Let (x', y') be the external point P'; (x_1, y_1) and (x_2, y_2) the two points P_1 and P_2 where the tangents meet the parabola. Then P_1P_2 will be the chord of contact whose equation is required.

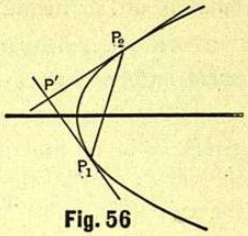

Fig. 56

The equations of the tangents at P_1 and P_2 (Art. 54) are

$$yy_1 = p(x + x_1) \quad (1)$$

$$yy_2 = p(x + x_2). \quad (2)$$

Since these tangents pass through $P'(x', y')$, the coordinates of P' must satisfy both equations.

$$\therefore y'y_1 = p(x' + x_1), \quad (3)$$

$$y'y_2 = p(x' + x_2). \quad (4)$$

But we see that (3) and (4) are the conditions that the two points (x_1, y_1), (x_2, y_2) may lie on the line whose equation is

$$yy' = p(x + x'). \quad (5)$$

Hence (5) is the equation of the line through the two points (x_1, y_1) and (x_2, y_2). Therefore it is the required equation of the chord of contact P_1P_2.

60. *Through any fixed point a chord is drawn to a parabola, and tangents to the parabola are drawn at the extremities of the chord; to find the equation of the locus of the intersection of the tangents when the chord is turned about the fixed point.*

Let (x', y') be the fixed point P' through which the chord passes; and (x'', y'') the point P'' in which the two tangents drawn at the extremities Q and R of one position of the chord intersect. It is required to find the locus of P'' as the chord turns about P'.

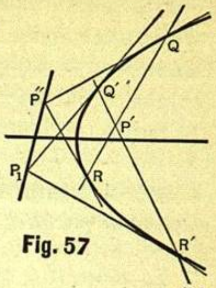

Fig. 57

The equation of the chord of contact (Art. 59) is

$$yy'' = p(x + x''). \quad (1)$$

Since this chord passes through P', we have

$$y'y'' = p(x' + x''). \quad (2)$$

Now, (2) is the condition that the point P'' (x'', y'') lies on the right line whose equation is

$$yy' = p(x + x'), \quad (3)$$

and this is true for any position of the chord P'QR passing through P'. Thus, if P'QR be turned about P', the point P'' will move along the fixed line $P''P_1$ whose equation is (3). Therefore (3) is the equation required, and the locus is a right line.

SCH.—The line $yy' = p(x + x')$ is called the **Polar** of the point (x', y') with regard to the parabola $y^2 = 2px$, and the point (x', y') is called the **Pole** of the line.

It will be seen (Art. 60), that if (x', y') be *any point whatever*, the equation $yy' = p(x + x')$ represents the locus of the intersection of the tangents at the extremities of the chord through (x', y').

61. The statements in Art. 49 with respect to the circle may all be applied to the parabola. Thus, we see that the equations of the *tangent*, of the *chord of contact*, and of the *locus of the intersection of tangents at the extremities of chords that pass through a fixed point*, are all identical in form; and inasmuch as the fixed point (x', y'), in the case of the chord of contact, is restricted to being *without* the curve, and in that of the tangent to being *on* the curve. while in the case of the locus just described it is *any point whatever*, it follows that the tangent and chord of contact in the parabola are particular cases of the locus, due to bringing the point (x', y') *on* the curve, or *outside* of it.

62. *A* **Diameter** *of a curve is the locus of the middle points of parallel chords.*

To find the equation of any diameter.

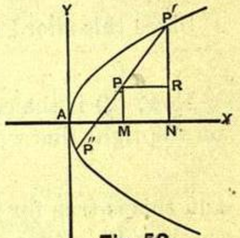

Fig. 58.

Let (x, y) be the middle point P of the chord P′P″; (x', y') the point P′ or P″; θ the inclination of P′P″ to the axis of x, the axis of the curve; r the length of PP′, half the line P′P″. Then

$$x' = \text{AM} + \text{MN} = x + r\cos\theta;$$

$$y' = \text{NR} + \text{RP}' = y + r\sin\theta.$$

Now as P′ is on the curve, its co-ordinates x', y' will satisfy the equation of the curve $y^2 = 2px$, giving

$$(y + r\sin\theta)^2 = 2p\,(x + r\cos\theta),$$

or $\quad r^2\sin^2\theta + 2r\,(y\sin\theta - p\cos\theta) + y^2 - 2px = 0, \quad (1)$

from which quadratic we can determine the two values of r. But as (x, y) is the middle point P of the chord, the two values of r are numerically equal with contrary signs; therefore (Alg. Art. 135), the coefficient of the first power of r vanishes, giving us

$$y\sin\theta - p\cos\theta = 0,$$

which represents the locus of the middle point P of the chord P′P″. Hence the required equation of any diameter is

$$y = p\cot\theta. \qquad (2)$$

Since p is fixed for any given parabola, and θ is constant for any given system of parallel chords, the second member of (2) is constant; and therefore it is a right line parallel to the axis of x (Art. 22, I, Cor. 2). Hence, every diameter is a right line parallel to the axis of the parabola. By giving to θ a suitable value, equation (2) may be made to represent *any* right line parallel to the axis. Hence it follows that every right line parallel to the axis of the parabola is a diameter; that is, it bisects some system of parallel chords.

SCH.—To draw a diameter of a parabola, draw any two parallel chords, bisect them; the line passing through the points of bisection is a diameter.

63. *To find the equation of the parabola referred to any diameter and the tangent at its vertex.*

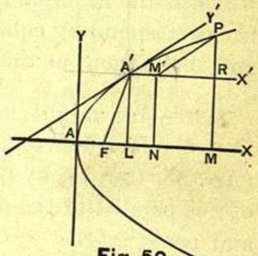

Fig. 59.

Let (m, n) be any point A′ on the parabola; take this point for the new origin, and draw through it the diameter A′X′ and the tangent A′Y′ for the new axes of co-ordinates. Let X′A′Y′ $= \beta$; then (Art. 54),

$$\tan\beta = \frac{p}{y'} = \frac{p}{n}.$$

Let (x, y) be any point P on the curve referred to the old axes AX and AY. Draw PM′ parallel to A′Y′, and draw PM, M′N, and A′L parallel to AY; then

$$x = \text{AM} = \text{AL} + \text{A}'\text{M}' + \text{M}'\text{R} = m + x' + y'\cos\beta. \quad (1)$$

$$y = \text{MP} = \text{LA}' + \text{RP} \qquad = n + y'\sin\beta. \quad (2)$$

Substitute these values of x and y in the equation

$$y^2 = 2px,$$

and obtain $(n + y'\sin\beta)^2 = 2p\,(m + x' + y'\cos\beta)$, or $y'^2 \sin^2\beta + 2y'(n\sin\beta - p\cos\beta) + (n^2 - 2pm) = 2px'$. (3)

But $$n = \frac{p}{\tan\beta} = p\frac{\cos\beta}{\sin\beta};$$

$$\therefore \quad n\sin\beta - p\cos\beta = 0;$$

also, since A′ is on the curve, its co-ordinates m, n will satisfy $y^2 = 2px$, giving us,

$$n^2 = 2pm.$$

Hence (3) becomes $y'^2\sin^2\beta = 2px'$,

or $$y'^2 = \frac{2p}{\sin^2\beta}x'.$$

Putting $\frac{2p}{\sin^2\beta} = 2p'$, and dropping the accents from x and y, since they are general variables, we have

$$y^2 = 2p'x, \tag{4}$$

which is the required equation, and is of the same form as the corresponding equation referred to the axis of the curve and the tangent at the principal vertex.

SCH.—We might have obtained equations (1) and (2) from the formulæ to pass from rectangular axes to oblique (Art. 35, Cor. 1), by remembering that, since the new axis of x is parallel to the old, $\alpha = 0$, and therefore $\sin\alpha = 0$, and $\cos\alpha = 1$.

COR. 1.—Solving equation (4) for y, we have

$$y = \pm\sqrt{2p'x},$$

which shows that, for every positive value of x, there are two *real* values for y, numerically equal, but with contrary signs. These two values, taken together, make up a chord parallel to the axis of y, and which is bisected by the axis of x. Hence the axis of x bisects all chords of the curve parallel to the axis of y; that is, the system of chords bisected by any diameter (Art. 62), is parallel to the tangent at the vertex.

The quantity $2p'$, or its equal $\frac{2p}{\sin^2\beta}$, is called the **Parameter** of the diameter that is taken as the axis of x.

COR. 2.—If (x', y') and (x'', y'') be any two points on the curve, we have from (4),

$$y'^2 = 2p'x'; \qquad y''^2 = 2p'x'';$$

therefore, $$y'^2 : y''^2 :: x' : x''.$$

That is, the squares of the ordinates to any diameter are to each other as the corresponding abscissas.

T Ex T N.18

64. *The parameter of any diameter is equal to four times the distance from the vertex of that diameter to the focus.*

By Art. 56 we have, in Fig. 59,

note 18

$$\text{FA}' = \text{AL} + \text{AF} = m + \tfrac{1}{2}p;$$

and by Art. 63,

$$m = \frac{n^2}{2p} = \tfrac{1}{2}p\cot^2\beta \text{ (since } n = p\cot\beta);$$

therefore $$m + \tfrac{1}{2}p = \tfrac{1}{2}p\cot^2\beta + \tfrac{1}{2}p = \frac{p}{2\sin^2\beta} = \text{FA}'.$$

hence, $$\frac{2p}{\sin^2\beta} = 4\text{FA}'.$$

But (Art. 63, Cor. 1), $\frac{2p}{\sin^2\beta}$ is the parameter of the diameter A′X′, which was represented by $2p'$; therefore the parameter of any diameter is equal to 4FA′.

65. *To find the equation of a tangent to a parabola referred to any diameter and the tangent at its vertex.*

The equation of a right line referred to oblique axes is of the same form (Art. 22, IV) as when referred to rectangular axes; also the equation of the parabola referred to any diameter and the tangent at its vertex is of the same form (Art. 63) as when referred to the axis of the curve and tangent at the principal vertex. Hence, the investigation of Art. 54 will apply without any change to the equation

$$y^2 = 2p'x,$$

giving us the required equation,

$$yy' = p'(x + x').$$

COR.—Making $y = 0$ in this equation, we get $x = -x'$, which shows that the tangent cuts any diameter on the left of its vertex, at a distance equal to the abscissa of the point of contact. Hence, *the subtangent to any diameter of a parabola is bisected at the vertex.*

66. *To find the polar equation of the parabola, the focus being the pole.*

Let $FP = r$, $XFP = \theta$; then we have, from the definition of Art. 52,

$$FP = OM = OF + FM;$$

that is, $$r = p + r\cos\theta;$$

therefore, $$r = \frac{p}{1 - \cos\theta}. \qquad (1)$$

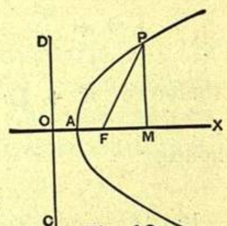

Fig. 60.

which is the required equation.

COR.—When $\theta = 0$, $r = \dfrac{p}{1 - 1} = \infty$, which shows that the radius-vector which coincides with the axis does not meet the curve, or rather meets it at an infinite distance. For any value of $\theta > 0$, however small, r is finite, which shows that if a line be drawn from the focus making any angle, however small, with the axis of the curve, it will meet the curve at a finite distance.

When $\theta = 90°$, $r = p$, as it should. When $\theta = 180°$, $r = \frac{1}{2}p$, as it should, since $AF = \frac{1}{2}p$ (Art. 53). When $\theta = 270°$, $r = p$, as it should. The two values of r corresponding to 90° and 270°, taken together, make the *parameter* of the axis of the curve, which is again seen to be equal to $2p$, as it was shown to be in Art. 53, Cor. 3.

67. *A chord passing through the focus of a conic section is called a* **Focal Chord.**

If tangents are drawn at the extremities of any focal chord of a parabola:

I. *The tangents will intersect on the directrix.*
II. *The tangents will meet at right angles.*
III. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

I. *The tangents will intersect on the directrix.* If the tangents to a parabola meet at the point (x', y'), the equation of the chord of contact (Art. 59) is

$$yy' = p(x + x').$$

If the chord pass through the focus, its co-ordinates, $x = \frac{1}{2}p$, $y = 0$, must satisfy this equation, giving

$$0 = p(\tfrac{1}{2}p + x'); \qquad \therefore\ x' = -\tfrac{1}{2}p;$$

that is, the point of intersection of the tangents is on the directrix.

II. *The tangents will meet at right angles.* The equation of the tangent to a parabola (Art. 54, Cor. 2) is

$$y = ax + \frac{p}{2a}. \qquad (1)$$

If the tangents meet at (x', y'), we have

$$y' = ax' + \frac{p}{2a};$$

or
$$a^2 - \frac{y'}{x'}a + \frac{p}{2x'} = 0, \tag{2}$$

a quadratic for determining the two values of a, which are the tangents of the angles that the two tangent lines through (x', y') make with the axis of the parabola.

Call the two roots of (2), a_1 and a_2, and we have from Algebra, Art. 140,

$$a_1 a_2 = \frac{p}{2x'}. \tag{3}$$

From (1) we have $x' = -\frac{1}{2}p$, which in (3) gives

$$a_1 a_2 = -1, \quad \text{or} \quad a_1 = -\frac{1}{a_2};$$

that is, the two tangents are perpendicular to each other (Art. 27, Cor. 1).

III. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.* The equation of the right line passing through the focus and the point (x', y'), by Art. 26, is

$$y = \frac{y'}{x' - \frac{1}{2}p}(x - \tfrac{1}{2}p). \tag{4}$$

From (I), $x' = -\frac{1}{2}p$, which in (4) gives

$$y = -\frac{y'}{p}(x - \tfrac{1}{2}p). \tag{5}$$

The equation of the chord of contact (Art. 59) is

$$yy' = p(x + x'),$$

which becomes for the focal chord,

$$y = \frac{p}{y'}(x - \tfrac{1}{2}p), \tag{6}$$

which is perpendicular to (5), by Art. 27, Cor. 1.

Notes 19+20.

EXAMPLES.

1. Find the intersections of the parabola $y^2 = 8x$ and the line $3y - 2x - 8 = 0$. *Ans.* (2, 4) and (8, 8).

2. Find the equation of the right line passing through the focus of the parabola $y^2 = 4x$, and making an angle of 45° with the axis of the curve. *Ans.* $y = x - 1$.

3. Find the points in which the focal chord, $y = x - 1$, intersects the parabola, $y^2 = 4x$.

Ans. $(3 \pm 2\sqrt{2}, 2 \pm 2\sqrt{2})$.

4. Find the equation of the right line passing through the vertex of any parabola and the extremity of the focal ordinate. *Ans.* $y = 2x$.

5. Find the equation of the circle which passes through the vertex of any parabola and the extremities of the double ordinate through the focus. *Ans.* $y^2 = \frac{5}{2}px - x^2$.

6. Find the equation of the circle which passes through the vertex of the parabola $y^2 = 12x$ and the extremities of the double ordinate through the focus.

Ans. $y^2 = 15x - x^2$.

7. Find the equations of the tangent and normal to any parabola at the extremity of the positive ordinate through the focus. *Ans.* $y = x + \frac{1}{2}p$ and $y + x = \frac{3}{2}p$.

8. Find the equations of the tangent and normal to the parabola $y^2 = 4x$, at the extremity of the positive ordinate through the focus. *Ans.* $y = x + 1$; $y + x = 3$.

9. Find the point where the normal in Ex. 7 meets the curve again, and the length of the intercepted chord.

Ans. $(\frac{9}{2}p, -3p)$; length of chord $= 4p\sqrt{2}$.

10. Find the point where the normal in Ex. 8 meets the curve again, and the length of the intercepted chord.

Ans. $(9, -6)$; length of chord $= 8\sqrt{2}$.

11. Find the point in a parabola where the tangent is inclined at an angle of 30° to the axis of x.

Ans. $(\frac{3}{2}p, p\sqrt{3})$.

12. Prove that the normal at any point of a parabola bisects the angle between the focal line and the diameter passing through that point. [See Art. 56.]

13. Prove that the quantity $2p'$, in equation (4) of Art. 63, which in the corollary of that article was called the *parameter*, is equal to the double ordinate passing through the focus. [See Art. 64.]

14. On a parabola whose latus rectum is 10, a tangent is drawn at the point whose ordinate is 6, the origin being at the principal vertex; determine where the tangent cuts the two co-ordinate axes. *Ans.* $(-3.6, 0)$ and $(0, 3)$.

15. Determine where the normal in the preceding example, at the same point, if produced, will cut the two axes.

Ans. $(8.6, 0)$ and $(0, 10.3)$.

16. Find the angle which the tangent in Ex. 14 makes with the axis of x. *Ans.* 39° 48′ 20″.

17. In the parabola $y^2 = 12x$, find the length of the perpendicular from the focus to the tangent at the point whose abscissa is 9. *Ans.* 6.

18. In the parabola $y^2 = 8x$, find the length of the normal at the point whose abscissa is 6. *Ans.* 8.

19. Prove that the circle described on a focal chord as a diameter is tangent to the directrix.

20. Prove that the tangent at any point of a parabola will meet the directrix and latus rectum produced, at two points equally distant from the focus.

21. Prove that a right line drawn from the point of the parabola of which the abscissa is $4p$, and cutting the axis at the point $x = 2p$, will, if produced, meet the curve again at the point $x = p$, and be a normal at that point, $2p$ being the latus rectum.

22. In the parabola $y^2 = 2px$ find the equation of the chord which passes through the vertex and is bisected by the diameter $y = a$. *Ans.* $ay = px$.

23. Show that the tangent to the parabola $y^2 = 4ax$ at the point (x', y') is perpendicular to the tangent at the point

$$\left(\frac{a^2}{x'}, \frac{-4a^2}{y'}\right).$$

24. Show that the line $y = 2x + \frac{a}{2}$ cuts $y^2 = 4ax$ in coincident points.

Show that it also cuts $20x^2 + 20y^2 = a^2$ in coincident points.

25. Show that the line $7x + 6y = 13$ is a tangent to the curve $y^2 - 7x - 8y + 14 = 0$.

26. Tangents are drawn from the point $(-2, 5)$ to the parabola $y^2 = 6x$: find the equation of the chord of contact.

Ans. $3x - 5y = 6$.

27. Show that the equation $y^2 - 8y - 6x + 28 = 0$ represents a parabola whose vertex is at the point $(2, 4)$, whose latus rectum is 6, and whose axis is parallel to the axis of x.

28. Show that the equation $x^2 + 4ax + 2ay = 0$ represents a parabola whose vertex is at the point $(-2a, 2a)$, whose latus rectum is $2a$, and whose axis is parallel to the axis of y.

29. Find the co-ordinates of the focus and the equation of the directrix of each of the following parabolas:

$$(1)\ y^2 = 5x + 10;\quad (2)\ x^2 - 4x + 2y = 0;$$

$$(3)\ (y - 2)^2 = 5(x + 4).$$

$$Ans. \begin{cases} (1)\ (-\frac{3}{4}, 0),\ 4x + 13 = 0; \\ (2)\ (2, \frac{3}{2}),\ 2y = 5; \\ (3)\ (-\frac{11}{4}, 2),\ 4x + 21 = 0. \end{cases}$$

30. If perpendiculars be let fall on any tangent to a parabola from two given points on the axis equidistant from the focus, show that the difference of their squares will be constant.

31. Show that two tangents to a parabola which make equal angles respectively with the axis and directrix, but are not at right angles, intersect on the latus rectum.

32. From any point on the latus rectum of a parabola perpendiculars are drawn to the tangents at its extremities: show that the line joining the feet of these perpendiculars touches the parabola.

33. Show that if tangents be drawn to the parabola $y^2 = 4ax$ from a point on the line $x + 4a = 0$, their chord of contact will subtend a right angle at the vertex.

34. Show that the locus of the middle point of a chord of a parabola which passes through a fixed point is a parabola.

35. The extremities of any chord of a parabola being (x', y'), (x'', y''), and the abscissa of its intersection with the axis of the curve being x, to prove that

$$x'x'' = x^2, \qquad y'y'' = -2px.$$

36. Two tangents of a parabola meet the curve in (x', y') and (x'', y''), their point of intersection being (x, y); show that

$$x = \sqrt{x'x''}, \qquad y = \frac{y' + y''}{2}.$$

37. The latus rectum of a parabola is 10, and the radius vector is 25; find the variable angle. *Ans.* 36° 52′ 12″.

38. The latus rectum of a parabola is 10, and the variable angle is 144°, the pole being at the focus; determine the radius vector. *Ans.* 2.76.

In the elipse and hyperbola tangent line may be obtained by simply changing x^2 into $x x$ and y^2 into $y y$ without [illegible] b-

CHAPTER VI.

THE ELLIPSE.

68. The **Ellipse** is the locus of a point moving in a plane, so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line, the ratio being less than unity.*

From this definition the ellipse may be constructed by points, thus:

Let F be the fixed point, DD′ the fixed right line, and e the given ratio. Draw through F the line OA perpendicular and EE′ parallel to DD′. Take

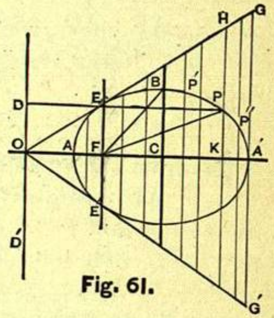

Fig. 61.

FE (= FE′) : FO :: e : 1,

and draw OE and OE′ produced indefinitely. Draw parallels to EE′, meeting the lines OG and OG′. With the half of any one of these lines, as KH, for a radius, and the fixed point F for a centre, describe an arc cutting KH at P; this is a point of the curve; for, joining P and F, and drawing PD perpendicular to DD′, we have KH (= FP) : KO (= PD) :: FE : FO.

That is, by construction we have, FP : PD :: e : 1.

In the same way any required number of points in the curve may be found. If A and A′ be found so that

AF : AO :: e : 1,

and A′F : A′O :: e : 1,

then A and A′ are points of the curve. Connecting all

* Todhunter's Conic Sections, p. 143.