which shows that as y increases, x diminishes, and when $y = \infty$, $x = 0$; that is, the curve approaches the axis of y, and finally touches it at an infinite distance from the centre.

Similarly, the curve approaches the axis of x, and finally touches it at an infinite distance from the centre.

Sch.—The second member of (7) is essentially positive, and of (8) essentially negative; hence, both x and y have the same sign in (7) and contrary signs in (8); therefore one branch of the given hyperbola lies wholly in the first angle and the other in the third; while one branch of the *conjugate* hyperbola lies wholly in the second and the other in the fourth angle. (See Fig. 96.)

In the case of *equilateral* hyperbolas (Art. 105, Sch. 2), the angle between the asymptotes, which (Art. 121, Sch.) is equal to $\sin^{-1}\dfrac{2ab}{a^2+b^2} = \sin^{-1} 1$, becomes a right angle; therefore, the *equilateral* hyperbola is also called the **Rectangular** hyperbola.

134. *To find the equation of the tangent at any point of an hyperbola referred to the asymptotes as axes.*

Let (x', y') and (x'', y'') be any two points, P and P′, on the curve. The equation of the secant through these points (Art. 26), is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \quad (1)$$

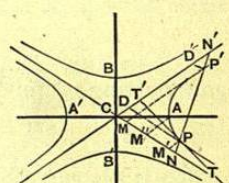

Fig. 97.

Since (x', y') and (x'', y'') are on the curve, we have (Art. 133),

$$x'y' = m^2 = x''y'', \qquad \text{or} \qquad y'' = \frac{x'y'}{x''},$$

which in (1) gives $\quad y - y' = -\dfrac{y'}{x''}(x - x'), \qquad (2)$

which is the equation of the *secant* to the hyperbola.

When the points become consecutive, we have $x'' = x'$; hence (2) becomes

$$y - y' = -\frac{y'}{x'}(x - x'). \quad (3)$$

Clearing (3) of fractions, transposing, and uniting, we have

$$x'y + y'x = 2x'y',$$

or

$$\frac{x}{x'} + \frac{y}{y'} = 2, \quad (4)$$

which is the equation of the tangent required.

Cor. 1.—Making y and x successively $= 0$ in (4) we get

$$x = 2x' = \text{CT}, \qquad \text{and} \qquad y = 2y' = \text{CT}'.$$

Hence, P is the middle point of TT′; therefore, *the portion of the tangent included between the asymptotes is bisected at the point of contact.*

Cor. 2.—From Cor. 1, we have,

$$\text{CT} \times \text{CT}' = 4x'y' = a^2 + b^2 \text{ (Art. 133).}$$

That is, *the rectangle of the intercepts cut off upon the asymptotes by any tangent is constant, and equal to the sum of the squares on the semi-axes.*

Cor. 3.—The area of the triangle TCT′, Fig. 97, is

$$= \tfrac{1}{2}\text{CT} \times \text{CT}' \sin \text{TCT}'$$

$$= 2x'y' \times \frac{2ab}{a^2 + b^2} \text{ (Cor. 1, and Art. 121, Sch.)},$$

$$= \frac{a^2 + b^2}{2} \times \frac{2ab}{a^2 + b^2} \text{ (Art. 133)}$$

$$= ab = \textit{constant}.$$

Therefore, *the triangle included between any tangent and the asymptotes is constant, and equal to the rectangle of the semi-axes.*

135. *To prove that the intercepts of a secant between the hyperbola and its asymptotes are equal.*

In equation (2) of Art. 134, make $y = 0$, and get

$$x = x'' + x'$$
$$= \mathrm{CN}\ (\text{Fig. } 97).$$

Hence, $\quad \mathrm{CN} - x' = x'',$

or $\quad \mathrm{M'N} = \mathrm{D'P'};$

therefore, $\quad \mathrm{NP} = \mathrm{N'P'};$

that is, *the intercepts of the secant are equal.*

SCH.—This proposition affords a convenient method of constructing the curve. If the axes are given, construct the rectangle on them, the diagonals of which are the asymptotes. Then through the extremity of the transverse axis, draw a right line intercepted by the asymptotes; lay off on this line from one asymptote a distance equal to the extremity of the axis from the other asymptote; the point thus found will be a point of the curve. In this manner, find any number of points, and draw a line through them; *this will be the required curve.*

136. *To prove that the parallelogram formed by drawing lines from any point of an hyperbola parallel to and terminating in the asymptotes, is equal to one-eighth the rectangle on the axes.*

Call ϕ the angle TCT′ (Fig. 97); the area of CM′PD

$$= x'y' \sin \phi$$
$$= \frac{a^2 + b^2}{4} \times \frac{2ab}{a^2 + b^2} \text{ (Art. 133, and Art. 121, Sch.)}$$
$$= \tfrac{1}{2}ab = \tfrac{1}{8}(2a \cdot 2b),$$

which proves the proposition.

137. *To find the equations of two conjugate diameters of an hyperbola referred to its asymptotes.*

The diameter which passes through the origin and the point P (x', y') is represented (see Art. 26, Cor. 4) by

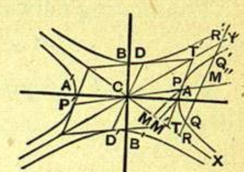

Fig. 98.

$$y = \frac{y'}{x'}x,$$

or $$\frac{x}{x'} - \frac{y}{y'} = 0. \tag{1}$$

The diameter conjugate to this one, CD, is parallel to the tangent at (x', y'), and therefore (Art. 134, Eq. 4) its equation is

$$y = -\frac{y'}{x'}x,$$

or $$\frac{x}{x'} + \frac{y}{y'} = 0. \tag{2}$$

COR.—When the diameters PP′ and DD′ become the axes, AA′ and BB′, we have, since the axes bisect the angle between the asymptotes,

$$\mathrm{CM'} = \mathrm{M'A}, \qquad \text{or} \qquad x' = y';$$

therefore (1) and (2) become

$$x - y = 0, \qquad \text{and} \qquad x + y = 0,$$

which are the equations of the axes referred to the asymptotes.

138. *Given the co-ordinates of the extremity of a diameter, to find those of the extremity of its conjugate.*

Let (x', y') be the point P (Fig. 98), and (x'', y'') the point D.

The equation of DD′ (Art. 137) is

$$\frac{x}{x'} + \frac{y}{y'} = 0. \tag{1}$$

The equation of the conjugate hyperbola (Art. 133) is

$$xy = -m^2. \tag{2}$$

Eliminating between (1) and (2), we get

$$x'' = \mp x', \qquad y'' = \pm y'. \tag{3}$$

Cor. 1.—The equation of the tangent at P (x', y') (Art. 134) is

$$\frac{x}{x'} + \frac{y}{y'} = 2. \tag{4}$$

The equation of the tangent at D (x'', y''), the extremity of the conjugate diameter (Art. 134) is

$$\frac{x}{x''} + \frac{y}{y''} = 2,$$

or from (3),

$$\frac{x}{x'} - \frac{y}{y'} = -2. \tag{5}$$

Adding (4) and (5), we get $x = 0$ as the locus of the intersection of the tangents (4) and (5), which is the equation of the axis of y, or the asymptote CR′. Therefore, *tangents at the extremities of conjugate diameters meet on the asymptotes.*

Cor. 2.—Since T′ is a vertex of the parallelogram formed on the conjugate diameters PP′ and DD′, we have

$$PT' = CD;$$

therefore,

$$TT' = 2PT' = DD';$$

that is, *the portion of the tangent at any point of an hyperbola, included between the asymptotes, is equal to the diameter conjugate to that which passes through the point of contact.*

139. *If a chord be drawn parallel to any diameter, it will be bisected by the conjugate diameter produced.*

Let QQ′ be drawn parallel to DD′ (Fig. 98); then will it be bisected at M″ by CP produced.

Since QQ′ is parallel to DD′, its equation will differ from that of DD′ only by a constant term; therefore [Art. 138, (1)]

$$\frac{x}{x'} + \frac{y}{y'} = c \tag{1}$$

is the equation of QQ′.

Combine (1) with the equation of PP′ (Art. 137), which is

$$\frac{x}{x'} - \frac{y}{y'} = 0, \tag{2}$$

and we get $\quad x = \frac{1}{2}cx', \quad y = \frac{1}{2}cy',$

as the co-ordinates of M″. But from (1) we have

$$CR = cx', \quad \text{and} \quad CR' = cy';$$

therefore M″ is the middle point of RR′. But (Art. 135),

$$RQ = R'Q';$$

therefore, QM″ = M″Q′, *which proves the proposition.*

EXAMPLES.

1. Find the axes of the hyperbola whose equation is $3y^2 - 2x^2 + 12 = 0$; also the eccentricity of the given and the conjugate hyperbola, and the parameter.

Ans. $a = \sqrt{6}$, $b = 2$; $e = \sqrt{\frac{5}{3}}$; $e' = \sqrt{\frac{5}{2}}$; $2p = \dfrac{8}{\sqrt{6}}$.

2. Find the intersection of the hyperbola $3y^2 - 2x^2 + 12 = 0$ and the circle $x^2 + y^2 = 16$. *Ans.* $(\pm 2\sqrt{3}, \pm 2)$.

3. Find whether the line $y = \frac{3}{4}x$ cuts the hyperbola $5y^2 - 2x^2 = -15$, or its conjugate.

Ans. It cuts the conjugate.

4. Find the equation of an hyperbola of given transverse axis, whose vertex bisects the distance between the centre and the focus. *Ans.* $y^2 - 3x^2 = -3a^2$.

5. If the ordinate MP (Fig. 95) of an hyperbola be produced to Q, so that MQ = F′P, find the locus of Q.

Ans. A right line.

6. If an ellipse and an hyperbola have the same foci, prove that their tangents at the point of intersection are at right angles. (See Art. 75, Cor. 2, and Art. 107, Cor.)

7. Find the condition that the line $\left(\frac{x}{m}+\frac{y}{n}=1\right)$ shall touch the hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right)$. *Ans.* $\frac{a^2}{m^2}-\frac{b^2}{n^2}=1$.

[To obtain this, compare $\frac{x}{m}+\frac{y}{n}=1$ with equation of tangent (Art. 107), which is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1,$$

and we have $\frac{x'}{a}=\frac{a}{m}$ and $\frac{y'}{b}=-\frac{b}{n}$,

which in the equation of curve gives the answer.]

8. Find where the tangents from the foot of the directrix will meet the hyperbola, and what angle they will make with the transverse axis.

Ans. The extremity of the *latus rectum*; $\tan^{-1} \pm e$.

9. Find the angle included between the asymptotes of the hyperbola $16y^2 - 9x^2 = -25$. *Ans.* 73° 44′.

10. Find the perpendicular from the focus of any hyperbola to its asymptotes. *Ans.* The semi-conjugate axis.

11. If 3AC = 2CF′ (Fig. 95), find the inclination of the asymptotes to the transverse axis. *Ans.* $\tan^{-1}\frac{\sqrt{5}}{2}$.

12. If the asymptotes of the hyperbola are axes, show that the equation of one directrix is $x + y - a = 0$.
[See Art. 137, Cor.]

13. Prove that if a circle be described with the focus of an hyperbola for its centre and with a radius equal to the semi-conjugate axis, it will touch the asymptotes in the points where they are cut by the directrix.

14. Prove that the radius of a circle which touches an hyperbola and its asymptotes is equal to that part of the latus rectum produced which is intercepted between the curve and the asymptote.

15. Find the length of the normal NP and of RP (Fig. 88). [See Art. 108, Cor. 1.] *Ans.* $NP = \frac{bb'}{a}$, $RP = \frac{ab'}{b}$.

16. Prove that the product of the two perpendiculars let fall from any point of an hyperbola upon the asymptotes is constant and equal to $\frac{a^2b^2}{a^2+b^2}$.

17. Tangents to an hyperbola are drawn from any point on either branch of the conjugate curve; prove that their chord of contact touches the opposite branch of the conjugate curve.

[Take the diameter passing through the *point* for axis of y, and the conjugate diameter for axis of x; equation of chord of contact is

$$\frac{xx'}{a'^2}-\frac{yy'}{b'^2}=1,$$

which soon reduces to $y = \pm b'$; ∴ etc.]

18. In any equilateral hyperbola, let ϕ = the inclination of a diameter, passing through any point P, and ϕ' = that of the polar of P, the transverse axis being the axis of x; prove that $\tan\phi \tan\phi' = 1$.

[Equation of diameter is $y=\frac{y'}{x'}x$; ∴ $\frac{y'}{x'}=\tan\phi$;

polar of P is $xx'-yy'=a^2$; ∴ $\frac{x'}{y'}=\tan\phi'$; ∴ etc.]

19. Prove that the middle points of a series of parallels intercepted between an hyperbola and its conjugate, lie on the curve whose equation is

$$4\left(\frac{x^2}{a'^2}-\frac{y^2}{b'^2}\right)=\frac{b'^2}{y^2}.$$

[Take for axis of y the diameter parallel to the lines, and for axis of x the conjugate diameter.]

20. Between the sides of a given angle ϕ, a right line moves so as to enclose a triangle of constant area $= k^2$; prove that the locus of the centre of gravity of the triangle is the hyperbola whose equation is $9xy \sin\phi = 2k^2$.

[Take the sides of the angle for the axes.]

21. A tangent at the extremity of the latus rectum of an hyperbola meets any ordinary MP produced in R; prove that FP = MR, where F is the focus through which the latus rectum passes.

22. If from a point P in an hyperbola PK be drawn parallel to the transverse axis, cutting the asymptotes in I and K, prove that $PK \times PI = a^2$; or, if parallel to the conjugate, $PK \times PI = b^2$.

[Combine equation of line through P (x', y') with equations of asymptotes, etc.]

23. AOB, COD are two straight lines which bisect each other at right angles: show that the locus of a point which moves so that $PA \cdot PB = PC \cdot PD$ is a rectangular hyperbola. Take OA and OC for axes of x and y respectively.

24. A right line has its extremities on two fixed right lines, and passes through a fixed point: show that the locus of the middle point of the line is an hyperbola, and find its equation. Take the fixed right lines for axes.

25. A right line has its extremities on two fixed right lines, and cuts off from them a triangle of constant area: show that the locus of the middle point of the line is an hyperbola, and find its equation.

Take the fixed right lines for axes; and let the constant area $= c^2$.

Ans. $4xy = c^2$.

26. If e and e' be the eccentricities of an hyperbola and of the conjugate hyperbola, prove that $\dfrac{1}{e^2}+\dfrac{1}{e'^2}=1$.

27. The distance of any point from the centre of a rectangular hyperbola varies inversely as the perpendicular distance of its polar from the centre.

28. If a parallelogram be constructed with its sides parallel to the asymptotes of an hyperbola, and one of its diagonals be a chord of the hyperbola; show that the direction of the other will pass through the centre.

29. PN is the ordinate of a point P on an hyperbola, PG is the normal meeting the axis in G: if NP be produced to meet the asymptote in Q, prove that QG is at right angles to the asymptote.

30. A series of chords of the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$ are tangents to the circle described on the right line joining the foci of the hyperbola as diameter: prove that the locus of their poles with respect to the hyperbola is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}.$$