But this is also the condition that the second diameter bisects all chords parallel to the first.

SCH.—Two diameters are **Conjugate** when each bisects all chords parallel to the other.

Because the conjugate of any diameter is parallel to the chords which the diameter bisects, therefore the inclinations of two conjugates must be connected in the same way as those of a diameter and its bisected chords. Hence, if θ and θ' are the inclinations, the *equation of condition for conjugate diameters* is (Art. 80, Cor.),

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

Since this condition shows that the tangents of inclination of any two conjugate diameters have opposite signs, it indicates that one of the two conjugates makes an *acute* angle with the major axis, and the other an *obtuse* angle. The minor axis makes a *right angle* with the major axis; therefore, *conjugate diameters of an ellipse lie on opposite sides of the minor axis.*

82. *The tangent at either extremity of any diameter is parallel to its conjugate diameter.*

Let (x', y') be either extremity of any diameter; θ the inclination of its conjugate to the major axis. Since (x', y') is on the diameter, its co-ordinates will satisfy its equation, giving us (Art. 80),

$$y' = -\frac{b^2}{a^2}\cot\theta\cdot x';$$

therefore,
$$\tan\theta = -\frac{b^2x'}{a^2y'}. \qquad (1)$$

But, Art. 74, Eq. (7), the equation of the tangent at (x', y') is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}; \qquad (2)$$

therefore, the tangent at the extremity of any diameter is parallel to its conjugate diameter.

83. *Given the co-ordinates x', y' of one extremity of a diameter, to find the co-ordinates x'', y'' of either extremity of the conjugate diameter.*

Since the conjugate diameter passes through the origin, and is parallel to the tangent at (x', y'), by Art. 82, therefore its equation (Art. 74) is

$$y = -\frac{b^2x'}{a^2y'}x. \qquad (1)$$

Substitute this value of y in the equation of the ellipse,

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and, after reducing, we obtain

$$x'' = \pm\frac{a}{b}y'; \qquad (2)$$

and from (1) we have $y'' = \mp\frac{b}{a}x'. \qquad (3)$

The upper sign in each of these values belongs to the extremity at the *right* of the origin, and the lower sign to the left extremity.

Ex T.

84. *To express the lengths of a semi-diameter (a'), and its conjugate (b'), in terms of the abscissa of the extremity of the diameter.*

Let (x', y') and (x'', y'') be the extremities of the diameters (a') and (b'); then we have

$$a'^2 = x'^2 + y'^2 = x'^2 + \frac{b^2}{a^2}(a^2 - x'^2) \text{ (Art. 71)}$$

$$= b^2 + \frac{a^2 - b^2}{a^2}x'^2 = b^2 + e^2x'^2 \text{ (Art. 70, Cor.)}. \qquad (1)$$

Also, $b'^2 = x''^2 + y''^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2$ (Art. 83)

$$= a^2 - x'^2 + \frac{b^2}{a^2}x'^2 \text{ (Art. 71)};$$

hence, $b'^2 = a^2 - e^2x'^2. \qquad (2)$

COR.—Adding (1) and (2), we get

$$a'^2 + b'^2 = a^2 + b^2;$$

that is, the sum of the squares of any pair of conjugate diameters is equal to the sum of the squares of the axes. [See Salmon's Conic Sections, p. 163.]

85. *To find the length of the perpendicular from the centre to the tangent at any point.*

Let (x', y') be the point, and p the perpendicular. The equation of the tangent at (x', y') is (Art. 74),

$$a^2yy' + b^2xx' = a^2b^2.$$

Therefore (Art. 24),

$$p = \frac{a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{ab}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}} = \frac{ab}{b'}. \quad \text{(Art. 84.)}$$

86. *To find the angle between any pair of conjugate diameters.*

Let ϕ be the required angle $=$ SCD in the figure. The angle between the two conjugate diameters is equal to the angle between either diameter and the tangent parallel to the other.

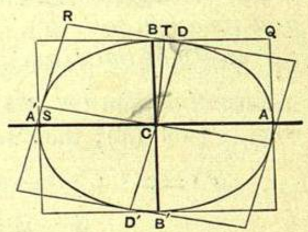

Fig. 74.

From the figure,

$$\sin\phi = \sin \text{SCD} = \sin \text{CDR} = \frac{\text{CT}}{\text{CD}} = \frac{p}{a'} = \frac{ab}{a'b'}. \quad \text{(Art. 85.)}$$

That is, $$\sin\phi = \frac{ab}{a'b'}. \qquad (1)$$

COR.—Clearing (1) of fractions, we have

$$a'b' \sin\phi = ab, \qquad (2)$$

which shows that the parallelogram CDRS is equal to the rectangle CAQB. Hence, *the area of the parallelogram whose sides touch the ellipse at the ends of any pair of conjugate diameters is constant, and equal to the rectangle of the axes.*

SCH.—Since the sum of the squares of any pair of conjugate diameters, (a') and (b'), of an ellipse is constant (Art. 84, Cor.), the rectangle $a'b'$ is the greatest when $a' = b'$,* and therefore $\sin\phi$ is the least; or, *the obtuse angle ϕ is the greatest when the conjugate diameters are of equal lengths.* These diameters are called **Equi-conjugate** diameters. Their length is found by making $a' = b'$ in the equation of Art. 84, Cor., giving

$$a'^2 = \tfrac{1}{2}(a^2 + b^2) = b'^2;$$

therefore, $$a' = b' = \tfrac{1}{2}\sqrt{2} \cdot \sqrt{a^2 + b^2}.$$

Therefore, by (1), $\sin\phi = \dfrac{ab}{a'b'} = \dfrac{2ab}{a^2 + b^2}.$

87. *To prove that the eccentric angles of the vertices of two conjugate diameters differ from each other by 90°.*

Let ϕ be the eccentric angle corresponding to the point D (x', y'), and ϕ' the eccentric angle corresponding to the point S (x'', y''), in Fig. 74.

---

* Let $$a' - b' = x. \qquad (1)$$

Squaring (1), $$a'^2 - 2a'b' + b'^2 = x^2.$$

$$\therefore\; a'b' = \frac{a'^2 + b'^2 - x^2}{2}$$

$$= \frac{a^2 + b^2 - x^2}{2}. \qquad \text{(Art. 84, Cor.)}$$

which is greatest when $x = 0$.

$\therefore$ from (1) $a'b'$ is greatest when $a' = b'$.

Then (Art. 73, Cor. 2), we have

$$x' = a\cos\phi, \qquad (1)$$

$$y' = b\sin\phi. \qquad (2)$$

Also (Art. 83), $\quad x'' = -\dfrac{a}{b}y' = a\cos\phi'$ (Art. 73, Cor. 2)

and $\quad y'' = \dfrac{b}{a}x' = b\sin\phi'$;

or, $$y' = -b\cos\phi', \qquad (3)$$

and $$x' = a\sin\phi'. \qquad (4)$$

Divide (2) by (1), and get

$$\frac{y'}{x'} = \frac{b}{a}\tan\phi. \qquad (5)$$

Divide (4) by (3), and get

$$\frac{x'}{y'} = -\frac{a}{b}\tan\phi'. \qquad (6)$$

Multiply (5) by (6), and get

$$\tan\phi\tan\phi' = -1,$$

or, $$\tan\phi\tan\phi' + 1 = 0.$$

Therefore (Art. 27, Cor. 1), the two angles ϕ and ϕ' differ from each other by 90°.

88. Two chords which join the extremities of any diameter to any point on the ellipse are called **Supplemental Chords.** If that diameter be the major axis, the chords are called **Principal** supplemental chords.

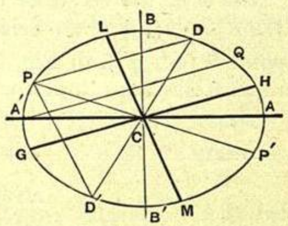

Fig. 75.

Thus, DP and D′P are supplemental with respect to the diameter DD′; AQ and A′Q with respect to the major axis.

89. *If a chord and diameter of an ellipse are parallel, the supplemental chord and the conjugate diameter are parallel.*

Let DD′ (Fig. 75) be a diameter of the ellipse; PD and PD′ two supplemental chords, the first parallel to the diameter GH. Let (x', y') be the point D, and therefore $(-x', -y')$ will be the point D′. Let ϕ be the inclination of the chord DP to the major axis, and ϕ' the inclination of the supplemental chord D′P. Then the equation of DP (Art. 25) is

$$y - y' = \tan\phi\,(x - x'), \qquad (1)$$

and the equation of D′P is

$$y + y' = \tan\phi'\,(x + x'). \qquad (2)$$

Since these lines are to intersect at the point P (x, y), we combine (1) and (2), and get

$$y^2 - y'^2 = \tan\phi\tan\phi'\,(x^2 - x'^2). \qquad (3)$$

And since (x, y) and (x', y') are points on the ellipse, their co-ordinates must satisfy the equation of the ellipse, giving

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and $$a^2y'^2 + b^2x'^2 = a^2b^2;$$

from which we get $\quad y^2 - y'^2 = -\dfrac{b^2}{a^2}(x^2 - x'^2)$;

which in (3) gives $\tan\phi\tan\phi' = -\dfrac{b^2}{a^2}$, which is the condition that two chords shall be supplemental.

But (Art. 81, Sch.) the condition that two diameters are conjugate to each other is

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2};$$

therefore, if $\phi = \theta$, we have $\phi' = \theta'$. But, as the chord PD and diameter GH are parallel, by hypothesis, ϕ does equal θ (if we call θ the inclination of GH); therefore, $\phi' = \theta'$; or, the supplemental chord PD′ and the conjugate diameter LM are parallel.

EXAMPLES.

Find the equations of the tangents to the ellipse $16x^2 + 25y^2 = 400$ from the point (3, 4).

Ans. $y = 4$ and $3x + 2y = 17$.

2. Find (1) the distance from the centre of the ellipse $16x^2 + 25y^2 = 400$ to the tangent making the angle of 30° with the major axis, and (2) the distance from the centre of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ to the tangent making the angle ϕ with the major axis.

Ans. (1) $\frac{1}{2}\sqrt{73}$; (2) $a\sqrt{1 - e^2 \cos^2\phi}$.

3. Find the value of the eccentric angle at the end of the latus rectum in Ex. 2.

Ans. (1) $\tan\phi = \frac{4}{3}$; (2) $\tan\phi = \frac{b}{ae}$.

4. Find the distance from the centre of the ellipse $x^2 + 4y^2 = 16$ to the directrix. *Ans.* $\frac{8}{3}\sqrt{3}$.

5. If tangents are drawn to the ellipse $x^2 + 4y^2 = 4$ from the point (2, 3), find the equation (1) of the chord of contact, and (2) of the line through (2, 3) and the middle of the chord. *Ans.* (1) $x + 6y = 2$; (2) $2y = 3x$.

6. Show that the lines $y = x$, and $3x + 4y = 0$ are conjugate diameters in the ellipse $3x^2 + 4y^2 = 1$.

7. Find the equation of a diameter parallel to the normal at the point (2, 3) in the ellipse $2x^2 + 3y^2 = 6$.

Ans. $9x = 4y$.

8. Find the eccentricity of the ellipse, the angle between the equi-conjugate diameters being 120°. *Ans.* $\frac{1}{3}\sqrt{6}$.

9. Find the equation of the ellipse when the co-ordinate axes are the major axis and right-hand latus rectum.

Ans. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2ex}{a} = \frac{b^2}{a^2}$.

90. *To find the equation of the ellipse referred to any pair of conjugate diameters.*

To do this we must transform the equation of the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2, \qquad (1)$$

from rectangular to oblique axes, having the same origin.

Let DD′ and SS′ be two conjugate diameters. Take CD for the new axis of x, and CS for the new axis of y. Denote the angles ACD and ACS by θ and θ' respectively. Let x, y be the co-ordinates of any point P of the ellipse referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes.

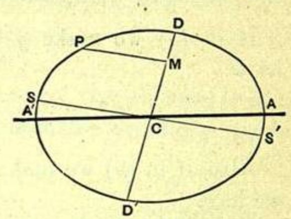

Fig. 76.

The formulæ for transformation (Art. 35, Cor. 1) are,

$$x = x' \cos\theta + y' \cos\theta',$$

$$y = x' \sin\theta + y' \sin\theta',$$

since m and n are 0.

Squaring, substituting in (1), and arranging, we have

$$\left.\begin{array}{r} (a^2 \sin^2\theta + b^2 \cos^2\theta)\, x'^2 \\ + (a^2 \sin^2\theta' + b^2 \cos^2\theta')\, y'^2 \\ + 2\,(a^2 \sin\theta \sin\theta' + b^2 \cos\theta \cos\theta')\, x'y' \end{array}\right\} = a^2b^2, \qquad (2)$$

which is the equation of the ellipse when the oblique axes make *any angles* θ and θ' with the major axis.

But since the new axes CD and CS are conjugate diameters, we have (Art. 81, Sch.),

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2};$$

or, $$a^2 \sin\theta \sin\theta' + b^2 \cos\theta \cos\theta' = 0;$$

hence the coefficient of $x'y'$ in (2) vanishes, and it becomes,

$$(a^2\sin^2\theta + b^2\cos^2\theta)\,x'^2 + (a^2\sin^2\theta' + b^2\cos^2\theta')\,y'^2 = a^2b^2, \quad (3)$$

which is the equation of the ellipse referred to any two conjugate diameters.

In this equation, the coefficients are still in terms of the axes of the ellipse; we may obtain the equation in terms of the conjugate diameters lying on the new axes; thus:

If in (3) we make $y' = 0$, and represent CD by a', we have

$$x'^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} = a'^2. \quad (4)$$

Also, if in (3) we make $x' = 0$, and represent CS by b', we have

$$y'^2 = \frac{a^2b^2}{a^2\sin^2\theta' + b^2\cos^2\theta'} = b'^2. \quad (5)$$

From (4) we get $\quad a^2\sin^2\theta + b^2\cos^2\theta = \dfrac{a^2b^2}{a'^2}. \quad (6)$

From (5) we get $\quad a^2\sin^2\theta' + b^2\cos^2\theta' = \dfrac{a^2b^2}{b'^2}. \quad (7)$

Substituting (6) and (7) in (3), dividing by a^2b^2, and omitting the accents from the variables, we have

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad (8)$$

or,
$$a'^2y^2 + b'^2x^2 = a'^2b'^2, \quad (9)$$

which is the required equation, and is of the same form as when referred to the major and minor axes (Art. 71).

91. *To find the equation of a tangent to the ellipse referred to any pair of conjugate diameters.*

The equation of a right line referred to oblique axes is of the same form (Art. 22, IV) as when referred to rectangular axes; also, the equation of the ellipse, referred to any pair of conjugate diameters, is of the same form (Art. 90), as

when referred to the axes of the ellipse. Hence, the investigation of Art. 74 will apply without any change to the equation $a'^2y^2 + b'^2x^2 = a'^2b'^2$, giving us the required equation,

$$a'^2yy' + b'^2xx' = a'^2b'^2. \quad (1)$$

Cor.—To find where the tangent cuts the axis of x make $y = 0$ in (1), and get

$$x = \frac{a'^2}{x'}.$$

92. *To prove that tangents at the extremities of any chord of an ellipse meet on the diameter which bisects that chord.*

Take the diameter CD, which bisects the chord PP′, for the axis of x, and the conjugate diameter CS for the axis of y.

Let (x', y') be the point P; then $(x', -y')$ will be the point P′.

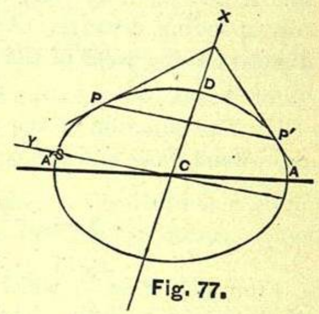

Fig. 77.

The equation of the tangent at P (Art. 91) is

$$a'^2yy' + b'^2xx' = a'^2b'^2. \quad (1)$$

The equation of the tangent at P′ is

$$-a'^2yy' + b'^2xx' = a'^2b'^2. \quad (2)$$

By Art. 91, Cor., both tangents cut the axis of x at the point $\left(\dfrac{a'^2}{x'},\ 0\right)$, which proves the proposition.

93. *If tangents are drawn at the extremities of any focal chord of an ellipse:*

I. *The tangents will intersect on the corresponding directrix.*

II. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

I. If the tangents to an ellipse meet at the point (x', y'), the equation of the chord of contact (Art. 78) is

$$a^2yy' + b^2xx' = a^2b^2.$$

If the chord passes through the right-hand focus, its co-ordinates $(x = ae,\ y = 0)$ must satisfy this equation, giving

$$b^2aex' = a^2b^2;$$

therefore $$x' = \frac{a}{e}; \qquad (1)$$

that is, the point of intersection of the tangents is on the corresponding directrix (Art. 71, Cor. 1), showing that the directrix is the polar of the focus. (Art. 79, Sch.)

II. The equation of the right line passing through the right-hand focus and the point (x', y') is, by (Art. 26),

$$y = \frac{y'}{x' - ae}(x - ae). \qquad (2)$$

From (1), $x' = \frac{a}{e}$, which in (2) gives

$$y = \frac{y'e}{a - ae^2}(x - ae)$$

$$= \frac{aey'}{b^2}(x - ae). \quad \text{(Art. 70, Cor.)} \qquad (3)$$

The equation of the chord of contact (Art. 78) is

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'};$$

which becomes, for the focal chord [since $x' = \frac{a}{e}$, from (1)],

$$y = -\frac{b^2}{aey'}x + \frac{b^2}{y'}, \qquad (4)$$

which is perpendicular to (3), by Art. 27, Cor. 1.

94. *Find the locus of the point of intersection of two tangents at right angles to each other.*

The equation of any tangent to the ellipse, by Art. 74, Cor. 1, is

$$y = mx + \sqrt{a^2m^2 + b^2}. \qquad (1)$$

The equation of the tangent at right angles to (1) is

$$y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} + b^2}. \qquad (2)$$

Clearing (2) of fractions, and transposing in both (1) and (2), we get,

from (1), $$y - mx = \sqrt{a^2m^2 + b^2}, \qquad (3)$$

and from (2), $$ym + x = \sqrt{a^2 + b^2m^2}. \qquad (4)$$

Adding the squares of (3) and (4) together, and dividing by the factor $(1 + m^2)$, we get

$$x^2 + y^2 = a^2 + b^2, \qquad (5)$$

which is the locus required. Hence, the locus is a circle with its centre at C, and $\sqrt{a^2 + b^2}$ for its radius. [See O'Brien's Co-ordinate Geometry, p. 118.]

95. *The rectangle of the focal perpendiculars upon any tangent is constant, and equal to the square of the semi-minor axis.*

Let p and p' be the perpendiculars, and b the semi-minor axis.

The equation of the tangent at any point (x', y') is

$$a^2yy' + b^2xx' = a^2b^2.$$

By Art. 24, we have

$$p = -\frac{b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{b(a - ex')}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}}$$

$$= \frac{b}{b'}(a - ex'). \quad \text{(Art. 84.)}$$

Similarly, $\qquad p' = \dfrac{b}{b'}(a + ex').$

Hence,

$$pp' = \frac{b^2}{b'^2}(a^2 - e^2x'^2) = b^2 \text{ (since } a^2 - e^2x'^2 = b'^2\text{, Art. 84).}$$

96. *To find the polar equation of the ellipse, the focus being the pole.*

Let $F'P = r$; $AF'P = \theta$; then, by definition, Art. 68, we have,

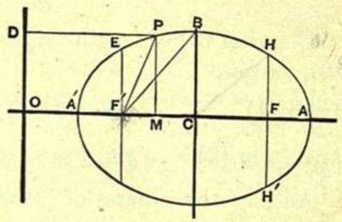

Fig. 78.

$$\begin{aligned} F'P &= e \cdot PD \\ &= e\,(OF' + F'M) \\ &= e \cdot OF' + e \cdot F'M \\ &= a\,(1 - e^2) + e \cdot F'P \cos AF'P \end{aligned}$$

$$\left[\text{since } OF' = \frac{a\,(1 - e^2)}{e}, \text{ by Art. 71, Cor. 1}\right];$$

or $\qquad r = a\,(1 - e^2) + er \cos\theta;$

therefore, $\qquad r = \dfrac{a\,(1 - e^2)}{1 - e\cos\theta}, \qquad (1)$

which is the required equation, the pole being at the left-hand focus.

Cor.—When $\theta = 0$, $r = \dfrac{a\,(1 - e^2)}{1 - e} = a + ae$; which makes $F'A = a + ae$, as it should do. (Art. 71, Cor. 1.)

For the point B, at the extremity of the minor axis,

$$\cos\theta = \frac{F'C}{F'B} = \frac{ae}{r};$$

which substituted for $\cos\theta$ in (1), gives

$$r = \frac{a\,(1 - e^2)}{1 - \dfrac{ae^2}{r}};$$

therefore, $\qquad r = a;$

that is, $F'B = a$, as it should (Art. 71, Cor. 4).

When $\theta = 90°$, $r = \dfrac{a\,(1 - e^2)}{1} = \dfrac{b^2}{a}$ (by Art. 70, Cor.), as it should; that is, F'E, the *semi latus rectum* $= \dfrac{b^2}{a}$ (which agrees with Art. 71, Cor. 4).

When $\theta = 180°$, $r = \dfrac{a\,(1 - e^2)}{1 + e} = a - ae$, which is the value of A'F', as it should be (Art. 71, Cor. 1).

6th day

97. *To find the polar equation of the ellipse when the pole is at the centre.*

The formulæ for passing from a rectangular to a polar system (Art. 36), are

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Substituting these values of x and y in the equation

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and solving for r^2 we find

$$r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{b^2}{1 - \dfrac{a^2 - b^2}{a^2}\cos^2\theta};$$

or $\qquad r^2 = \dfrac{b^2}{1 - e^2\cos^2\theta}$ (Art. 70, Cor.). $\qquad (1)$

Sch.—Equation (1) shows that, for every value of θ, r has two values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter; hence, every diameter of the ellipse is bisected at the centre (see Art. 71, Sch.).

Also, it is evident from (1) that the value of r is the same for θ and $(\pi - \theta)$.

It is equally evident, from (1), that equal diameters make supplemental angles with the major axis.

Therefore, in the case of equi-conjugate diameters,

$$\theta' = \pi - \theta; \quad \text{and hence} \quad \tan \theta' = -\tan \theta;$$

which, in the equation of condition for conjugate diameters (Art. 81, Sch.), gives

$$\tan^2 \theta = \frac{b^2}{a^2} \quad \text{or} \quad \tan \theta = \pm \frac{b}{a}.$$

Hence, *the equi-conjugate diameters of an ellipse are the diagonals of the rectangle constructed on its two axes.*

COR.—When $\theta = 0$ or $180°$, $r = \pm a$; when $\theta = 90°$ or $270°$, $r = \pm b$.

It is evident from equation (1) that r is the greatest possible when $\theta = 0$, giving $r = \pm a$; and the *least* possible when $\theta = 90°$, giving $r = \pm b$. Hence, in every ellipse the transverse axis is the *greatest*, and the conjugate axis is the *least* diameter. For this reason, the transverse and conjugate axes of an ellipse are called the *major* and *minor axes* respectively. (See Art. 71, Sch.)

98. *Any chord which passes through the focus of an ellipse is a third proportional to the major axis and the diameter parallel to the chord.*

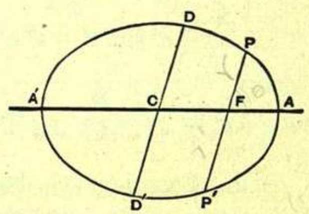

Fig. 79.

Let PP′ be any chord of the ellipse passing through the focus F; and DD′ the diameter parallel to PP′. Put PF $= r$, P′F $= r'$, and AFP $= \theta$.

Then (Art. 96),

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}, \quad \text{and} \quad r' = \frac{a(1 - e^2)}{1 - e \cos \theta}.$$

Therefore $\quad \text{PP}' = r + r' = \dfrac{2a(1 - e^2)}{1 - e^2 \cos^2 \theta}$

$$= \frac{2b^2}{a(1 - e^2 \cos^2 \theta)} \text{ (Art. 70, Cor.).} \qquad (1)$$

From Art. 97, $\quad \overline{\text{CD}}^2 = \dfrac{b^2}{1 - e^2 \cos^2 \theta}. \qquad (2)$

Dividing (1) by (2), we get

$$\frac{\text{PP}'}{\overline{\text{CD}}^2} = \frac{2}{a}, \quad \text{or} \quad \text{PP}' \cdot 2a = 4\overline{\text{CD}}^2;$$

therefore, $\quad 2a : 2\text{CD} :: 2\text{CD} : \text{PP}'.$

EXAMPLES.

1. Find the semi-axes of the ellipse $3y^2 + 2x^2 = 6$.

Comparing this equation with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we find

$$a = \sqrt{3}, \quad \text{and} \quad b = \sqrt{2}, \quad \textit{Ans}$$

2. Find the semi-axes of the ellipse $4y^2 + 3x^2 = 19$.
Ans. $a = \sqrt{\frac{19}{3}}$, $b = \sqrt{\frac{19}{4}}$.

3. Find the points of intersection of the parabola $y^2 = 4x$ and the ellipse $3y^2 + 2x^2 = 14$. *Ans.* (1, 2) and (1, −2).

4. Find the equation of a tangent to the ellipse

$$3y^2 + 2x^2 = 35,$$

at the point whose abscissa is 2. *Ans.* $9y + 4x = 35$.

5. Find the eccentricity of the ellipse $2x^2 + 3y^2 = d^2$.
Ans. Eccentricity $= \sqrt{\frac{1}{3}}$.

6. Find the equation of the tangent to the ellipse at the end of the latus rectum; also, find the lengths of the intercepts of this tangent on the two axes.

Ans. $y + ex = a$; the intercepts are $\dfrac{a}{e}$ on the axis of x, and a on the axis of y.

7. Write the equation of the normal at the end of the latus rectum.

$$Ans.\ y + ae^2 = \frac{x}{e}.$$

8. Find the equation of the line through A′B, and also through CH (Fig. 78); and find the eccentricity of the ellipse if these two lines are parallel.

$$Ans. \begin{cases} y = \frac{b}{a}(x + a); \quad y = \frac{b^2x}{a^2e}; \\ \text{the lines are parallel if } 2e^2 = 1. \end{cases}$$

9. Find a point on the ellipse such that the tangent at the point is equally inclined to the two axes.

$$Ans.\ x = \frac{a^2}{\sqrt{a^2 + b^2}}, \quad y = \frac{b^2}{\sqrt{a^2 + b^2}}.$$

10. Find a point on the ellipse such that the tangent at the point makes intercepts on the two axes that are proportional to the axes.

$$Ans.\ x = \frac{a}{\sqrt{2}}, \quad y = \frac{b}{\sqrt{2}}.$$

11. Express the equation of the tangent at any point of an ellipse in terms of the eccentric angle at that point.

$$Ans.\ \frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

12. Find the angle (θ) at which the focal radius F′P (Fig. 78) is inclined to the major axis, when F′P is a mean proportional between the semi-axes of the ellipse, when $a = 50$ and $b = 30$.

$$Ans.\ \cos\theta = \frac{5\sqrt{5} - 3\sqrt{3}}{4\sqrt{5}}.$$

13. Show that the equations of the tangents to the ellipse $3x^2 + y^2 = 3$, and inclined at an angle of 45° to the major axis, are $y = x + 2$, $y = x - 2$.

14. If the semi-axes of an ellipse are 5 and 4, find the angle at which CP is inclined to the major axis, when it is an arithmetic mean between a and b.

$$Ans.\ \cos\theta = \pm \tfrac{5}{27}\sqrt{17}.$$

15. Find the length of the normal NP, in Fig. 69, and of RP. [See Art. 75, Cor.]

Ans. $\mathrm{NP} = \frac{bb'}{a}$, and $\mathrm{RP} = \frac{ab'}{b}$ (where a and b are the semi-axes, and b' is the semi-diameter conjugate to the diameter passing through the point P).

16. Prove that the equi-conjugate semi-diameter is to the semi-diagonal on the axes as 1 is to $\sqrt{2}$.

17. In the ellipse whose axes are 8 and 6, find the altitude of the circumscribed parallelogram whose sides are parallel to the equi-conjugate diameters.

[Find a' by Art. 86, Sch.; then alt. = area ÷ $2a'$, Cor.]

Ans. 6.79 nearly.

18. In an ellipse whose axes are 12 and 8, what is the length of the diameter from the point whose eccentric angle is 60°?

Ans. $2\sqrt{21}$.

19. If from the vertex of any diameter right lines are drawn to the foci, prove that their product is equal to the square of half the conjugate diameter.

[This follows immediately from Arts. 72 and 84.]

20. Find the locus of the intersection of tangents at the extremities of conjugate diameters.

$$Ans.\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

[This is easily obtained by squaring and adding the equations of the two tangents, observing the relations of Art. 83. (See Salmon's Conic Sections, p. 198.)]

21. Find the locus of the intersection of a tangent with a perpendicular on it from either focus. *Ans.* $x^2 + y^2 = a^2$.

[This is readily obtained by writing the equation of the tangent in the form of (11) in Art. 74, and adding the square of it to the square of the equation of the perpendicular on it from either focus.]

22. Find the locus of the vertex of a triangle, having given the base $= 2m$, and the product of the tangents of the base angles $= \dfrac{l^2}{n^2}$.

[Take the base and a perpendicular to it at the middle point for the axes.] *Ans.* The locus is $l^2x^2 + n^2y^2 = l^2m^2$.

23. Find the polar of either focus of the ellipse; also, of either vertex of the minor axis. *Ans.* $x = \pm \dfrac{a}{e}$, $y = \pm b$.

24. Find the equations of the tangent and normal at the extremity of the latus rectum; and determine the eccentricity of the ellipse in which the normal mentioned passes through the extremity of the minor axis.

$$Ans. \begin{cases} \text{Equation of tangent is } ex + y = a; \\ \quad \text{“} \quad \text{“ normal “} \; x - ey = ae^3; \\ e = \sqrt{\dfrac{b}{a}}. \end{cases}$$

25. The ordinate of any point P on an ellipse is produced to meet the circumscribed circle at P′; prove that the focal perpendicular upon the tangent at P′ is equal to the focal distance of P.

[Use Equation 7, Art. 42, for the tangent at P′; then, Art. 24, $p =$ (after a little reduction) $a - ex = r$. (Art. 72.)]

26. In an ellipse, prove that the rectangle of the central perpendicular on any tangent, and the part of the corresponding normal intercepted between the axes, is constant, and equal to $a^2 - b^2$.

[By Art. 85, $p = \dfrac{ab}{b'}$; by Ex. 15, NR $= \dfrac{b'\,(a^2 - b^2)}{ab}$; $\therefore$ etc.]

27. Find the sum of the focal perpendiculars on the polar of (x', y').

[By Arts. 78 and 24,

$$p + p' = \frac{2a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{2ab}{\sqrt{\dfrac{a^2}{b^2}y'^2 + \dfrac{b^2}{a^2}x'^2}};$$

if (x', y') be *on* the ellipse, this value

$$= \frac{2ab}{\sqrt{a^2 - e^2x'^2}} = \frac{2ab}{b'} \text{ (Art. 84)};$$

if (x', y') be at the right focus, it equals

$$\frac{2ab}{be} = \frac{2a}{e}.$$

(Compare with Art. 71, Cor. 1.)]

28. Prove that the sum of the reciprocals of two focal chords at right angles to each other is constant.

[Find the focal chord PP′ by Art. 98, and the one perpendicular to it by putting sine for cosine; adding the reciprocals, we get $\dfrac{a\,(2 - e^2)}{2b^2} =$ a constant.]

29. If the axes of an ellipse be in the proportion of $\sqrt{2} : 1$, *any* parabola described on the minor axis as axis, and having its vertex at the centre, will cut the ellipse at right angles.

The equations of the ellipse and parabola are

$$a^2y^2 + b^2x^2 = a^2b^2, \tag{1}$$

and

$$x^2 = 2py, \tag{2}$$

respectively. Call θ and ϕ the angles which the tangents to the two curves at their point of intersection make with the axis of x; and ϕ' the angle which the tangent at (2) makes with the axis of y.

Then $$\tan\theta = -\frac{b^2x'}{a^2y'};$$

$$\tan\phi' = \frac{p}{x'} = \cot\phi;$$

hence, $\tan\phi = \dfrac{x'}{p}$;

$$\therefore\ \tan\theta\tan\phi = -\frac{b^2x'^2}{a^2py'} = -\frac{2b^2}{a^2}\ (\text{since } x'^2 = 2py'). \quad (3)$$

Now, as $a : b :: \sqrt{2} : 1$, we have $a^2 = 2b^2$, which in (3) gives

$$\tan\theta\tan\phi = -1;$$

therefore the two tangents, and hence the two curves, at their point of intersection, cut each other at right angles. [See O'Brien's Co-ordinate Geometry, p. 128, where this example is incorrectly solved.]

30. Putting ρ and ρ' to denote the focal radii of any point on an ellipse, and ϕ for its eccentric angle, prove that

$$\rho = a(1 - e\cos\phi),$$
$$\rho' = a(1 + e\cos\phi).$$

31. From the centre of an ellipse, two radii-vectores are drawn at right angles to each other, and tangents to the curve are formed at their extremities; prove that the tangents intersect on the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

32. Express the equation of the normal at any point of an ellipse in terms of the eccentric angle of the point.

Ans. $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1.$

33. Show that the equation of the locus of the poles of normal chords of an ellipse is

$$x^2y^2(a^2 - b^2)^2 = a^6y^2 + b^6x^2.$$

34. Show that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by the constant 2α is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\alpha.$$

CHAPTER VII.

THE HYPERBOLA.

99. The **Hyperbola** is the locus of a point moving in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line, the ratio being greater than unity.*

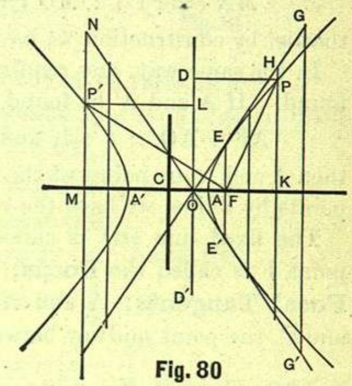

Fig. 80

From this definition the hyperbola may be constructed by points, thus:

Let F be the fixed point, DD′ the fixed right line, and e the given ratio. Draw through F the line OAF perpendicular and EE′ parallel to DD′. Take

$$\text{FE } (= \text{FE}') : \text{FO} :: e : 1,$$

and draw OE and OE′ produced indefinitely. Draw parallels to EE′, meeting the lines OG and OG′. With the half of any one of these parallels, as KH, for a radius, and the fixed point F for a centre, describe an arc cutting KH at P; this is a point of the curve. For, joining P and F, and drawing PD perpendicular to DD′, we have

$$\text{KH } (= \text{FP}) : \text{KO } (= \text{PD}) :: \text{FE} : \text{FO};$$

that is, by construction we have

$$\text{FP} : \text{PD} :: e : 1.$$

In the same way, any required number of points in the curve may be found.

* See Todhunter's Conic Sections, p. 188.