hence, $$\tan\phi = \frac{x'}{p};$$

$$\therefore \ \tan\theta\tan\phi = -\frac{b^2x'^2}{a^2py'} = -\frac{2b^2}{a^2}\ (\text{since } x'^2 = 2py'). \quad (3)$$

Now, as $a : b :: \sqrt{2} : 1$, we have $a^2 = 2b^2$, which in (3) gives

$$\tan\theta\tan\phi = -1;$$

therefore the two tangents, and hence the two curves, at their point of intersection, cut each other at right angles. [See O'Brien's Co-ordinate Geometry, p. 128, where this example is incorrectly solved.]

30. Putting ρ and ρ' to denote the focal radii of any point on an ellipse, and ϕ for its eccentric angle, prove that

$$\rho = a(1 - e\cos\phi),$$
$$\rho' = a(1 + e\cos\phi).$$

31. From the centre of an ellipse, two radii-vectores are drawn at right angles to each other, and tangents to the curve are formed at their extremities; prove that the tangents intersect on the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

32. Express the equation of the normal at any point of an ellipse in terms of the eccentric angle of the point.

Ans. $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$

33. Show that the equation of the locus of the poles of normal chords of an ellipse is

$$x^2y^2(a^2 - b^2)^2 = a^6y^2 + b^6x^2.$$

34. Show that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by the constant 2α is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\alpha.$$

CHAPTER VII.

THE HYPERBOLA.

99. The **Hyperbola** is the locus of a point moving in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line, the ratio being greater than unity.*

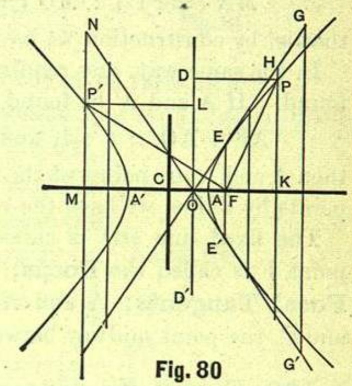

Fig. 80

From this definition the hyperbola may be constructed by points, thus:

Let F be the fixed point, DD′ the fixed right line, and e the given ratio. Draw through F the line OAF perpendicular and EE′ parallel to DD′. Take

$$\text{FE} (= \text{FE}') : \text{FO} :: e : 1,$$

and draw OE and OE′ produced indefinitely. Draw parallels to EE′, meeting the lines OG and OG′. With the half of any one of these parallels, as KH, for a radius, and the fixed point F for a centre, describe an arc cutting KH at P; this is a point of the curve. For, joining P and F, and drawing PD perpendicular to DD′, we have

$$\text{KH} (= \text{FP}) : \text{KO} (= \text{PD}) :: \text{FE} : \text{FO};$$

that is, by construction we have

$$\text{FP} : \text{PD} :: e : 1.$$

In the same way, any required number of points in the curve may be found.

* See Todhunter's Conic Sections, p. 188.

Since $e > 1$, the distance from F to any point in the curve is greater than the distance from the same point to the line DD′; therefore there are points in the curve on the opposite side of DD′, which are found in the same way as those to the right of DD′, thus: with the half of any of the parallels, to the left of DD′, as MN, for a radius, and F for a centre, describe an arc cutting MN at P′; this is a point of the curve. For, joining P′ and F, and drawing P′L perpendicular to DD′, we have

$$\text{MN} \; (=\text{FP}') : \text{MO} \; (=\text{P}'\text{L}) :: \text{FE} : \text{FO};$$

that is, by construction we have $\text{FP}' : \text{P}'\text{L} :: e : 1$.

In the same way, any required number of points may be found. If A and A′ be found so that

$$\text{AF} : \text{AO} :: e : 1, \text{ and } \text{A}'\text{F} : \text{A}'\text{O} :: e : 1,$$

then A and A′ are points of the curve. Connecting all these points by a line, we have the required hyperbola.

The fixed line DD′ is called the **Directrix;** the fixed point F is called the **Focus;** OG and OG′ are called the **Focal Tangents;** A and A′ are called the **Vertices;** and C, the point midway between them, is the **Centre.**

note 31

100. *To find the distances from the centre of the hyperbola to the focus and the directrix.*

Represent AA′ by $2a$, and the given ratio by e.

Then we have, from definition,

$$\text{AF} : \text{AO} :: \text{A}'\text{F} : \text{A}'\text{O} :: e : 1. \qquad (1)$$

$$\therefore \; \text{AF} : \text{AO} :: \text{AF} + \text{A}'\text{F} : \text{AO} + \text{A}'\text{O},$$

or

$$e : 1 :: 2\text{CF} : 2a;$$

$$\therefore \; \text{CF} = ae. \qquad (2)$$

Also from (1), we have

$$\text{AF} : \text{AO} :: \text{A}'\text{F} - \text{AF} : \text{A}'\text{O} - \text{AO}$$
$$:: \text{AA}' : \text{AA}' - 2\text{AO},$$

or

$$e : 1 :: 2a : 2\text{CO};$$

$$\therefore \; \text{CO} = \frac{a}{e}. \qquad (3)$$

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101. *To find the equation of the hyperbola.*

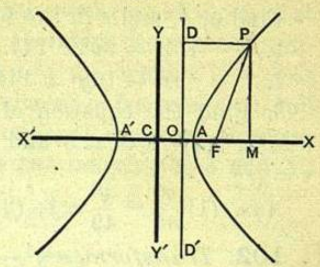

Fig. 81

Let F be the focus, DD′ the directrix, A and A′ the vertices, and C the centre. Take AA′ as the axis of x, and the perpendicular through C as the axis of y.

Let (x, y) be any point P on the locus; join FP; draw PM and PD respectively perpendicular to CX and CY.

Represent AA′ by $2a$, and the given ratio by e.

Then we have, from definition,

$$\text{FP} = e\,\text{PD},$$

or

$$\overline{\text{FP}}^2 = e^2\,\overline{\text{PD}}^2;$$

$$\therefore \; \overline{\text{FM}}^2 + \overline{\text{MP}}^2 = e^2\,\overline{\text{OM}}^2.$$

But $\text{FM} = \text{CM} - \text{CF} = x - ae$; (Art. 100)

and $\text{OM} = \text{CM} - \text{CO} = x - \dfrac{a}{e}$. "

$$\therefore \; (x - a\,e)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2;$$

or

$$y^2 = (1 - e^2)\,(a^2 - x^2), \qquad (1)$$

which is the required equation.

Cor.— When $x = 0$, equation (1) becomes

$$y^2 = (1 - e^2)\,a^2 = -\,b^2 \text{ [by putting } (e^2 - 1)\,a^2 = b^2],$$

$$\therefore \; (e^2 - 1) = \frac{b^2}{a^2},$$

which in (1) gives

$$y^2 = \frac{b^2}{a^2}\,(x^2 - a^2), \qquad (2)$$

or

$$a^2\,y^2 - b^2\,x^2 = -\,a^2\,b^2, \qquad (3)$$

which may be written in the symmetric form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (4)$$

Note.—Since $e > 1$, $a^2\,(1 - e^2)$ is *negative;* and therefore we put it equal to $-\,b^2$ above.

note 32 - ext l

EXAMPLES.

Find a, b, and e in the following hyperbolas:

1. $16x^2 - 9y^2 = 144$. *Ans.* $3, 4, \frac{5}{3}$.

2. $9x^2 - 16y^2 = 144$. *Ans.* $4, 3, \frac{5}{4}$.

3. Find the equation of an hyperbola (1) if $a = 8$ and $b = 7$; (2) if $2a = 5$ and $2ae = 13$; (3) if $ae = 1$ and $e = \sqrt{2}$.

Ans. (1) $\frac{x^2}{64} - \frac{y^2}{49} = 1$; (2) $\frac{4x^2}{25} - \frac{y^2}{36} = 1$; (3) $2x^2 - 2y^2 = 1$.

102. *Transform* $a^2y^2 - b^2x^2 = -a^2b^2$, (1)

to the vertex A. The formulæ for this transformation become $x = x' + a$, $y = y'$,

which in (1) give, after suppressing accents, and solving for y^2,

$$y^2 = \frac{b^2}{a^2}(2ax + x^2). \tag{2}$$

Cor. 1.—We have from (2) and (3) of Art. 100,

$$\text{CF} = ae, \text{ and } \text{CO} = \frac{a}{e}.$$

$$\therefore \text{AF} = a(e-1), \quad \text{OA} = \frac{a(e-1)}{e}, \quad \text{OF} = \frac{a(e^2-1)}{e}.$$

Cor. 2.—When $y = 0$ in (1), $x = \pm a$, which shows that the curve cuts the axis of x at two points equally distant from the origin, and on opposite sides of it. When $x = 0$, $y = \pm b\sqrt{-1}$; hence the curve cuts the axis of y in two *imaginary* points on opposite sides of the origin. We may, however, take two points B and B′, on different sides of C, making $\text{CB} = \text{CB}' = b$, as we shall have occasion to use them hereafter.

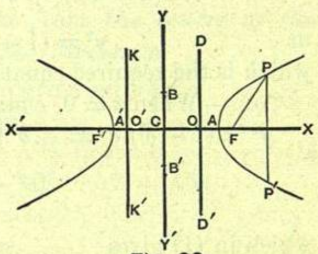

Fig. 82.

Cor. 3.—Solving (1) for y, we get

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2},$$

which shows that for every value of $x > +a$ or $< -a$ there are two *real* values of y, numerically equal, with contrary signs; hence, for every point P on one side of the axis of x, there is a point P′ on the other side of the axis, at the same distance from it; and therefore the curve is symmetrical with respect to the axis of x. When x is $+a$ or $-a$, $y = \pm 0$; and for every value of x between $+a$ and $-a$, the two values of y are imaginary; therefore the curve is limited towards the centre by two tangents at A and A′.

Similarly, solving (1) for x, we get

$$x = \pm\frac{a}{b}\sqrt{y^2 + b^2},$$

which shows that for every value of y from $-\infty$ to $+\infty$ there are two real values of x, numerically equal with contrary signs; hence the curve is symmetrical with respect to the axis of y, and is unlimited in the direction of this axis.

Sch.—Because the curve is symmetrical with respect to the line BB′, it follows that if we take $\text{CF}' = \text{CF}$ (Fig. 82), and $\text{CO}' = \text{CO}$, and draw KK′ perpendicular to OO′, the point F′ and the line KK′ will form respectively a second focus and directrix.

AA′ is called the **Transverse** axis of the hyperbola; BB′ is called the **Conjugate** axis of the hyperbola. In the *ellipse*, the conjugate axis is always less than the transverse axis (see Art. 70, Cor.), and therefore the former was called the *minor* and the latter the *major* axis. In the *hyperbola*, the conjugate axis may be greater than the transverse, since $b^2 = a^2(e^2 - 1)$ (Art. 101, Cor.), and e is > 1; therefore we do not call the axes in the hyperbola the *major* and *minor* axes.

The ratio e (Art. 99) is called the **Eccentricity** of the hyperbola.

The point C is called the **Centre** of the hyperbola, because it bisects every chord of the hyperbola which passes through it. This may be shown in the same way as in the case of the ellipse (Art. 71, Sch.), Cor. 4.—*To find the latus rectum* (Art. 53, Cor. 3).

Make $x = \text{CF} = ae$ (Cor. 1); denote the corresponding value of y by p; we have from Eq. (1) (Art. 102),

$$p^2 = \frac{b^2}{a^2}(a^2e^2 - a^2) = b^2(e^2 - 1) = \frac{b^4}{a^2} \text{ (Art. 101, Cor.).}$$

Therefore, $\quad 2p = \frac{2b^2}{a} = \frac{4b^2}{2a} = $ *latus rectum.*

Forming a proportion from this equation, we have

$$2a : 2b :: 2b : 2p.$$

That is, *the latus rectum is a third proportional to the transverse axis and the conjugate.*

Since $b^2 = (e^2 - 1)a^2$ (Art. 101, Cor.), we have

$$a^2 + b^2 = a^2e^2;$$

that is, $\quad a^2 + b^2 = \overline{\text{CF}}^2$ (Art. 102, Cor. 1).

But $\quad a^2 + b^2 = \overline{\text{AB}}^2$ (see Fig. 82).

Therefore, $\quad \text{AB} = \text{CF}.$

Hence, *the conjugate axis of the hyperbola is a perpendicular to the transverse axis at its centre, and is limited by an arc described with the vertex of the transverse axis as a centre, and with a radius equal to the distance from the focus to the centre.*

Cor. 5.—Comparing equation (1), Art. 102, with (1) of Art. 71, we see that the equation of the hyperbola may be derived from that of the ellipse, by changing $+b^2$ into $-b^2$. Hence, we infer that *any function of b, expressing a property of the ellipse, will be converted into one expressing a corresponding property of the hyperbola, by changing b into* $b\sqrt{-1}$; therefore, in obtaining the properties of the hyperbola that are similar to those which have been proved for the ellipse, we shall, in most cases, either change the sign of b^2, or else refer the student to the corresponding demonstration in the ellipse.

By a process similar to that of Art. 71, Cor. 5, the details of which the student must supply, we obtain

$$y'^2 : y''^2 :: (x' + a)(x' - a) : (x'' + a)(x'' - a);$$

that is, *the squares of any two ordinates to the transverse axis of an hyperbola are to each other as the rectangles of the segments into which they divide the transverse axis.*

Cor. 6.—A point is *outside, on,* or *inside* the hyperbola, according as $a^2y^2 - b^2x^2 + a^2b^2 >, =,$ or < 0. The proof is similar to that given in Art. 71, Cor. 6, for the ellipse.

A point is said to be *outside* the hyperbola if it lies in the space between the branches, so that no right line can be drawn through it to a focus without cutting the curve.

103. *To find the distance of any point in the hyperbola from the focus, in terms of the abscissa of the point.*

From the figure we have

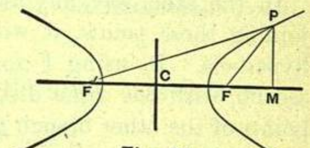

Fig. 83.

$$\overline{\text{FP}}^2 = (x - ae)^2 + y^2$$

$$= (x - ae)^2 + \frac{b^2}{a^2}x^2 - b^2.$$

(Art. 102.)

$$= a^2 - 2aex + e^2x^2 \text{ (since } a^2e^2 - b^2 = a^2\text{);}$$

therefore, $\quad \text{FP} = ex - a.$

[We take only the *positive* value of the root, for the reason given in Art. 72.]

In like manner we find, by writing $-ae$ for $+ae$,

$$\overline{\text{F'P}}^2 = (x + ae)^2 + y^2 = a^2 + 2aex + e^2x^2;$$

therefore, $\quad \text{F'P} = ex + a.$

Hence, $\quad \text{F'P} - \text{FP} = 2a;$

or, *the difference of the distances of any point in an hyperbola from the foci is equal to the transverse axis.*

COR.—This result furnishes two other methods of constructing an hyperbola, having given the axes.

I. With C as a centre and BA as a radius, describe an arc cutting AA′ produced at F and F′; these points are the foci (Art. 102, Cor. 4). Now, with F′ as a centre and a radius greater than F′A, describe an arc; then with F as a centre, and a radius equal to that used before, diminished by the transverse axis AA′, describe another arc cutting the first at the point P; this will be a point of the curve, since

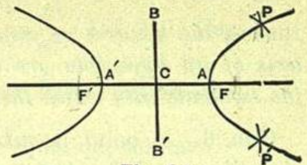

Fig. 84.

$$FP = F'P - 2a,$$

or
$$F'P - FP = 2a.$$

In the same way, any number of points may be found; joining these points, it will be a branch of the required hyperbola. By using F for the first centre and F′ for the second, with the same distances as before, any number of points of the other branch may be found.

II. Take a ruler, and fasten one end of it at F′ so it can revolve about F′ as a centre. Take a string whose length is less than that of the ruler by AA′, and fasten one end of it at F and the other end at B, the end of the ruler; then press the string against the edge of the ruler with the point of a pencil P, and revolve the ruler about F′, keeping the string tight; the pencil will describe one branch of an hyperbola, since, in every position of it, we shall have

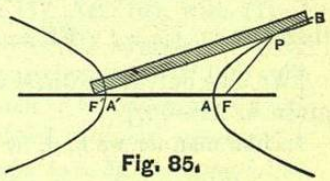

Fig. 85.

$$F'P - FP = AA'.$$

104. A **Conjugate Hyperbola** is one having the conjugate axis of a given hyperbola for its transverse axis, and the transverse axis of the given hyperbola for its conjugate axis. Either of two hyperbolas thus related is conjugate to the other. Thus, the hyperbola whose transverse axis is BB′ (Fig. 86) is the conjugate of the hyperbola whose transverse axis is AA′, and *conversely*, the latter is the conjugate of the former. They are often distinguished as the x **Hyperbola** and the y **Hyperbola**, each taking the name of the coordinate axis upon which its transverse axis lies; and when spoken of together are called **Conjugate Hyperbolas.**

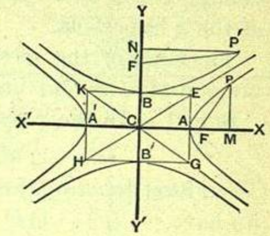

Fig.86

105. *To find the equation of an hyperbola conjugate to a given hyperbola.*

By Art. 102, the equation of the given hyperbola is

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

or, Fig. 86,
$$\overline{PM}^2 = \frac{\overline{CB}^2}{\overline{CA}^2}(\overline{CM}^2 - \overline{CA}^2).$$

Hence, since P′ is a point on the conjugate hyperbola, having BB′ for its transverse axis and AA′ for its conjugate axis, we have,

$$\overline{NP'}^2 = \frac{\overline{CA}^2}{\overline{CB}^2}(\overline{CN}^2 - \overline{CB}^2),$$

or
$$x^2 = \frac{a^2}{b^2}(y^2 - b^2), \qquad (1)$$

which is the equation of the conjugate hyperbola, and is the same expression we would obtain from the equation of the given hyperbola by putting $-b^2$ for $+b^2$, and $-a^2$ for $+a^2$.

Or, since the second hyperbola holds the same relation to the axis of y that the first does to the axis of x, we might have deduced the equation of the y hyperbola at once by changing a to b and b to a, x to y and y to x in the equation of the x hyperbola.

The sides of the rectangle described on the axes are the tangents to the four branches at the vertices.

Sch. 1.—In the x hyperbola we have (Art. 101, Cor.),

$$(e^2-1)\,a^2 = b^2; \quad \therefore \quad a^2e^2 = a^2 + b^2.$$

Therefore, denoting the eccentricity of the y hyperbola by e', we have $\quad (e'^2-1)\,b^2 = a^2; \quad \therefore \quad b^2e'^2 = a^2+b^2.$

Hence $\quad a^2e^2 = b^2e'^2$; or $\overline{\mathrm{CF}}^2 = \overline{\mathrm{CF}'}^2$.

(See Art. 102, Cor. 1.) Therefore the foci of the y hyperbola are at the same distance from the centre as the foci of the x hyperbola, but the *eccentricity* of the former has a different value from that of the latter.

Sch. 2.—The equations of the diagonals CE and CG are respectively

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

If in the equations of the two conjugate hyperbolas we make $b = a$, we have (Art. 102),

$$y^2 - x^2 = -a^2, \tag{2}$$

and (1) of the present Art. becomes

$$y^2 - x^2 = a^2. \tag{3}$$

These hyperbolas are called **Equilateral** hyperbolas, from the equality of the axes. The equilateral hyperbola corresponds to the case in which the ellipse becomes a circle. (See Art. 71, Cor. 7.) The peculiarity in the figure of the equilateral hyperbola is that *the curve is identical in form with its conjugate.* From Art. 101, Cor., we have

$$e^2 - 1 = \frac{b^2}{a^2};$$

therefore, in the *equilateral* hyperbola we have $e = \sqrt{2}$.

106. *To construct a pair of conjugate hyperbolas whose axes are given.*

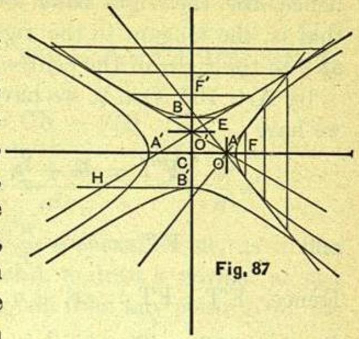

Fig. 87

Draw the axes AA$'$ and BB$'$ at right angles to each other; construct the x hyperbola as in Art. 99. Now take CF$'$ = CF, which equals AB (Art. 105, Sch.), and F$'$ is the focus of the y hyperbola. Take BE = BF$'$, and B$'$H = B$'$F$'$; draw through E and H a right line; it is one of the focal tangents. Through O$'$ draw a line perpendicular to BB$'$; this is the directrix corresponding to the focus F$'$ of the y hyperbola. The construction is now the same as in Art. 99.

107. *To find the equation of the tangent at any point of an hyperbola.*

To obtain this equation for the hyperbola, we change b^2 into $-b^2$ in equations (6), (7), and (11) of Art. 74, and get

$$a^2yy' - b^2xx' = -a^2b^2, \tag{1}$$

$$y = \frac{b^2x'}{a^2y'}x - \frac{b^2}{y'}, \tag{2}$$

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{3}$$

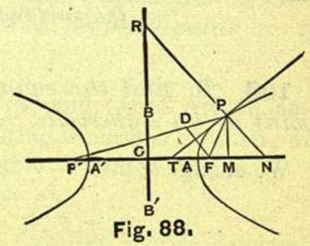

Fig. 88.

Cor.—To find the point in which the tangent cuts the axis of x, make $y = 0$ in (1), and get

$$x = \frac{a^2}{x'} = \mathrm{CT},$$

which is the same value we found for the abscissa of the point at which the tangent cuts the axis of x in the ellipse.

(Art. 74, Cor. 2). This value of x has the same sign as x'; hence, for the right-hand branch, it is always positive; that is, the tangent to the right-hand branch cuts the axis of x to the right of the centre.

By Art. 102, Cor. 2, we have $F'C = FC = ae$; therefore we have

$$F'T = ae + \frac{a^2}{x'} = \frac{a}{x'}(ex' + a),$$

and

$$FT = ae - \frac{a^2}{x'} = \frac{a}{x'}(ex' - a).$$

Hence, $F'T : FT :: ex' + a : ex' - a :: F'P : FP$

(by Art. 103). That is, *the tangent of an hyperbola cuts the distance between the foci in segments proportional to the adjacent focal radii of contact; and therefore it bisects the internal angle between these focal radii.*

This principle affords a method of drawing a tangent to an hyperbola at a given point.

Let P be the given point (see Fig. 88). Draw the focal radii F'P and FP to the given point P. On the longer, F'P, lay off PD = PF, and join DF. Through P draw PT perpendicular to DF; PT will be the tangent required, for it bisects the angle FPF'.

The *subtangent* $MT = CM - CT = x' - \frac{a^2}{x'}$. That is,

$$\textit{the subtangent} = \frac{x'^2 - a^2}{x'}.$$

108. *To find the equation of the normal at any point of an hyperbola.*

We change b^2 into $-b^2$ in (2) of Art. 75, and get

$$y - y' = -\frac{a^2y'}{b^2x'}(x - x'), \tag{1}$$

which is the required equation of the normal at (x', y').

COR. 1.—To find the point in which the normal cuts the axis of x, we make $y = 0$ in (1), and get, after reduction,

$$x = \frac{a^2 + b^2}{a^2}x' = CN \text{ (Fig. 88)} = e^2x' \text{ (Art. 105, Sch. 1).}$$

The subnormal $MN = CN - CM$

$$= \frac{a^2 + b^2}{a^2}x' - x' = \frac{b^2}{a^2}x'.$$

SCH.—The expression $CN = e^2x'$ enables us, as in the case of the ellipse (Art. 75, Sch.), to draw a normal at any point P of the hyperbola, or one from any point N of the transverse axis.

COR. 2.—By Art. 102, Cor. 1,

$$F'C = FC = ae;$$

therefore we have

$$F'N = e(ex' + a),$$

and

$$FN = e(ex' - a).$$

Hence, $F'N : FN :: ex' + a : ex' - a :: F'P : FP$

(Art. 103). That is, *the normal of an hyperbola cuts the distance between the foci in segments proportional to the adjacent focal radii of contact; and hence it bisects the external angle between the focal radii of contact.*

109. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from either focus.*

Changing the sign of b^2 in (3) and (4) of Art. 76, and adding the squares of the resulting equations together, we get

$$x^2 + y^2 = a^2,$$

for the required locus, which is therefore a circle described on the transverse axis.

EXAMPLES.

1. Find the equation of an hyperbola if the distance between the foci = twice the transverse axis.
Ans. $y^2 - 3x^2 + 3a^2 = 0$.

2. Find the equation of the hyperbola conjugate to the hyperbola $9x^2 - 4y^2 = 36$, the axes, and the distance between its foci.
Ans. $\begin{cases} 4y^2 - 9x^2 = 36; \text{ transverse} = 6, \text{ conjugate} = 4; \\ \text{distance between foci} = 2\sqrt{13}. \end{cases}$

3. Find the equation of the hyperbola if the distance between the foci = 6 and the transverse axis = 4.
Ans. $5x^2 - 4y^2 = 20$.

4. If the vertex of an hyperbola bisects the distance from the centre to the focus, and the transverse axis = 10, find the equation of the hyperbola. *Ans.* $3x^2 - y^2 = 75$.

5. If the distance from the focus of an hyperbola to the nearest vertex is 1 and the eccentricity is $1\frac{2}{3}$, find (1) the equation of the hyperbola, and (2) its latus rectum.
Ans. (1) $16x^2 - 9y^2 = 36$; (2) $5\frac{1}{3}$.

6. Find the equations of the tangent and the normal to the hyperbola $4x^2 - 9y^2 = 36$ at the point of contact $(4\frac{1}{2}, \sqrt{5})$.
Ans. $2x - \sqrt{5}y - 4 = 0$; $4y + 2\sqrt{5}x = 13\sqrt{5}$.

7. Find the perpendicular distance from the origin to the tangent at the end of the latus rectum of the equilateral hyperbola $x^2 - y^2 = 9$. *Ans.* $\sqrt{3}$.

8. Find the equations of the tangents to $9x^2 - 4y^2 = 36$ which are parallel to $y = 3x - 4$. *Ans.* $y = 3x \pm 3\sqrt{3}$.

9. Find the equations of the tangents to the equilateral hyperbola at the positive end of the latus rectum.
Ans. $y = \pm x\sqrt{2} - a$.

110. *To find the co-ordinates of the point of contact of a tangent to an hyperbola from a fixed point.*

Let (x', y') be the required point of contact, and (x'', y'') the fixed point through which the tangent passes.

Changing $+ b^2$ to $- b^2$ in the results of Art. 77, we get

$$x' = \frac{a^2b^2x'' \mp a^2y''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2}}{b^2x''^2 - a^2y''^2},$$

$$y' = \frac{a^2b^2y'' \mp b^2x''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2}}{b^2x''^2 - a^2y''^2}.$$

These values indicate that from any fixed point *two* tangents can be drawn to an hyperbola, *real*, *coincident*, or *imaginary*, according as

$$a^2y''^2 - b^2x''^2 + a^2b^2 >, =, \text{ or } < 0;$$

that is, according as the point (x'', y'') is *outside*, *on*, or *inside* the curve (Art. 102, Cor. 6).

Cor.—It is clear that if any two real tangents be drawn from a given point to touch the *same branch*, their abscissas of contact will have *like* signs; and *unlike*, if they touch *different* branches. Hence, since the values of x in the former case must have the same signs, we have, regarding only their *numerical* values,

$$a^2b^2x'' > a^2y''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2};$$

or squaring, transposing, and reducing, we have

$$y'' < \frac{b}{a}x''. \qquad (1)$$

But (Art. 105, Sch. 2) $y = \frac{b}{a}x$ is the equation of the diagonal of the rectangle formed upon the axes of the hyperbola; therefore, the ordinate of the point from which two tangents can be drawn to the *same branch* of an hyperbola must be less than the corresponding ordinate of the diagonal; that is, the point itself must be somewhere between the diagonals (CE, CG) or (CH, CK) produced, and the adjacent branch of the curve (Fig. 86). These diagonals produced are called **Asymptotes** of the hyperbola, which we shall consider in Art. 113. Hence, generally, the two tangents which can be drawn to an hyperbola from any external point, will both touch the *same branch*, if the external point be between that branch and the adjacent portions of the asymptotes; but if the external point be so placed that we cannot pass from it to the curve without crossing an asymptote, the two tangents touch *different branches* of the curve.

111. *Tangents are drawn to an hyperbola from a given external point; to find the equation of the chord of contact* (Art. 77).

Change b^2 into $-b^2$ in (5) of Art. 78, and get

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (1)$$

which is the equation of the chord of contact.

112. *Through any fixed point a chord is drawn to an hyperbola, and tangents to the hyperbola are drawn at the extremities of the chord; to find the equation of the locus of the intersection of the tangents, when the chord is turned about the fixed point.*

Change b^2 into $-b^2$ in (3) of Art. 79, and get

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (1)$$

which is the equation required, and the locus is a right line.

SCH.—The line (1) is called the **Polar** of the point (x', y') with regard to the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$, and the point (x', y') is called the **Pole** of the line.

The statements in Art. 49 with respect to the circle may all be applied to the hyperbola as they were to the parabola (Art. 61), and the same conclusions arrived at that were reached in Arts. 49 and 61, and referred to in the ellipse (Art. 79, Sch.).

113. An **Asymptote** of a curve is a line which continually approaches the curve, and becomes tangent to it only at an infinite distance, while it passes within a finite distance of the origin. We have called the diagonals produced of the rectangle on the axes (Art. 110, Cor.), the *asymptotes* of the hyperbola; we now proceed to show that they are such, that is, that they meet the curve only at infinity.

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114. *To prove that the diagonals of the rectangle on the axes are asymptotes to both the given and conjugate hyperbolas.*

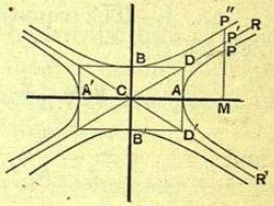

Fig. 89.

Produce the ordinate MP of any point P in the given hyperbola, to meet the diagonal CR and the conjugate hyperbola, in the points P′ and P″ respectively. The distance of the point P from CR = PP′ sin PP′C, and therefore it varies as PP′. Now, if CM, the common abscissa $= x$, PM $= y$, P′M $= y'$, and P″M $= y''$, we have, from the equations of the given hyperbola, the diagonal, and the conjugate hyperbola,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \qquad (1)$$

$$y'^2 = \frac{b^2}{a^2}x^2, \qquad (2)$$

$$y''^2 = \frac{b^2}{a^2}(x^2 + a^2). \qquad (3)$$

Subtracting (1) from (2), we have

$$y'^2 - y^2 = b^2, \quad \text{or} \quad y' - y = \frac{b^2}{y + y'}. \qquad (4)$$

Subtracting (2) from (3), we have

$$y''^2 - y'^2 = b^2, \quad \text{or} \quad y'' - y' = \frac{b^2}{y'' + y'}. \qquad (5)$$

If now we suppose the abscissa CM to increase continually, and the line MP to move parallel to itself, the ordinates y, y', and y'' will increase continually, and therefore, from (4) and (5), $y' - y$ and $y'' - y'$ will diminish continually; and when x (CM), and therefore y, y', and y'' become infinitely great, $y' - y$ and $y'' - y'$ will become infinitely small; that is, as x increases indefinitely, the two curves continually approach the diagonal CR, and become tangent to it and to each other only at infinity. Hence the diagonals are asymptotes to both curves.

COR. 1.—The equations of CR and CR′ are (Art. 105, Sch. 2),

$$y = \frac{b}{a}x \quad \text{or} \quad \frac{x}{a} - \frac{y}{b} = 0;$$

and

$$y = -\frac{b}{a}x \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 0;$$

therefore the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ includes both asymptotes.

COR. 2.—Let ACR $= \theta$, ACR′ $= \theta'$; then

$$\tan\theta = \frac{b}{a}, \qquad \tan\theta' = -\frac{b}{a};$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}, \qquad \cos\theta = \frac{a}{\sqrt{a^2+b^2}} = \frac{1}{e};$$

$$\sin\theta' = -\frac{b}{\sqrt{a^2+b^2}}, \qquad \cos\theta' = \frac{a}{\sqrt{a^2+b^2}} = \frac{1}{e}.$$

115. *To find the equation of any diameter.* (Def. of Art. 62.)

Change b^2 into $-b^2$ in (2) of Art. 80, and get

$$y = \frac{b^2}{a^2}\cot\theta \cdot x \tag{1}$$

for the required equation.

Since a^2 and b^2 are constant for any given hyperbola, and θ is constant for any given system of parallel chords, (1) is the equation of a right line passing through the origin, that is, through the centre of the hyperbola. Hence, every diameter of the hyperbola passes through the centre. By giving to θ a suitable value, (1) may be made to represent *any* right line passing through the centre. Hence, *every* right line that passes through the centre of an hyperbola is a diameter; that is, it bisects some system of parallel chords.

SCH.—To draw a diameter of an hyperbola, draw any two parallel chords, and bisect them; the line passing through the points of bisection is a diameter. The intersection of two diameters will be the centre of the hyperbola.

COR. 1.—Let $\theta' =$ the inclination of the diameter itself to the transverse axis; then we have

$$\tan\theta' = \frac{y}{x};$$

which in (1) gives

$$\tan\theta\tan\theta' = \frac{b^2}{a^2},$$

as the relation between θ and θ' when they are the angles which a system of parallel chords and their diameter respectively make with the axis of x.

COR. 2.—Writing the equation of the diameter in the form

$$y = \tan\theta \cdot x, \tag{1}$$

and eliminating y between this equation and that of the given hyperbola, to find the abscissas of the points of intersection of (1) and the curve, we obtain

$$x = \pm\frac{ab}{\sqrt{b^2 - a^2\tan^2\theta}}. \tag{2}$$