Now, eliminating y between (1) and the equation of the conjugate hyperbola (Art. 105), to find the abscissas of the points of intersection of (1) and the conjugate curve, we obtain

$$x = \pm \frac{ab}{\sqrt{a^2 \tan^2 \theta - b^2}}. \qquad (3)$$

If $a^2 \tan^2 \theta < b^2$, that is, if $\tan \theta < \pm \frac{b}{a}$, the values of x in (2) are *real*, showing that (1) intersects the given hyperbola at finite distances from the centre; while the values of x in (3) are *imaginary*, showing that (1) does not cut the y hyperbola.

If $a^2 \tan^2 \theta > b^2$, that is, if $\tan \theta > \pm \frac{b}{a}$, the values of x in (2) are *imaginary*, showing that (1) does not cut the given hyperbola; while the values of x in (3) are *real*, showing that (1) cuts the y hyperbola at finite distances from the centre.

If $a^2 \tan^2 \theta = b^2$, that is, if $\tan \theta = \pm \frac{b}{a}$, the values of x in (2) and (3) are infinite, showing that (1) does not cut either the x or the y hyperbola. In this case, (1) coincides with the diagonals of the rectangle described on the axes of the two conjugate hyperbolas (Art. 105, Sch. 2), that is, with the asymptotes (Art. 113).

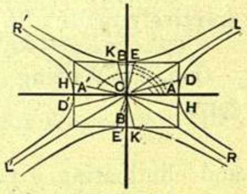

Fig. 90.

We learn, then, that diameters which cut the given hyperbola in real points, must either make with the transverse axis an angle *less* than is made by the first of these diagonals, or *greater* than is made by the second, as DD′ and HH′. If they cut the *conjugate* hyperbola in real points, they must either make with the transverse axis an angle *greater* than is made by the first of these diagonals, or *less* than is made by the second, as EE′ and KK′. If they *coincide* with these diagonals, as LL′ and RR′, they will intersect the hyperbolas at an infinite distance. Hence, every right line drawn through the centre of an hyperbola must meet the hyperbola or its conjugate, unless it coincides with one of the asymptotes.

116. *If one diameter of an hyperbola bisects all chords parallel to a second diameter, the second will bisect all chords parallel to the first.*

Let θ and θ' be the respective inclinations of any two diameters to the transverse axis. Then the condition that the first diameter shall bisect all chords parallel to the second diameter (Art. 115, Cor. 1) is

$$\tan \theta \tan \theta' = \frac{b^2}{a^2}. \qquad (1)$$

But this is also the condition that the second diameter bisects all chords parallel to the first.

SCH.—Two diameters are **Conjugate** when each bisects all chords parallel to the other.

Because the conjugate of any diameter is parallel to the chords which that diameter bisects, therefore the inclinations of two conjugates must be connected in the same way as those of a diameter and its bisected chords. Hence, if θ and θ' are the inclinations, the *equation of condition for conjugate diameters* in the hyperbola (Art. 115, Cor. 1) is

$$\tan \theta \tan \theta' = \frac{b^2}{a^2}. \qquad (2)$$

This condition shows that the tangents of inclination of any two conjugate diameters have *like* signs; therefore it indicates that the angles made with the transverse axis by the two conjugates are either *both acute* or *both obtuse*. Therefore, *conjugate diameters of an hyperbola lie on the same side of the conjugate axis*, as CD and CE, or CK and CH′ (see Fig. 90).

COR.—From (2), if

$$\tan\theta < \frac{b}{a}, \qquad \tan\theta' > \frac{b}{a};$$

and if

$$\tan\theta > -\frac{b}{a} \qquad \tan\theta' < -\frac{b}{a}.$$

Therefore (Art. 115, Cor. 2), *if one of two conjugates,* DD′, *meets an hyperbola, the other,* EE′, *meets the conjugate hyperbola.*

117. *The tangent at either extremity of any diameter is parallel to its conjugate diameter.*

[For demonstration, see Art. 82.]

118. *Given the co-ordinates* x', y' *of one extremity of a diameter, to find the co-ordinates* x'', y'' *of either extremity of the conjugate diameter.*

By the **Extremities** of the conjugate diameter, we mean the points in which the conjugate cuts the conjugate hyperbola.

Let (x', y') be the point D (Fig. 90), and (x'', y'') the point E or E′. Since the conjugate diameter EE′ is parallel to the tangent at (x', y') (Art. 117), and passes through the origin, therefore its equation (Art. 107) is

$$y = \frac{b^2x'}{a^2y'}x;$$

which, combined with the equation of the conjugate hyperbola (Art. 105) gives

$$x'' = \pm\frac{a}{b}y'; \qquad y'' = \pm\frac{b}{a}x'.$$

We see that the upper signs of the co-ordinates are both positive and the lower signs both negative, while in the ellipse (Art. 83), the upper signs are unlike and the lower also. This agrees with the properties of the two curves developed in Arts. 81 and 116, Sch.

119. *To express the length of a semi-diameter* (a'), *and its conjugate* (b'), *in terms of the abscissa of the extremity of the diameter.*

Let (x', y') and (x'', y'') be the extremities D and E of the diameters DD′ and EE′; then we have

$$a'^2 = x'^2 + y'^2 = x'^2 + \frac{b^2}{a^2}(x'^2 - a^2) \quad \text{(Art. 102)}$$

$$= \frac{a^2+b^2}{a^2}x'^2 - b^2;$$

therefore

$$a'^2 = e^2x'^2 - b^2 \quad \text{(Art. 105, Sch. 1).} \qquad (1)$$

Also,

$$b'^2 = x''^2 + y''^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2 \quad \text{(Art. 118)}$$

$$= x'^2 - a^2 + \frac{b^2}{a^2}x'^2 \quad \text{(Art. 102);}$$

therefore,

$$b'^2 = e^2x'^2 - a^2 \quad \text{(Art. 105, Sch. 1).} \qquad (2)$$

COR. (1) — (2) gives

$$a'^2 - b'^2 = a^2 - b^2;$$

that is, the difference of the squares of any two conjugate diameters of an hyperbola is equal to the difference of the squares of the axes.

120. *To find the length of the perpendicular from the centre to the tangent at any point.*

Note 34

Let (x', y') be the point, and p the perpendicular. The equation of the tangent at (x', y') is (Art. 107),

$$a^2yy' - b^2xx' = -a^2b^2. \qquad (1)$$

Therefore (Art. 24),

$$p = \frac{a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{ab}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}} = \frac{ab}{b'}. \quad \text{(Art. 119.)}$$

EXAMPLES.

1. Find (1) the foci and (2) the asymptotes of the hyperbola $4x^2 - 9y^2 = 36$.

Ans. (1) $(\pm\sqrt{13}, 0)$; (2) $y = \pm \frac{2}{3}x$.

2. Prove that in an equilateral hyperbola the length of a normal is equal to the distance of the point of contact from the centre.

3. Find the polar of the point (3, 4) with respect to the hyperbola $4x^2 - 9y^2 = 36$. *Ans.* $12x - 36y = 36$.

4. Find the pole of the line $4x + 5y - 12 = 0$ with respect to the hyperbola $16x^2 - 9y^2 = 144$. *Ans.* $(3, -6\frac{2}{3})$.

5. Find the equation of the diameter conjugate to the diameter $16y - 75x = 0$ in the hyperbola $25x^2 - 16y^2 = 400$.

Ans. $3y = x$.

6. Find the equation of the chord of the hyperbola $16x^2 - 9y^2 = 144$ which is bisected at the point (12, 3).

Ans. $64x - 9y = 741$.

7. Find the equation of a chord of the hyperbola

$$a^2y^2 - b^2x^2 + a^2b^2 = 0$$

in terms of its middle point (x_1, y_1).

Ans. $a^2yy_1 - b^2xx_1 = a^2{y_1}^2 - b^2{x_1}^2$.

8. Find the common tangents to the curves

$$y^2 = 4ax, \text{ and } x^2 - 12y^2 = 24a^2.$$

Make the tangent to the parabola (Art. 54, Cor. 2) cut the hyperbola in two coincident points, Art. 45.

Ans. $\pm 2y = x + 4a$.

9. The line $y = mx + \frac{p}{m}$ touches the parabola $y^2 = 4px$ (Art. 54, Cor. 2); find the condition that this line shall also touch the hyperbola $a^2y^2 - b^2x^2 + a^2b^2 = 0$.

Compare with (3) of Art. 107. *Ans.* $m^2(a^2m^2 - b^2) = p^2$.

121. *To find the angle between any pair of conjugate diameters.*

Let ϕ be the required angle ECD in Fig. 90. By the same process as in the ellipse (Art. 86), we find

$$\sin\phi = \frac{ab}{a'b'}. \tag{1}$$

COR.—Clearing (1) of fractions, we have

$$a'b' \sin\phi = ab, \tag{2}$$

which shows that *the area of the parallelogram whose sides touch the hyperbola at the ends of any pair of conjugate diameters is constant, and equal to the rectangle of the axes.*

SCH.—By Art 119, Cor., $a'^2 - b'^2 =$ a constant; therefore, a' and b' increase or decrease together; hence, by causing D to move along the hyperbola from A, E also will move along from B (Fig. 90). But any diameter CD tends towards an *infinite length,* as its inclination tends towards the limit $\theta = \tan^{-1}\frac{b}{a}$ (Art. 115, Cor. 2); therefore its semi-conjugate CE tends towards infinity; and, as $a'b' \sin\phi$ is *constant,* and a' and b' tend towards infinity, $\sin\phi$ tends towards 0; or, the angle between two conjugates of an hyperbola diminishes without limit. When the two conjugates approach infinity in length, they tend to coincide with the diagonals of the rectangle constructed on the axes; but they are never equal, since $a'^2 - b'^2$ is always equal to $a^2 - b^2$ (Art. 119, Cor.), unless the curve is equilateral. Therefore, the infinite diameters which form the limit of the conjugates, are not *equal* infinites, and hence we do not, as in the ellipse, have *equi-conjugates.* We may, however, call these conjugates in their limit, when they coincide with each other and with either of the asymptotes, **Self Conjugates,*** since each is a diameter conjugate to itself.

* See Howison's Analytic Geometry, p. 381.

The inclinations of the self-conjugate diameters to the transverse axis are determined by the equation

$$\tan\theta = \pm\frac{b}{a}\cdot \quad \text{(Art. 115, Cor. 2.)}$$

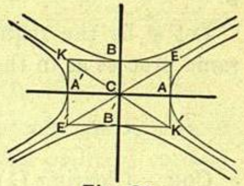

Fig. 91.

The first value corresponds to the angle ACE, and the second value to the angle ACK (Fig. 91).

The inclination of these self-conjugates to each other, as ECK or ECK′, is determined by

$$\sin\phi = 2\sin BCE\cos BCE$$

$$= 2\frac{a}{\sqrt{a^2+b^2}}\times\frac{b}{\sqrt{a^2+b^2}};$$

that is, $\quad \sin\phi = \dfrac{2ab}{a^2+b^2},$

where $\quad \phi = ECK$ or ECK'.

122. *If a chord and diameter of an hyperbola are parallel, the supplemental chord and the conjugate diameter are parallel.* (See Def., Art. 88.)

Let DD′ be a diameter of the hyperbola; PD and PD′ two supplemental chords, the first parallel to the diameter EE′; then will the supplemental chord PD′ be parallel to the conjugate diameter KK′.

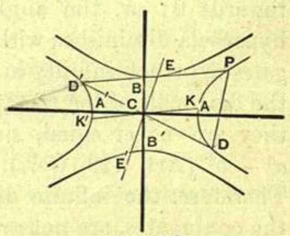

Fig. 92.

Let (x', y') be the point D, and therefore $(-x', -y')$ will be the point D′. Let ϕ and ϕ' be the inclinations of the two chords DP and D′P. Then, by the same process as in Art. 89, or simply by changing b^2 into $-b^2$ in that Art., we get

$$\tan\phi\tan\phi' = \frac{b^2}{a^2},$$

as the condition that the two chords DP and D′P shall be supplemental.

Now, from Art. 116, Sch., we have

$$\tan\theta\tan\theta' = \frac{b^2}{a^2},$$

as the condition that two diameters shall be conjugate to each other; the rest of the argument of Art. 89 applies directly to the hyperbola. Therefore, the supplemental chord PD′ is parallel to the conjugate diameter KK′.

123. *To find the equation of the hyperbola referred to any pair of conjugate diameters.*

To do this we must transform the equation of the hyperbola

$$a^2y^2 - b^2x^2 = -a^2b^2, \tag{1}$$

from rectangular to oblique axes, having the same origin.

Let DD′ and SS′ be two conjugate diameters. Take CD for the new axis of x, and CS for the new axis of y. Denote the angle ACD by θ and ACS by θ'. Let x, y be the co-ordinates of any point P of the hyperbola referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes.

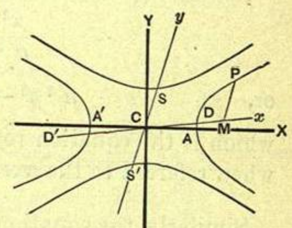

Fig. 93.

Now we may use the same process employed in Art. 90; or, we may simply change b^2 into $-b^2$ in (3) of Art. 90 (see Art. 102, Cor. 5), and get

$$(a^2\sin^2\theta - b^2\cos^2\theta)x'^2 + (a^2\sin^2\theta' - b^2\cos^2\theta')y'^2 = -a^2b^2. \tag{1}$$

Let a' and b' denote the lengths of the semi-diameters CD and CS. If we make $y' = 0$ in (1), we get

$$x'^2 = \frac{-a^2b^2}{a^2\sin^2\theta - b^2\cos^2\theta} = a'^2. \qquad (2)$$

Also, if in (1) we make $x' = 0$, we get

$$y'^2 = \frac{-a^2b^2}{a^2\sin^2\theta' - b^2\cos^2\theta'} = -b'^2. \qquad (3)$$

We put this latter equal to $-b'^2$, because we have supposed the new axis of x to meet the given hyperbola, as in Fig. 93; therefore we know (Art. 116, Cor.) that the new axis of y will *not* meet the given hyperbola; hence $\dfrac{-a^2b^2}{a^2\sin^2\theta' - b^2\cos^2\theta'}$ is a negative quantity.

From (2) we get $\quad a^2\sin^2\theta - b^2\cos^2\theta = -\dfrac{a^2b^2}{a'^2}. \qquad (4)$

From (3) we get $\quad a^2\sin^2\theta' - b^2\cos^2\theta' = \dfrac{a^2b^2}{b'^2}. \qquad (5)$

Substitute (4) and (5) in (1), divide by $-a^2b^2$, omit accents from the variables, and we get

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1, \qquad (6)$$

or, $$a'^2y^2 - b'^2x^2 = -a'^2b'^2, \qquad (7)$$

which is the equation required, and is of the same form as when referred to the axes of the curve (Art. 102).

Similarly, the equation of the *conjugate* hyperbola referred to the same pair of conjugate diameters is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1, \qquad (8)$$

or $$a'^2y^2 - b'^2x^2 = a'^2b'^2. \qquad (9)$$

[Let the student give the demonstration.]

Cor.—Comparing (7) with (9) of Art. 90, we see that the equation of the hyperbola may be derived from that of the ellipse by changing b'^2 into $-b'^2$. Hence, we infer that *any function of b' expressing a property of the ellipse will be converted into one expressing a corresponding property of the hyperbola by changing b' into $b'\sqrt{-1}$.*

124. *To find the equation of a tangent to the hyperbola referred to any pair of conjugate diameters.*

By reasoning exactly as in Art. 91, using the term "hyperbola" for "ellipse," or, by changing b'^2 into $-b'^2$ in (1) of Art. 91, according to Art. 123, Cor., we get

$$a'^2yy' - b'^2xx' = -a'^2b'^2, \qquad (1)$$

which is the required equation.

Cor.—To find where the tangent cuts the axis of x, make $y = 0$ in (1), and get

$$x = \frac{a'^2}{x'}.$$

125. *To prove that tangents at the extremities of any chord of an hyperbola meet on the diameter which bisects that chord.*

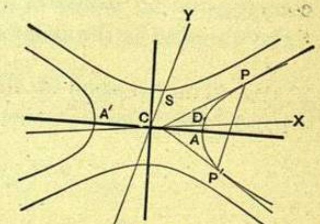

Fig. 94.

Take the diameter CD, which bisects the chord PP′, for the axis of x, and the conjugate diameter CS for the axis of y.

Now reason as in Art. 92, or change b'^2 into $-b'^2$ in (1) and (2) of Art. 92, according to Art. 123, Cor., and get

$$a'^2yy' - b'^2xx' = -a'^2b'^2, \qquad (1)$$

and $$-a'^2yy' - b'^2xx' = -a'^2b'^2, \qquad (2)$$

which are the equations of the tangents at the extremities of the chord PP′ referred to the diameter CD which bisects PP′, and the conjugate diameter CS. Now, by Art. 124, Cor., both of these tangents cut the axis of x at the point $\left(\frac{a'^2}{x'}, 0\right)$, which proves the proposition.

126. *If tangents are drawn at the extremities of any focal chord of an hyperbola:*

I. *The tangents will intersect on the corresponding directrix.*

II. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

I. Reasoning as in Art. 93, we find for the equation of the chord of contact (Art. 111),

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (1)$$

which, for the right-hand focus $(ae, 0)$, becomes

$$-b^2aex' = -a^2b^2;$$

or

$$x' = \frac{a}{e}; \qquad (2)$$

that is, the point of intersection of the tangents is on the corresponding directrix (Art. 102, Cor. 1), showing that the directrix is the polar of the focus. (Art. 79, Sch.)

II. The equation of the line passing through the right-hand focus and the point (x', y') is, by (Art. 26),

$$y = \frac{y'}{x' - ae}(x - ae). \qquad (3)$$

From (2), $x' = \frac{a}{e}$, which in (3) gives

$$y = \frac{y'e}{a - ae^2}(x - ae)$$

$$= -\frac{aey'}{b^2}(x - ae). \text{ (Art. 101, Cor.)} \qquad (4)$$

The equation of the chord of contact [see (1) above] is

$$y = \frac{b^2x'}{a^2y'}x - \frac{b^2}{y'};$$

which becomes, for the focal chord [since $x' = \frac{a}{e}$, from (2)],

$$y = \frac{b^2}{aey'}x - \frac{b^2}{y'}, \qquad (5)$$

which is perpendicular to (4), by Art. 27, Cor. 1.

127. *Find the locus of the point of intersection of two tangents to an hyperbola at right angles to each other.*

Reason as in Art. 94, or change b^2 into $-b^2$ in equation (5) of that Art., and get

$$x^2 + y^2 = a^2 - b^2 \qquad (1)$$

as the required locus. Hence, the locus is a circle with its centre at C, and with $\sqrt{a^2 - b^2}$ for its radius, *unless* $b^2 > a^2$, in which case the locus is impossible; that is, two tangents cannot be drawn at right angles to each other when b^2 is greater than a^2.

128. *The rectangle of the focal perpendiculars upon any tangent is constant, and equal to the square of the semi-conjugate axis.*

Call p and p' the perpendiculars. The equation of the tangent at any point (x', y') is

$$a^2yy' - b^2xx' = -a^2b^2.$$

By Art. 24,

$$p = +\frac{b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{b\,(ex' - a)}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}}$$

$$= \frac{b}{b'}(ex' - a). \text{ (Art. 119.)}$$

Also, $p' = -\dfrac{-b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \dfrac{b(ex' + a)}{\sqrt{\dfrac{a^2}{b^2}y'^2 + \dfrac{b^2}{a^2}x'^2}}$;

$= \dfrac{b}{b'}(ex' + a)$. (Art. 119.)

Hence, $\quad pp' = \dfrac{b^2}{b'^2}(e^2x'^2 - a^2) = b^2$. (Art. 119.)

129. *To find the polar equation of the hyperbola, the left focus being the pole.*

Let $F'P = r$; $AF'P = \theta$; then (Def. of Art. 99) we have,

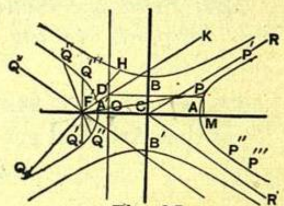

Fig. 95.

$$F'P = e \cdot PD$$
$$= e\,(F'M - F'O)$$
$$= e \cdot MF' - e \cdot F'O$$
$$= e \cdot F'P \cos AF'P - a(e^2 - 1) \text{ (Art. 102, Cor. 1)},$$

or $\qquad r = er\cos\theta - a\,(e^2 - 1)$;

therefore, $$r = \frac{a\,(e^2 - 1)}{e\cos\theta - 1}, \qquad (1)$$

which is the equation required.

Cor.—When $\theta = 0$, $r = ae + a = F'C + CA = F'A$, as it should do. (Art. 102, Cor. 1.)

When $e\cos\theta - 1 = 0$, that is, when $\theta = \cos^{-1}\dfrac{1}{e}$,

$$r = \frac{a\,(e^2 - 1)}{0} = \infty.$$

But in this case r, or F′K, is parallel to the asymptote CR (See Art. 114, Cor. 2, and Fig. 89). That is, while θ increases from 0 to $\cos^{-1}\dfrac{1}{e}$, r increases from $ae + a$ to ∞, tracing the branch APP′.

When θ passes the value $\cos^{-1}\dfrac{1}{e}$, $e\cos\theta - 1$ becomes negative, and therefore r becomes negative, and the left-hand branch is generated, the negative end of r tracing QQ′Q″; thus, when $\theta = AF'H$, r, being negative, is reckoned backwards to Q.

When $\theta = 90°$, $r = -a\,(e^2 - 1) = -\dfrac{b^2}{a}$ (Art. 101, Cor.), which equals the *semi latus rectum*, p, with a negative sign (Art. 102, Cor. 4), and Q′ is located.

When $\theta = 180°$, $r = -a\,(e - 1) = a - ae = -F'A'$, as it should.

While θ increases from 90° to 270°, the arc Q′Q″A′Q‴ is traced with the negative end of r.

When $\theta = 270°$, $r = -a\,(e^2 - 1) = -\dfrac{b^2}{a} = -p$, and the point Q‴ is *located.*

While θ increases from 270° to $\cos^{-1}\dfrac{1}{e}$, r remains negative, and increases numerically from p to ∞, its negative end tracing Q‴, Q^{iv}.

At $\theta = \cos^{-1}\dfrac{1}{e}$ in the fourth quadrant, $r = \infty$, and is parallel to the asymptote CR′.

While θ increases from $\cos^{-1}\dfrac{1}{e}$ to 360°, r is positive, and diminishes from ∞ to $a + ae$, and the arc P‴, P″, A is traced.

130. *To find the polar equation of the hyperbola when the pole is at the centre.*

Changing $a^2y^2 - b^2x^2 = -a^2b^2$ into a system of polar co-ordinates (as in Art. 97), we have

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{b^2}{e^2\cos^2\theta - 1} \text{ (Art. 101, Cor.)} \quad (1)$$

Similarly, the polar equation of the conjugate hyperbola is

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}. \qquad (2)$$

Cor.—Equation (1) shows that for every value of θ between $-\cos^{-1}\frac{1}{e}$ and $+\cos^{-1}\frac{1}{e}$, r has two *real* values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter; hence, every diameter of the hyperbola is bisected at the centre (Art. 102, Sch.).

When $\theta = \cos^{-1}\frac{1}{e}$, $r = \infty$; but in this case, r or CR, Fig. 89, coincides with the asymptote.

While θ increases from $\cos^{-1}\frac{1}{e}$ to $\left(180° - \cos^{-1}\frac{1}{e}\right)$, r is imaginary, showing that it does not reach either branch of the given hyperbola.

Equation (2) shows that for every value of θ between $\cos^{-1}\frac{1}{e}$ and $\left(180° - \cos^{-1}\frac{1}{e}\right)$, r has two real values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter of the conjugate hyperbola; hence, every diameter of the conjugate hyperbola is bisected at the centre.

When $\theta = -\cos^{-1}\frac{1}{e}$, $r = \infty$; in this case, r coincides with the asymptote.

For every value of θ between $-\cos^{-1}\frac{1}{e}$ and $+\cos^{-1}\frac{1}{e}$, r is imaginary, showing that it does not reach either branch of the given hyperbola.

In (1), r is least when $\theta = 0$, giving

$$r = \sqrt{\frac{b^2}{e^2 - 1}},$$

which equals a (Art. 101, Cor.). In (2), r is least when $\theta = 90°$, giving $r = b$. Hence, in the hyperbola, each axis is the minimum diameter of its own curve.

Also, it is evident from both (1) and (2) that the value of r is the same for θ and $(\pi - \theta)$. Therefore, diameters which make supplemental angles with the transverse axis of an hyperbola are equal.

131. The properties of the hyperbola hitherto established are similar to those of the ellipse. We have now to consider some properties *peculiar to the hyperbola*, arising from the presence of the *asymptotes*. (See Art. 113.)

132. *To prove that the asymptotes are the diagonals of every parallelogram formed on a pair of conjugate diameters.*

The equations of the hyperbola and its asymptotes, when referred to the axes of the curve, are respectively

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (1)$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \qquad (2)$$

When we transform the equation of the hyperbola to its conjugate diameters (Art. 123), equation (1) becomes

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1;$$

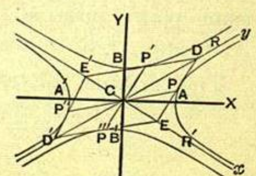

Fig. 96.

therefore we may at once infer that (2) transformed to the same conjugate diameters, becomes

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0;$$

that is, the equations of the asymptotes CR and CR′, referred to any pair of conjugate diameters, are

$$\frac{x}{a'} - \frac{y}{b'} = 0, \qquad (3)$$

and
$$\frac{x}{a'} + \frac{y}{b'} = 0. \qquad (4)$$

Take CP $= a'$, and CP$' = b'$.

Equation (3), or $y = \frac{b'}{a'}x$, is the equation of a line passing through the origin and the point (a', b') (see Art. 26, Cor. 4), that is, through C and D; and (4), or $y = -\frac{b'}{a'}x$, is a line passing through the origin and $(a', -b')$, that is, through C and E. Hence, (3) and (4), which are the asymptotes, are also the diagonals of the parallelogram EDE′D′ on the conjugate diameters PP″ and P′P‴.

133. *To find the equation of the hyperbola referred to its asymptotes as axes.*

To do this, we must transform the equation

$$a^2y^2 - b^2x^2 = -a^2b^2, \qquad (1)$$

from rectangular to oblique axes, having the same origin.

Let CX and CY be the old axes (Fig. 96). Take the lower asymptote CR′ for the new axis of x, and the other, CR, for the new axis of y.

Let x, y be the co-ordinates of any point P in the curve referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes. Denote the angles ACR and ACR′ by θ and θ' respectively.

The formulæ for transformation (Art. 35, Cor. 1) are

$$x = x' \cos\theta' + y' \cos\theta,$$
$$y = x' \sin\theta' + y' \sin\theta.$$

Squaring, substituting in (1), and arranging, we have

$$\left.\begin{aligned} &(a^2 \sin^2\theta' - b^2 \cos^2\theta')\, x'^2 \\ &+ (a^2 \sin^2\theta - b^2 \cos^2\theta)\, y'^2 \\ &+ 2\,(a^2 \sin\theta \sin\theta' - b^2 \cos\theta \cos\theta')x'y' \end{aligned}\right\} = -a^2b^2. \qquad (2)$$

From Art. 114, Cor. 2, we have

$$\tan^2\theta = \frac{b^2}{a^2} = \tan^2\theta';$$

from which we get

$$a^2 \sin^2\theta - b^2 \cos^2\theta = 0, \qquad (3)$$

and
$$a^2 \sin^2\theta' - b^2 \cos^2\theta' = 0. \qquad (4)$$

Also, from Art. 114, Cor. 2, we have

$$\sin\theta \sin\theta' = -\frac{b^2}{a^2+b^2},$$

and
$$\cos\theta \cos\theta' = \frac{a^2}{a^2+b^2};$$

therefore,

$$a^2 \sin\theta \sin\theta' - b^2 \cos\theta \cos\theta' = -\frac{2a^2b^2}{a^2+b^2}. \qquad (5)$$

Substituting (3), (4), and (5) in (2), we get

$$-\frac{4a^2b^2}{a^2+b^2}x'y' = -a^2b^2;$$

or suppressing accents from the variables and reducing, we have

$$xy = \frac{a^2+b^2}{4}, \qquad (6)$$

and putting m^2 for $\frac{a^2+b^2}{4}$, we have,

$$xy = m^2, \qquad (7)$$

which is the equation required.

COR.—The equation of the *conjugate* hyperbola, referred to the same axes, is (Art. 105)

$$xy = -m^2. \qquad (8)$$

If we solve (7) for x, we get

$$x = \frac{m^2}{y},$$