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CHAPTER VIII.

GENERAL EQUATION OF THE SECOND DEGREE.

140. It has been shown (Art. 23) that every equation of the first degree between two variables is the equation of a right line. We have seen that the equations of the circle, parabola, ellipse, and hyperbola are all of the second degree. We shall now show that *every* equation of the second degree between two variables is the equation of a circle, a parabola, an ellipse, an hyperbola, or two right lines, intersecting, parallel or coincident, or a point.

141. The most general form of the equation of the second degree is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \qquad (1)$$

where a, b, c, d, e, f are all constants.

Five relations between the coefficients are sufficient to determine a locus of the second degree, although (1) contains *six* constants. The nature of the locus depends, not on the *absolute magnitude* of the coefficients, but on their *mutual ratios*, for if we multiply or divide (1) by any constant, it will still clearly represent the same locus. We may, therefore, divide (1) by f, so as to make the absolute term $= 1$, and there will then remain but five constants to be determined.

If the locus passes through the origin, $f = 0$ (see Art. 41, Cor. 2), and (1) becomes

$$ax^2 + bxy + cy^2 + dx + ey = 0, \qquad (2)$$

which is the equation of the locus when it passes through the origin.

If the origin of co-ordinates be taken at the *centre* of the locus (Art. 71, Sch.), for every point (x', y') whose co-ordinates satisfy the equation, there will be a corresponding point $(-x', -y')$ whose co-ordinates also will satisfy the equation; hence, *when the centre is the origin, the equation will not be altered by writing* $-x$, $-y$ *for* x, y; therefore, the terms of the first degree must vanish from it. In order, then, to find the *centre* of the locus, we must transfer the origin to a point (x', y'), and then see what values of x', y' will make the new coefficients of x and y vanish. These values of x' and y' will be the co-ordinates of the *centre* of the locus referred to the original axes. In the following transformations, we shall suppose the co-ordinate axes rectangular; for if they were oblique, we might transform the equation to one in which the axes were rectangular, without affecting the *degree* or *form* of the equation.

FIRST TRANSFORMATION.

M **142.** The object of this transformation is to remove from

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \qquad (1)$$

the terms involving the *first* power of x and y. To do this we transform (1) to parallel axes passing through a new origin (x', y').

The formulæ for transformation to parallel axes through (x', y') are (Art. 33), $x = x' + x$, $y = y' + y$, where x' and y' are put for m and n. Substituting these values for x and y in (1), and arranging the terms of the resulting equation, we have

$$ax^2 + bxy + cy^2 \begin{array}{l|l|l|} + 2ax' & + 2cy' & + ax'^2 \\ + by' & + bx' & + bx'y' \\ + d & + e & + cy'^2 \\ & & + dx' \\ & & + ey' \\ & & + f \end{array} \hspace{-4em}\begin{array}{l} \\ \end{array}$$

$$\left.\begin{array}{l} \end{array}\right. ax^2 + bxy + cy^2 + (2ax' + by' + d)x + (2cy' + bx' + e)y + (ax'^2 + bx'y' + cy'^2 + dx' + ey' + f) = 0, \qquad (2)$$

or $$ax^2 + bxy + cy^2 + d'x + e'y + f' = 0, \tag{3}$$
from which we see that the coefficients of x^2, xy, and y^2 are, as before, a, b, c; that

the new d is $\quad d' = 2ax' + by' + d$;
the new e is $\quad e' = 2cy' + bx' + e$;
the new f is $\quad f' = ax'^2 + bx'y' + cy'^2 + dx' + ey' + f$.

Hence, *if the equation of a locus of the second degree be transformed to parallel axes through a new origin, the coefficients of the highest powers of the variables will remain unchanged, while the new absolute term will be the result of substituting in the original equation the co-ordinates of the new origin.*

Putting the coefficients of x and y in (2) equal to 0, we have
$$2ax' + by' + d = 0, \tag{4}$$
$$2cy' + bx' + e = 0, \tag{5}$$
which are the equations for the centre of (1).

Equations (4) and (5) may thus be obtained: *For* (4) *take only those terms of* (1) *which involve x; multiply each term by the exponent of x in it, and diminish that exponent by unity.* Equation (5) may be obtained similarly by substituting y for x in the above rule. Thus, the equations for the centre of the locus represented by
$$4x^2 + 3xy + 2y^2 - 14y + 17 = 0$$
are $\quad 8x + 3y = 0 \quad$ and $\quad 3x + 4y - 14 = 0$.

SCH.—Solving (4) and (5) for x' and y', we find them to be
$$x' = \frac{2cd - be}{b^2 - 4ac}, \tag{6}$$
and
$$y' = \frac{2ae - bd}{b^2 - 4ac}, \tag{7}$$
which are the co-ordinates of the *centre* with reference to the old axes.

It is plain that these values of x' and y' will always be finite, except when $b^2 - 4ac = 0$, in which case they will be infinite. Hence, loci of the second degree may be divided into two classes: I, those which *have a centre;* II, those which in general *have not a centre*, or rather, whose centre is infinitely distant. The first are often called **Central Curves**, while the second are called **Non-central Curves.** We shall first consider the case of central loci.

Substituting (4) and (5) in (2), and representing the absolute term by f', for shortness, we have
$$ax^2 + bxy + cy^2 + f' = 0. \tag{8}$$

We see that if (8) is satisfied by any values, x' and y' for x and y, it is also satisfied by the values $-x'$ and $-y'$. Hence, the origin of co-ordinates in (8) is *the centre* of the locus which (1) or (8) represents.

SECOND TRANSFORMATION.

M **143.** The object of this transformation is to remove from
$$ax^2 + bxy + cy^2 + f' = 0 \tag{1}$$
the term involving xy, and leave (1) in the form
$$a'x^2 + c'y^2 + f' = 0,$$
where if any value be given to one of the variables, the other will have two equal values, with contrary signs.

To effect this transformation, we revolve the axes of co-ordinates through the angle θ till they coincide with the axes of the locus. The formulæ for this transformation (Art. 35, Cor. 3), are
$$x = x'\cos\theta - y'\sin\theta,$$
$$y = x'\sin\theta + y'\cos\theta.$$

Substituting these values for x and y in (1), and arranging the terms, we have

$$\left.\begin{array}{r}a\cos^2\theta\\+b\sin\theta\cos\theta\\+c\sin^2\theta\end{array}\right|x'^2\left.\begin{array}{r}-2a\sin\theta\cos\theta\\+b\cos^2\theta\\-b\sin^2\theta\\+2c\sin\theta\cos\theta\end{array}\right|x'y'\left.\begin{array}{r}+a\sin^2\theta\\-b\sin\theta\cos\theta\\+c\cos^2\theta\end{array}\right|y'^2+f'=0. \qquad (2)$$

If we equate the new coefficient of $x'y'$ to 0, we obtain

$$2(c-a)\sin\theta\cos\theta+b(\cos^2\theta-\sin^2\theta)=0,$$

or $$(c-a)\sin 2\theta+b\cos 2\theta=0;$$

therefore, $$\tan 2\theta=\frac{b}{a-c}, \qquad (3)$$ learn

from which we may determine the angle θ through which the co-ordinate axes must be turned to remove the term containing xy.

As the tangent of an angle may have any value, positive or negative, from 0 to ∞, it follows that (3) will always give real values for 2θ; that is, there are two real lines at right angles with each other to which when the locus is referred, the term involving xy vanishes.

Substituting (3) in (2), we have, for the required transformation of (1),

$$\left\{\begin{array}{r}(a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta)\,x'^2\\+\,(a\sin^2\theta-b\sin\theta\cos\theta+c\cos^2\theta)\,y'^2\\+\,f'\end{array}\right\}=0, \qquad (4)$$

or, omitting the accents from the variables, and writing a' and c' for the coefficients of x^2 and y^2, we have

$$a'x^2+c'y^2+f'=0, \qquad (5)$$

which is the equation of the locus referred to its centre and axes.

To find the values of a' and c' in (5), we have

$$\begin{aligned}a'&=a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta\ \text{[from (4)]}\\&=\tfrac12[a\cos^2\theta+a(1-\sin^2\theta)+c\sin^2\theta+c(1-\cos^2\theta)\\&\qquad+2b\sin\theta\cos\theta]\\&=\tfrac12[a+c+(a-c)\cos 2\theta+b\sin 2\theta]. \qquad (4)'\end{aligned}$$

Similarly, $c'=\frac12[a+c-(a-c)\cos 2\theta-b\sin 2\theta]$. (5)'

From Trigonometry we have

$$\cos 2\theta=\frac{1}{\sqrt{1+\tan^2 2\theta}}=\frac{a-c}{\sqrt{b^2+(a-c)^2}}$$

[from (3)]; also

$$\sin 2\theta=\sqrt{1-\cos^2 2\theta}=\frac{b}{\sqrt{b^2+(a-c)^2}}.$$

Substituting these values of $\cos 2\theta$ and $\sin 2\theta$ in (4)' and (5)', we get

$$\begin{aligned}a'&=\tfrac12\left[a+c+\frac{(a-c)^2+b^2}{\sqrt{b^2+(a-c)^2}}\right]\\&=\tfrac12\left[a+c+\sqrt{b^2+(a-c)^2}\right] \qquad (6)\end{aligned}$$ learn

also, $$\begin{aligned}c'&=\tfrac12\left[a+c-\frac{(a-c)^2+b^2}{\sqrt{b^2+(a-c)^2}}\right]\\&=\tfrac12\left[a+c-\sqrt{b^2+(a-c)^2}\right]. \qquad (7)\end{aligned}$$ learn

Hence, we see that the general equation of the second degree given in (1) of Art. 141 can always be transformed to the form given in (5), *provided* that it is not subject to the condition $b^2-4ac=0$ (Art. 142, Sch.).

Cor. 1.—Multiplying (6) and (7) together, we have

$$a'c'=\tfrac14[(a+c)^2-b^2-(a-c)^2]=\tfrac14(4ac-b^2).$$

Hence, if a' and c' have *like* signs, $4ac-b^2$ will be *positive*, or b^2-4ac will be *negative;* but if a' and c' have *unlike* signs, b^2-4ac will be *positive*.

Cor. 2.—When $b^2-4ac<0$, a' and c' have the same sign (Cor. 1); if f' have a *contrary* sign from a' and c', (5) becomes

$$\frac{a'}{f'}x^2+\frac{c'}{f'}y^2=1, \qquad (8)$$

which is the equation of an ellipse [Art. 71, (3)] whose axes are $\sqrt{\frac{f'}{a'}}$ and $\sqrt{\frac{f'}{c'}}$.

If $c' = a'$, (8) becomes

$$x^2 + y^2 = \frac{f'}{a'}, \tag{9}$$

which is the equation of a circle whose radius is $\sqrt{\frac{f'}{a'}}$.

If $f' = 0$, (8) becomes

$$a'x^2 + c'y^2 = 0,$$

which is the equation of the two imaginary right lines

$$x\sqrt{a'} + y\sqrt{-c'} = 0, \quad \text{and} \quad x\sqrt{a'} - y\sqrt{-c'} = 0,$$

which meet in the real point $x = 0$, $y = 0$; or it is the equation of the origin, or the ellipse diminished indefinitely.

If f' have the *same* sign as a' and c', (5) becomes

$$\frac{a'}{f'}x^2 + \frac{c'}{f'}y^2 = -1,$$

which cannot be satisfied by any real value of x and y; therefore the locus is *imaginary*.

Hence, *if $b^2 - 4ac < 0$, the general equation of the second degree between two variables represents an ellipse, a circle, a point, or an imaginary locus.*

Cor. 3.—When $b^2 - 4ac > 0$, a' and c' have unlike signs (Cor. 1). Suppose c' and f' to be positive, and a' to be negative; (5) becomes

$$\frac{a'}{f'}x^2 - \frac{c'}{f'}y^2 = 1, \tag{10}$$

which is the equation of an hyperbola [Art. 102, (3)] whose axes are $\sqrt{\frac{f'}{a'}}$ and $\sqrt{\frac{f'}{c'}}$.

If a' and f' are positive, and c' negative, (5) becomes

$$\frac{a'}{f'}x^2 - \frac{c'}{f'}y^2 = -1, \tag{11}$$

which is the equation of an hyperbola [Art. 105, (1)] conjugate to (10).

If $f' = 0$, (10) or (11) becomes

$$a'x^2 - c'y^2 = 0,$$

which is the equation of the two lines

$$y = \pm x\sqrt{\frac{a'}{c'}}$$

intersecting at the origin.

If $c' = a'$, (10) and (11) become

$$x^2 - y^2 = \frac{f'}{a'}, \quad \text{and} \quad x^2 - y^2 = -\frac{f'}{a'},$$

which are equilateral hyperbolas [Art. 105, (2) and (3)].

Hence, *if $b^2 - 4ac > 0$, the general equation of the second degree between two variables represents an hyperbola or its conjugate, an equilateral hyperbola, or two right lines intersecting each other.*

144. We have shown (Art. 142) that the coefficients of the first three terms of the general equation of the second degree between two variables are not altered by a transfer of the origin; we shall now show that when the axes are turned through an angle θ, and the new coefficients of the first three terms are denoted by a', b', c', we have the relations $a' + c' = a + c$ and $b'^2 - 4a'c' = b^2 - 4ac$.

From (2) of Art. 143, we have

$$a' = a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta$$

$$= \tfrac{1}{2}\left[a + c + (a - c)\cos 2\theta + b\sin 2\theta\right] \text{ [from (4)′]}. \tag{1}$$

$$b' = 2(c - a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta)$$

$$= (c - a)\sin 2\theta + b\cos 2\theta, \tag{2}$$

$$c' = a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta$$

$$= \tfrac{1}{2}\,\{a+c-[(a-c)\cos 2\theta + b\sin 2\theta]\}\ \text{[from (5)']}. \quad (3)$$

Adding (1) and (3), we get

$$a' + c' = a + c. \quad (4)$$

Also, from (1), (2), and (3), we have

$$b'^2 - 4a'c' = \left\{\begin{array}{l} [(c-a)\sin 2\theta + b\cos 2\theta]^2 \\ -\{(a+c)^2 - [(a-c)\cos 2\theta + b\sin 2\theta]^2\} \end{array}\right\}$$

$$= \left\{\begin{array}{l} (a-c)^2(\sin^2 2\theta + \cos^2 2\theta) \\ \quad + b^2(\cos^2 2\theta + \sin^2 2\theta) - (a+c)^2 \end{array}\right\}$$

$$= (a-c)^2 + b^2 - (a+c)^2;$$

that is, $$b'^2 - 4a'c' = b^2 - 4ac. \quad (5)$$

Thus, the expression $b^2 - 4ac$ has the same value whether it be formed from the coefficients of the *general equation* of the second degree, as given in (1) of Art. 141, or after one or both transformations have been made, as in (8) of Art. 142, or (5) of Art. 143.

145. *To sum up briefly:*

1st. *In order to reduce the equation of a central locus to parallel axes through its centre, we have the following directions:*

1. The coefficients of the first three terms remain unaltered (Art. 142).

2. The co-ordinates of the centre of the locus are given by (6) and (7) of Art. 142, Sch.

3. The absolute term is replaced by a new one, which is the result of substituting in the original equation the co-ordinates of the centre (Art. 142).

The equation is now reduced to the form

$$ax^2 + bxy + cy^2 + f' = 0 \ \text{[Art. 142, (8)]}, \quad (1)$$

where the origin is at the centre of the locus.

2d. *To reduce* (1) *to the form* $a'x^2 + c'y^2 + f' = 0$ *by turning the axes through the angle* $\theta = \frac{1}{2}\tan^{-1}\dfrac{b}{a-c}$. (Art. 143.)

4. The coefficients a' and c' are given in (6) and (7) of Art. 143.

5. The absolute term, f', remains unaltered [Art. 143, (2)].

The equation is now reduced to the form

$$a'x^2 + c'y^2 + f' = 0 \ \text{[Art. 143, (5)]}. \quad (2)$$

146. We shall now consider the case in which

$$b^2 - 4ac = 0.$$

We saw (Art. 142, Sch.) that in this case the centre was infinitely distant, or, in other words, that there was *no centre*. We cannot, therefore, remove the terms dx and ey from the general equation by changing the origin to the centre, as we did in Art. 142; but we can remove the term xy from the equation by turning the axes through the angle θ, as we did in Art. 143, where θ is obtained from (3) of Art. 143.

Substituting $x'\cos\theta - y'\sin\theta$ for x, and $x'\sin\theta + y'\cos\theta$ for y in (1) of Art. 142, and arranging as in (2) of Art. 143, we have

$$\left.\begin{array}{r} (a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta)\ x'^2 \\ (-2a\sin\theta\cos\theta + b\cos^2\theta - b\sin^2\theta + 2c\sin\theta\cos\theta)\ x'y' \\ (+\,a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta)\ y'^2 \\ (+\,d\cos\theta + e\sin\theta)\ x' \\ (+\,e\cos\theta - d\sin\theta)\ y' \\ +\,f \end{array}\right\} = 0. \quad (1)$$

Now, for $\tan 2\theta = \dfrac{b}{a-c}$, the term containing $x'y'$ in (1) vanishes, by Art. 143, (2); and if we denote the coefficients of x'^2, y'^2, x', y', by a', c', d', e', (1) becomes

$$a'x'^2 + c'y'^2 + d'x' + e'y' + f = 0, \quad (2)$$