25. Find the species and situation of the following curves:

(1) $xy - 2x + y - 2 = 0$;
(2) $y^2 - 2ay + 4ax = 0$;
(3) $y^2 + ax + ay + a^2 = 0$;
(4) $(x + 2y)^2 + (y - 2x)^2 = 5a^2$;
(5) $y^2 - x^2 - 2ax = 0$.

Ans. (1) The two right lines $x + 1 = 0$, $y - 2 = 0$;
(2) Parabola, vertex $(\frac{1}{4}a, a)$;
(3) Parabola, vertex $(-\frac{3}{4}a, -\frac{1}{2}a)$;
(4) Ellipse, lengths of the major and minor axes $= 2a$ and a, respectively.
(5) Rectangular hyperbola, centre (a, o).

26. Transform the following equations to parallel axes through the centres of the curves:

(1) $3x^2 - 5xy + 6y^2 + 11x - 17y + 13 = 0$;
(2) $xy + 3ax - 3ay = 0$;
(3) $3x^2 - 7xy - 6y^2 + 3x - 9y + 5 = 0$.

Ans. (1) $3x^2 - 5xy + 6y^2 - 1 = 0$, centre $(-1, 1)$;
(2) $xy + 9a^2 = 0$, centre $(3a, -3a)$;
(3) $3x^2 - 7xy - 6y^2 + 5 = 0$, centre $(-\frac{9}{11}, -\frac{3}{11})$.

27. Transform $2x^2 + 4xy + 3y^2 + 3x + y + \frac{5}{8} = 0$ to parallel axes through the centre of the curve.

Ans. $2x^2 + 4xy + 3y^2 - \frac{3}{2} = 0$; centre $(-1\frac{3}{4}, 1)$.

28. Transform $2x^2 + 4xy + 3y^2 - 3 = 0$ to its axes.

Ans. $\frac{21}{60}x^2 + \frac{9}{60}y^2 = 1$, the axis of x coinciding with the *minor* axis of the ellipse. In this case we turned the old axes through $\frac{1}{2}\tan^{-1} - 4$; had we turned them through $-\frac{1}{2}\tan^{-1} 4$, and taken the *minus* value of the radical for a' in Art. 143, and *positive* value for c', we would have found, for the transformed equation, $\frac{9}{60}x^2 + \frac{21}{60}y^2 = 1$, the axis of x coinciding with the *major* axis of the curve. (See Remark, Ex. 3.)

CHAPTER IX.

HIGHER PLANE CURVES.

147. Higher Plane Curves are those whose equations are above the second degree, or which involve *transcendental* functions (Art. 17). It has been shown that every equation of the first degree between two variables represents a right line, and that every equation of the second degree between two variables represents a conic section; it follows that all other loci in a plane are *higher plane curves.*

An **Algebraic Curve** is one whose rectilinear equation contains only algebraic functions of the co-ordinates. Thus, $y = ax + b$, $x \cos \alpha + y \sin \alpha = p$ are algebraic curves. A **Transcendental Curve** is one whose rectilinear equation contains transcendental functions of one or more co-ordinates. Thus, $y = \sin x$, $y = \tan^{-1} x$ are transcendental curves.

Many of the *higher plane curves* possess historical interest, from the labor bestowed on them by ancient mathematicians. We shall consider only a few of them.

THE CISSOID OF DIOCLES.

148. This curve was invented by Diocles, a Greek geometer who lived about the sixth century of the Christian era; the purpose of its invention was the solution of the problem of finding two mean proportionals. It may be defined as follows: If pairs of equal ordinates be drawn to the diameter of a circle, and through one extremity of this diameter and the point of intersection of one of the ordinates with the circumference a line be drawn, the locus of the intersection of this line and the equal ordinate, produced if necessary, is the **Cissoid of Diocles.**

The curve is constructed as follows: Let AB (Fig. 103) be the diameter of a circle; draw two equal ordinates MR and M′R′; join AR′, cutting MR in P; then is P a point

of the locus. In the same way, any number of points may be found. In like manner, draw through A and R a line cutting M′R′ produced in P′; P′ will be a point of the locus. In the same way, points can be found below AB.

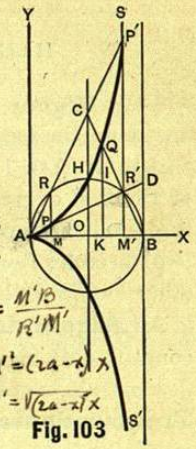

Fig. 103

149. *To find the equation of the Cissoid of Diocles.*

I. *The rectangular equation.*

Let AX and AY be the axes; AB $= 2a$; and let (x, y) be any point P of the locus. Then we have

$$\mathrm{AM} : \mathrm{PM} :: \mathrm{AM}' : \mathrm{R'M'},$$

$$\text{or} \quad \frac{y}{x} = \frac{\sqrt{(2a-x)\,x}}{2a-x} = \frac{\sqrt{x}}{\sqrt{2a-x}};$$

Squaring and reducing, we have

$$y^2 = \frac{x^3}{2a-x}, \tag{1}$$

which is the required equation.

SCH.—Solving (1) for y, we have

$$y = \pm\sqrt{\frac{x^3}{2a-x}},$$

which shows that, for every value of $x < 2a$, y has two real values, numerically equal, with contrary signs; that is, the curve is symmetrical with respect to the axis of x. When $x = 2a$, $y = \infty$; hence the branches are infinite in length, and BD is an asymptote to them. When $x > 2a$, or negative, y is imaginary; therefore the locus is limited by $x = 0$ and $x = 2a$.

Sir Isaac Newton has given the following elegant construction of this curve by continuous motion: A right angle

has the side GF of fixed length, $=$ AB, the point F moves along the fixed line CI, which is perpendicular to AB at its middle point, while the side GL always slides through the fixed point E such that AE $=$ AC; a pencil at the middle point P of GF will describe the Cissoid.

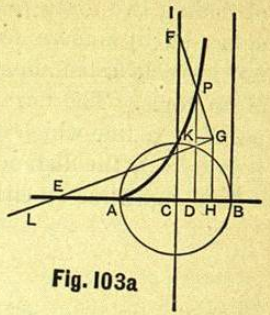

Fig. 103a

II. *The polar equation.*

Let A be the pole, and AB the initial line; let (r, θ) be any point P in the locus (see Fig. 103). Then, since AM $=$ BM′, we have AP $=$ DR′; therefore we have

$$\begin{aligned} r &= \mathrm{AD} - \mathrm{AR'} = \mathrm{AB}\sec\theta - \mathrm{AB}\cos\theta \\ &= 2a\,(\sec\theta - \cos\theta) \\ &= 2a\left(\frac{1-\cos^2\theta}{\cos\theta}\right) = 2a\,\frac{\sin^2\theta}{\cos\theta}; \end{aligned}$$

that is, $r = 2a\tan\theta\sin\theta$, which is the required equation.

SCH.—When $\theta = 0$, $r = 0$; when $\theta = 45°$, $r = a\sqrt{2}$; that is, H is the point in the curve. When $\theta = 90°$, $r = \infty$; when $\theta > 90°$ and $< 270°$, r is negative; while θ increases from 90° to 270°, the negative end of the radius-vector traces the branch AS′ and the branch AS a second time; while θ increases from 270° to 360°, r is positive, and AS′ is traced a second time; thus, the curve is traced twice by one revolution of the radius-vector.

THE CONCHOID OF NICOMEDES.*

150. This curve was invented by Nicomedes, who lived about the second century of our era, and was, like the preceding, first formed for the purpose of solving the problem

* See Gregory's Examples, p. 130.

of finding two mean proportionals, or the duplication of the cube; but it is more readily applicable to another problem not less celebrated among the ancients, that of the trisection of an angle. The curve may be defined as the locus of a point in a line which slides on and revolves about a fixed point, while the distance between the generating point and a fixed right line on either side of it is constant.

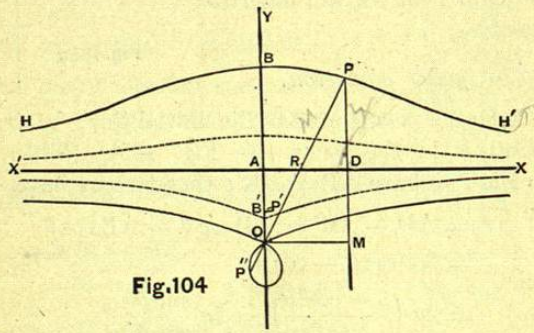

Fig.104

The curve is constructed as follows: Let O be the fixed point, XX′ the fixed right line, and AB the constant distance on the revolving line between the generating point and the fixed line. Draw through O any line, as OP; on OP, above XX′, lay off RP equal to AB; then will P be a point of the locus. In like manner, if we take AB′, below XX′, as a constant distance, and lay off RP′ equal to AB′, P′ will be a point of the locus.

151. *To find the equation of the Conchoid of Nicomedes.*

I. *The rectangular equation.*

Let XX′ and YY′ be the axes; OA $= p$; AB $= m$; and (x, y) any point P in the locus. Then we have, from the similar triangles PDR and PMO,

$$PD : DR :: PM : MO,$$

or, $$y : \sqrt{m^2 - y^2} :: y + p : x;$$

squaring and reducing, we have

$$x^2y^2 = (p + y)^2 (m^2 - y^2), \qquad (1)$$

which is the equation required.

Sch.—Solving (1) for x, we get

$$x = \pm \frac{p + y}{y} \sqrt{m^2 - y^2},$$

which shows that for every value of y, positive or negative, and numerically $< m$, x has two real values, numerically equal, with contrary signs; hence the curve has two branches, one above and one below the axis of x, both being symmetrical with respect to the axis of y. When y diminishes numerically, x increases and becomes ∞ when $y = 0$; hence the two branches are infinite in length, and the axis of x is an asymptote to them.

When $m > p$, for $y = -m$ or $-p$, $x = 0$; but for y *between* $-m$ and $-p$, x has two values, numerically equal, with contrary signs; hence the locus between these two limits is an oval symmetrical with respect to the axis of y. For y negative and less numerically than p, the values of x increase till they become $\pm \infty$ at $y = 0$.

When $m < p$, it is easily seen that there is no oval. The continuous line represents the case when $m > p$, and the broken line when $m < p$.

II. *The polar equation.*

Let O be the pole, OA the initial line, and (r, θ) any point P in the curve. Then we have

$$r = OP = OR + RP = OA \sec \theta + m;$$

that is, $r = p \sec \theta + m$, which is the required equation.

Sch.—When $\theta = 0$, $r = p + m$, and B is located; when $\theta = 90°$, $r = \infty$; when $\theta = 180°$, $r = -p + m$, and B′ is located; when $\theta > 90°$ and $< 270°$, $\sec \theta$ is negative, and

the lower branch is traced by the negative end of the radius vector; while θ increases from 270° to 360°, r is positive and the branch H′B is traced.

The fixed point O (Fig. 104) is called the **Pole,** the fixed right line XAX′ is called the **Directrix,** and the constant distance AB is the **Parameter.**

THE WITCH OF AGNESI.*

152. This curve was invented by Donna Maria Agnesi, an Italian lady, who lived in the eighteenth century. It may be defined as the locus of the extremity of an ordinate of a circle, produced till the produced ordinate is to the diameter of the circle as the ordinate itself is to one of the segments into which it divides the diameter.

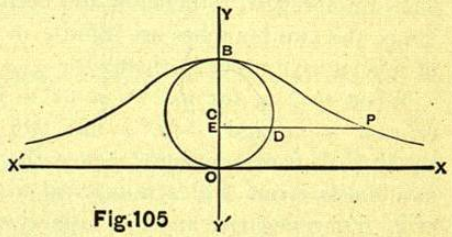

Fig.105

To construct the Witch, let OB be the diameter of the circle; draw the ordinate ED; find the point P in ED produced so that

$$PE : OB :: ED : OE,$$

and P will be a point of the locus. In the same way, any number of points may be found.

153. *To find the equation of the Witch of Agnesi.*

Let XX′ and YY′ be the axes of co-ordinates, and (x, y) any point P in the locus. Call the diameter $2a$; then we have, from the definition,

$$x : 2a :: \sqrt{(2a-y)\,y} : y;$$

therefore, $x^2y = 4a^2(2a-y)$, which is the required equation.

* See Gregory's Examples, p. 131.

Sch.—When $y = 0$, $x = \infty$; when $y = 2a$, $x = 0$; for every positive value of y between 0 and $2a$, x has two real values, numerically equal, with contrary signs, showing that the locus is symmetrical with respect to the axis of y, and is embraced between $y = 0$ and $2a$, and has the axis of x for an asymptote.

THE LEMNISCATE OF BERNOULLI.*

154. This curve was invented by James Bernoulli, who lived in the seventeenth century. It may be defined as the locus of the intersection of a tangent to an equilateral hyperbola with the perpendicular on it from the centre.

To find the equation of the Lemniscate.

I. *The rectangular equation.*

Let (x', y') be any point Q of the hyperbola at which the tangent is drawn; and let x and y be the current co-ordinates of the lines QP and OP. The equations of the hyperbola and the tangent are respectively

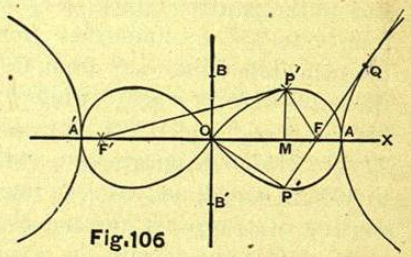

Fig.106

$$x'^2 - y'^2 = a^2, \qquad (1)$$

and

$$xx' - yy' = a^2, \qquad (2)$$

therefore the equation of OP is

$$y = -\frac{y'}{x'}x, \quad \text{or} \quad \frac{x}{x'} = -\frac{y}{y'}. \qquad (3)$$

Multiplying (2) and (3) together, we get

$$x^2 + y^2 = \frac{a^2x}{x'} = -\frac{a^2y}{y'};$$

* See Price's Calculus, Vol. I, p. 314.

therefore,

$$x' = \frac{a^2x}{x^2+y^2}, \quad \text{and} \quad y' = -\frac{a^2y}{x^2+y^2},$$

which in (1) gives,

$$\frac{a^4x^2}{(x^2+y^2)^2} - \frac{a^4y^2}{(x^2+y^2)^2} = a^2,$$

or,

$$(x^2+y^2)^2 = a^2(x^2-y^2), \tag{4}$$

which is the required equation.

On transforming to polar co-ordinates, (4) becomes

$$r^4 = a^2r^2(\cos^2\theta - \sin^2\theta),$$

or,

$$r^2 = a^2\cos 2\theta. \tag{5}$$

SCH.—When $\theta = 0$, $r = \pm a$; if we confine our attention to the positive values of r, we see that as θ increases from 0 to 45°, r diminishes from a to 0, and AP′O is traced; while θ increases from 45° to 135°, r is imaginary; when $\theta = 135°$, $r = 0$; while θ increases from 135° to 225°, r is real, and OA′O is traced; while θ increases from 225° to 315°, r is imaginary; while θ increases from 315° to 360°, r is real, and OPA is traced. The curve therefore consists of two ovals meeting at O; the tangents to the ovals at O coincide with the asymptotes of the equilateral hyperbola, and form angles of 45° with the axis of x (Art. 133, Sch.).

SCH. 2.—Take two points, F and F′, on opposite sides of O, at the distance $a\sqrt{\frac{1}{2}}$ from it, and take any point P′ in the curve; then we have

$$FP' = \sqrt{(a\sqrt{\tfrac{1}{2}} - x)^2 + y^2}, \tag{6}$$

and

$$F'P' = \sqrt{(a\sqrt{\tfrac{1}{2}} + x)^2 + y^2}. \tag{7}$$

Multiply (6) and (7), and we have

$$FP' \times F'P' = \sqrt{(a\sqrt{\tfrac{1}{2}} - x)^2 + y^2} \times \sqrt{(a\sqrt{\tfrac{1}{2}} + x)^2 + y^2}$$

$$= \sqrt{(x^2+y^2)^2 - a^2(x^2-y^2) + \frac{a^4}{4}}$$

$$= \frac{a^2}{2}, \text{ by (4); that is,}$$

$$FP' \times F'P' = \frac{a^2}{2}.$$

Hence we may define the **Lemniscate** as a curve such that the product of the distances of any point in it from two fixed points, called the foci, is constant, and equal to the square of half the distance between the foci. (See Gregory's Examples, p. 132.)

[Let the student find the equation of the curve from this definition.]

We may construct the curve, from this latter definition, by points. Let F and F′ be the foci. With F as a centre, and any convenient radius, as FP′, describe an arc; with F′ as a centre, and a third proportional to FP′ and F′O, as F′P′, describe a second arc cutting the former at P′; then will P′ be a point in the locus. In the same way any number of points may be found.

THE CYCLOID.

155. The invention of this curve is usually ascribed to Galileo; it is generated by the motion of a point in the circumference of a circle which rolls along a fixed right line. Thus, if the circle NPB (Fig. 107) be rolled along the line OX, any point P in the circumference will describe a cycloid. The circle NPB is called the **Generating Circle** or **Generatrix**, and the point P the **Generating Point**. OK is called the **Base**, and is equal to the cir-

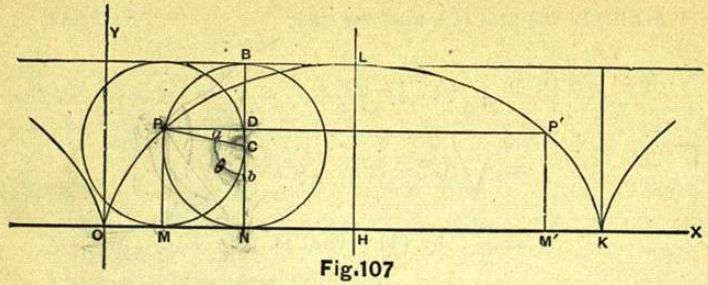

Fig.107

cumference of the generating circle. HL, perpendicular to the base at its centre, is the **Axis**, and is equal to the diameter of the generating circle, and L is the **Highest Point** of the cycloid.

156. *To find the equation of the cycloid referred to its base and a perpendicular at its left hand vertex.*

Let (x, y) be any point P in the cycloid OPLK, referred to the axis OX and OY; suppose that P has described the arc OP, while the generatrix has rolled from O to N, then ON = arc PN. Call the radius of the generatrix r. Then we have

$$x = \text{OM} = \text{ON} - \text{MN} = \text{arc PN} - \text{PD}$$

$$= r \text{ arc } ab - \sqrt{\text{ND} \times \text{DB}};$$

that is, $x = r \text{ vers}^{-1} \frac{y}{r} - \sqrt{2ry - y^2}$; (1)

which is the required equation of the cycloid, the arc ab being taken in the circle whose radius = 1.

Sch.—When y is negative, $\sqrt{2ry - y^2}$ is imaginary; therefore the curve lies only on the positive side of the base; when $y = 0$, $x = 0$, $2r\pi$, $4r\pi$, etc.; hence there is an infinite number of branches similar and equal to OLK, which is also evident from the mode of generation of the curve; when $y = 2r$, $x = r \text{ vers}^{-1} 2 = \pi r$, $3\pi r$, etc. For

any one value of y, x has an infinite number of values, OM, OM′, etc.

It is frequently convenient to refer the cycloid to its *highest point* as origin, and to its *axis* as the axis of x.

157. *To find the equation of the cycloid referred to its highest point as its origin and to its axis as the axis of x.*

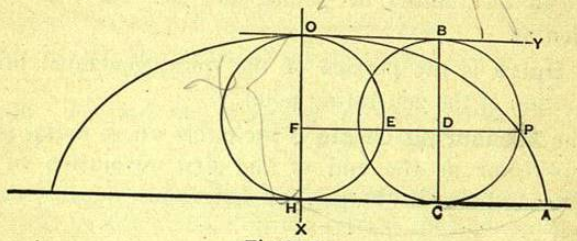

Fig.108

Let (x, y) be any point P in the locus referred to the axes OX and OY; then we have

$$y = \text{PF} = \text{PD} + \text{DF} = \text{PD} + \text{CH}$$

$$= \text{PD} + \text{AH} - \text{AC} = \text{PD} + \text{arc CPB} - \text{arc CP}$$

$$= \text{PD} + \text{arc PB} = \sqrt{\text{CD} \times \text{BD}} + \text{vers}^{-1} \text{BD};$$

that is,

$$y = r \text{ vers}^{-1} \frac{x}{r} + \sqrt{2rx - x^2} \text{ (see Art. 156)},$$

which is the required equation.

Sch.—When $x = 0$, $y = 0$; when $x = 2r$, $y = r \text{ vers}^{-1} 2 = \pi r$, $3\pi r$, etc.; when x is negative, y is imaginary; for any one value of x, y has an infinite number of values.

After the conic sections there is no curve in geometry which has more exercised the ingenuity of mathematicians than the cycloid; and their labors have been rewarded by the discovery of a multitude of interesting properties, important both in geometry and in dynamics. [See Gregory's Examples, p. 136.]

SPIRALS.

158. We shall conclude this chapter with a brief account of *spirals*, many of which have been treated at length by old geometers. A **Spiral** is the locus of a point revolving about a fixed point, and constantly receding from it in accordance with some law. A right line then meets the curve in an infinity of points, and the curve is transcendental.

A **Spire** is the portion of the spiral generated in one revolution of the generating point.

The **Measuring Circle** is the circle whose radius is the radius-vector at the end of the first revolution of the generating point in the positive direction.

THE SPIRAL OF ARCHIMEDES.

159. This spiral was invented by Conon, but its principal properties were discovered by the geometer whose name it bears; it is the locus of a point revolving uniformly about a fixed point, and at the same time receding uniformly from it.

To construct the spiral of Archimedes.

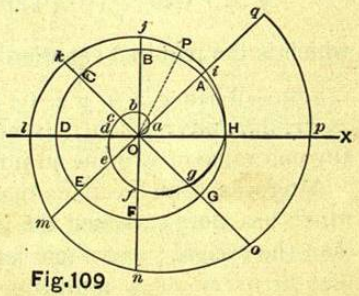

Fig.109

Let O be the fixed point and OX the initial line; with O as a centre and any radius as OH, describe the circumference HADG; divide this circumference into any number of equal parts; for example, eight. On the radius OA lay off $Oa = \frac{1}{8}OH$; on OB lay off $Ob = \frac{2}{8}OH$; on OC lay off $Oc = \frac{3}{8}$ OH, etc.; the curve

passing through these points, a, b, c, d, e, f, g, H, $i, \ldots p, q$, etc., will be the spiral of Archimedes, since the radius-vectors Oa, Ob, etc., increase uniformly, while the variable angle, estimated from OX, increases uniformly.

The circumference HADG is the measuring circle, O is the pole, O*abcdefg*H is the first spire, H*ijklmnop* is the second spire, etc. The distance between any two consecutive spires measured on the radius-vector is equal at all points to OH, the radius of the measuring circle.

160. *To find the equation of the spiral of Archimedes.*

Let O be the pole (Fig. 109) and OX the initial line, and let (r, θ) be any point P in the spiral; then we have, from the definition, $r = a\theta$, as the required equation, when a is the ratio of r to θ.

Otherwise, we have from the figure,

$$OP : OH :: \theta : 2\pi;$$

or, calling the radius of the measuring circle a', we have

$$r : a' :: \theta : 2\pi;$$

therefore
$$r = \frac{a'\theta}{2\pi};$$

or writing a instead of $\frac{a'}{2\pi}$,

$$r = a\theta,$$

is the required equation.

When $\theta = 0$, $r = 0$; when $\theta = 2\pi$, $r = a'$; when $\theta = 4\pi$, $r = 2a'$; when $\theta = 6\pi$, $r = 3a'$, etc. The curve, therefore, starts at the pole, and the radius-vector, which is o at the beginning, becomes equal to OH $(= a')$, when it has made one revolution; and this is the distance between the points at which any radius-vector is cut by two successive spires.

THE RECIPROCAL OR HYPERBOLIC SPIRAL.

161. This spiral may be defined as the locus of a point revolving uniformly about a fixed point, and continually approaching it so that the radius-vector varies inversely as the variable angle.

To construct the Hyperbolic Spiral.

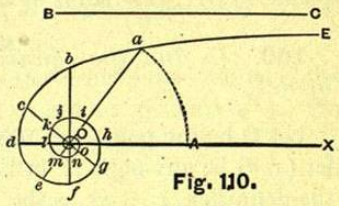

Fig. 110.

Let O be the pole and OX the initial line. Draw through O the lines Oa, Ob, Oc, etc., making equal angles with each other. Take a for a point of the spiral; lay off Ob = $\frac{1}{2}$Oa; Oc = $\frac{1}{3}$Oa, etc.; the curve passing through the points a, b, c, d, e, f, g, h, etc., will be the hyperbolic spiral, since the radius-vector, Oa, Ob, etc., vary inversely as the variable angle estimated from OA.

The equation of the hyperbolic spiral follows directly from the definition, and is

$$r = \frac{a}{\theta}, \quad \text{or} \quad r\theta = a.$$

When $\theta = 0$, $r = \infty$; that is, the curve approaches the initial line and touches it at infinity; when $\theta = 2\pi$, $r = \text{O}h = a'$, which is the radius of the measuring circle; when $\theta = 4\pi$, $r = \frac{a'}{2}$, etc.; when $\theta = \infty$, $r = 0$; therefore, the curve continually approaches the pole as the radius-vector revolves, and reaches it after an infinite number of revolutions. From the equation $r = \frac{a}{\theta}$, it is evident that the arc Aa of the circle described with the radius Oa to any point of the curve, is constant and equal to a. [See Salmon's Higher Plane Curves, p. 280.]

THE LITUUS.

162. Another spiral worth mentioning is the **Lituus**, which may be defined as the locus of a point revolving uniformly about a fixed point, and continually approaching it so that the radius-vector varies inversely as the square root of the variable angle. Its equation therefore is

$$r = \frac{a}{\theta^{\frac{1}{2}}}.$$

SCH.—These spirals belong to one family, included in the general equation $r = a\theta^n$. When $n = 1$, we have $r = a\theta$, which is the spiral of Archimedes. When $n = -1$, we have $r = \frac{a}{\theta}$, which is the hyperbolic spiral. When $n = -\frac{1}{2}$, we have $r = \frac{a}{\theta^{\frac{1}{2}}}$, which is the Lituus.

THE CHORDEL.

162*a*. The **Chordel** is a plane curve, every point of which terminates an arc which originates in a fixed line, is described with a fixed point as a centre, and subtends a given length the same number of times as a chord.

The fixed line is called the **Directrix**, the fixed point the **Focus**, and the given length the **Element**.

A chordel in which the element is subtended n times as a chord, whose directrix is a right line, and whose focus is on the directrix, is called

*A chordel of n elements, and rectilinear and focal directrix.**

* This curve was invented by Mr. J. Bruen Miller; for an account of it see Van Nostrand's Engineering Magazine for March, 1880, pp. 206–209, which contains Mr. Miller's investigation of the chordel given geometrically, including the construction of the curve and its application to the division of an angle into n equal parts.

To find its equation,

Let the focus O be the pole, and the directrix OX be the initial line. Let (r, θ) be any point P of the curve, and

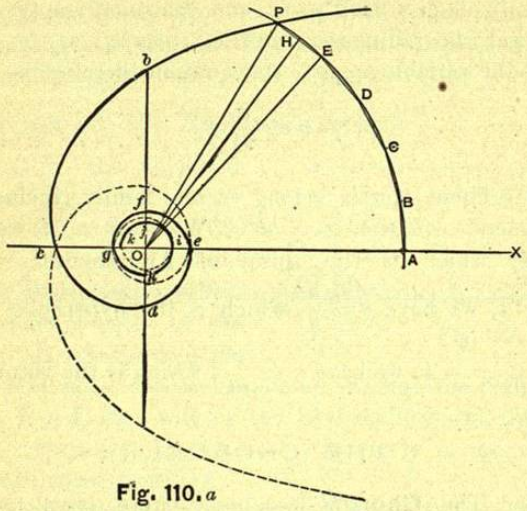

Fig. 110. *a*

$2a$ = an element AB = BC = etc. = EP, draw OH perdicular to EP. Then we have

$$\sin \text{HOP} = \frac{\text{HP}}{\text{OP}}; \quad \therefore \quad \sin\left(\frac{\theta}{2n}\right) = \frac{a}{r};$$

or
$$r = a \operatorname{cosec}\left(\frac{\theta}{2n}\right); \qquad (1)$$

which is the equation required.

Let $n = 5$ and $a = 1$; then (1) becomes $r = \operatorname{cosec}\left(\frac{\theta}{10}\right)$.

Letting $\theta = \frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi, \frac{5}{2}\pi, 3\pi, \frac{7}{2}\pi, 4\pi, \frac{9}{2}\pi, 5\pi$, successively we get $r = 6.39, 3.24, 2.20, 1.70, 1.41, 1.24, 1.12, 1.05, 1.01$, and 1.00. Locating these values we have the points $b, c, d, e, f, g, h, i, j, k$; when $\theta = 0$, $r = \infty$. Now letting θ continue to increase, becoming $\frac{11}{2}\pi$, 6π, and

so on to 10π, we get $r = 1.01, 1.05, 1.12$, and so on to infinity, the values being the same as those given above, when θ is increasing from 0 to 5π, except the order is inverted. Locating the second series of values we have the curve represented by the dotted line, which is the continuation of the part given in the full line, the two parts being symmetrical with respect to the line OX. While θ is increasing from 10π to 20π, r is negative and a second branch is traced by the negative end of the radius-vector, the two branches being symmetrically equal.

The essential merits of Mr. Miller's curve appear to be its mechanical construction, affording a mechanical multisection of any angle; and its very *general* definition, which will probably make the investigation of its properties rather fruitful. But such investigation would be out of place here.

THE LOGARITHMIC SPIRAL.

163. This spiral was invented by Descartes, and is the locus of a point so moving that the radius-vector increases in a *geometric*, while the variable angle increases in an *arithmetic* ratio. Its equation is therefore usually written

$$r = a^{\theta}.$$

To construct the Logarithmic Spiral.

Suppose $a = 2$, then $r = 2^{\theta}$; when $\theta = 0$, $r = 2^{0} = 1$, which gives the point a on the initial line OX, Fig. III. When $\theta = 1$, $r = 2^{1} = 2$; lay off the angle $XOb = 1$ = arc of $57^{\circ}.3$, and take $Ob = 2$; b will be a point of the curve. When $\theta = 2$, $r = 2^{2} = 4$; lay off $XOc = 2$ = arc of $114^{\circ}.6$, and take $Oc = 4$; c will be a point of the curve. The curve passing through a, b, c, etc., will be the logarithmic spiral.

When $\theta = -1 = XOb'$, $r = 2^{-1} = \frac{1}{2}$; lay off

$XOb' = -1 = -57°.3$, and take $Ob' = \frac{1}{2}$; b' will be a point of the curve. When $\theta = -2$, $r = 2^{-2} = \frac{1}{4}$; lay off $XOc' = -2 = -114°.6$, and take $Oc' = \frac{1}{4}$; c' will be a point of the curve; and so on for any number of points.

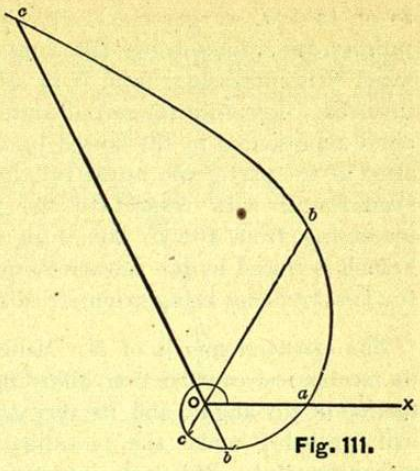

Fig. 111.

When $\theta = \infty$, $r = \infty$, hence the radius-vector will become infinite when it has made an infinite number of revolutions. When $\theta = -\infty$, $r = 0$; and therefore the spiral runs into its pole after an infinite number of revolutions in the negative direction.

The present chapter is but a brief sketch of Higher Plane Curves. The student who wishes to pursue the subject further, is referred to Salmon's Higher Plane Curves, Gregory's Examples, Price's Infinitesimal Calculus, and Cramer's Introduction to the Analysis of Curves.

EXAMPLES.

1. In Fig. 103 prove that M′R′ and AM′ are two mean proportionals between AM and M′P′; that is, prove that

$$AM : M'R' :: M'R' : AM',$$

and

$$M'R' : AM' :: AM' : M'P';$$

i. e.,

$$AM : M'R' : AM' : M'P'.$$

$\overline{M'R'}^2 = AM \cdot AM'$; and the similar $\triangle$s M′AR′ and M′P′A give M′R′ : AM′ :: AM′ : M′P′; ∴ etc.

2. In Fig. 103a find the equation of the locus of P.

Let AB be the axis of x, A the origin, $AB = 2a$, and (x, y) any point P of the locus. Draw GH and PD perpendicular to AB, and GK perpendicular to CI.

Then $GK = x - a$, $EH = 2x$, $KP = \sqrt{2ax - x^2}$, and $GH = y - \sqrt{2ax - x^2}$; from which we soon obtain

$$y^2 = \frac{x^3}{2a - x};$$

hence, the locus is a cissoid.

3. Find the edge of a cube whose volume shall be double that of a given cube.*

In Fig. 103, take OC = AB, draw BC cutting the cissoid in Q, and draw QK ⊥ to AB: then find $QK^3 = 2AK^3$. Let a be the edge of any given cube, find b so that

$$AK : KQ :: a : b. \quad \therefore b^3 = 2a^3.$$

4. Find the edge of a cube whose volume is three times, four times, or n times that of a given cube.

5. Show that $IK^3 = 2KB^3$; also that $AK^3 = 2IK^3$.

6. Trisect the angle AOB by means of the conchoid.

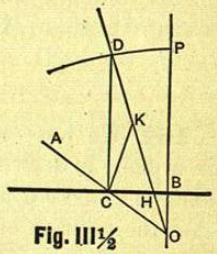

Fig. 111½

Through B draw BC ⊥ to OB, and take BP = 2OC. With O as pole, BC as directrix, and BP as parameter, construct a conchoid. Draw CD ⊥ to BC cutting the conchoid in D, and join DO. Then DOB = ⅓AOB. For, bisect DH at K; then CK = CO. ∴ etc.

* This is called *the duplication of the cube.*