PART II.

ANALYTIC GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

THE POINT.

164. We have seen (Art. 5) that the position of a point in a plane is determined by referring it to two co-ordinate axes, OX, OY, drawn in the plane. We shall now show that the position of a point in *space* may be determined by referring it to the three *co-ordinate planes*.

Let XOY, YOZ, ZOX be three planes of indefinite extent, intersecting each other in the three right lines OX, OY, OZ. Now, if through any point P in the surrounding space we draw PA parallel to OX, PB parallel to OY, and PC parallel to OZ, it is plain that the position of P with reference to the three planes is known, if the lengths of PA, PB, and PC are known. For example, if we have given PA $= a$, PB $= b$, PC $= c$, we can determine the position of the point P with reference to the three planes as follows: We measure OM $(= a)$ along OX, and ON $(= b)$ along OY, and draw the parallels MC and NC; then at the intersection C measure

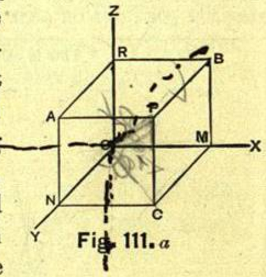

Fig. 111.

CP $(= c)$ on a line parallel to OZ; P will be the point whose position we wished to determine. Otherwise thus: having measured OM, ON, OR, equal respectively to a, b, c, pass through M the plane PCBM parallel to the plane yz; through N the plane PACN, parallel to zx; and through R the plane PABR, parallel to xy; the intersection of the three planes so drawn is the point P.

165. The three planes XOY, YOZ, ZOX, are called the **Co-ordinate Planes**, and are designated as the planes xy, yz, zx, respectively. The three lines OX, OY, OZ, in which these planes intersect, are called the **Co-ordinate Axes**; OX is called the axis of x, OY the axis of y, and OZ the axis of z. The point O in which the three axes intersect, and which is therefore common to the three co-ordinate planes, is called the **Origin**. The distances PA, PB, PC, or their equals, OM, ON, OR, are called the **Rectilinear Co-ordinates** of P, and are respectively represented by x, y, z. The co-ordinate axes may be inclined to each other at any angle whatever; and they are said to be rectangular or oblique, according as the angles at which they intersect are right or oblique angles. In this work we shall employ rectangular axes, as they are the most simple, and can always be secured by a proper transformation.

The co-ordinates of a point are the distances of the point from the three co-ordinate planes yz, zx, xy; hence, if the co-ordinates of a point are respectively denoted by a, b, c, we have for the point,

$$x = a, \qquad y = b, \qquad z = c,$$

which are the **Equations of the Point**. When these equations are given, the point is said to be given, and may be constructed as in Art. 164; the point whose position is defined by the above equation is commonly spoken of as the point (a, b, c).

166. The plane xy is supposed to be horizontal, as the plane of the floor on which the student is standing; the plane xz is perpendicular to the first, and in front of the student; the plane yz is perpendicular to both the others, and on the left of the student.

These co-ordinate planes divide the surrounding space into eight solid angles, which are numbered as follows: The **First** angle lies *above* the plane xy, in *front* of the plane xz, and to the *right* of the plane yz; the **Second** is to the left of the first; the **Third** is behind the second; the **Fourth** is behind the first; the **Fifth, Sixth, Seventh** and **Eighth** are below the first, second, third, and fourth, respectively.

167. In order that the equations $x = a$, $y = b$, $z = c$ should be satisfied by only one point, it is necessary to consider not only the absolute values of the co-ordinates, but also their signs. It is customary to consider lines measured upwards as positive, and hence those measured downwards must be considered negative; also those measured towards the right are considered positive, and hence those measured towards the left are negative; also those measured towards the front are considered positive, and hence those measured towards the rear are negative. Hence, by giving the coordinates their proper signs, we may represent a point in either of the eight angles by one of the following sets of equations:

First Angle, $\begin{cases} x = +a, \\ y = +b, \\ z = +c, \end{cases}$ or by (a, b, c).

Second " $\begin{cases} x = -a, \\ y = +b, \\ z = +c, \end{cases}$ " $(-a, b, c)$.

Third " $\begin{cases} x = -a, \\ y = -b, \\ z = +c, \end{cases}$ " $(-a, -b, c)$.

Fourth Angle, $\begin{cases} x = +a, \\ y = -b, \\ z = +c, \end{cases}$ or by $(a, -b, c)$.

Fifth " $\begin{cases} x = +a, \\ y = +b, \\ z = -c, \end{cases}$ " $(a, b, -c)$.

Sixth " $\begin{cases} x = -a, \\ y = +b, \\ z = -c, \end{cases}$ " $(-a, b, -c)$.

Seventh " $\begin{cases} x = -a, \\ y = -b, \\ z = -c, \end{cases}$ " $(-a, -b, -c)$.

Eighth " $\begin{cases} x = +a, \\ y = -b, \\ z = -c, \end{cases}$ " $(a, -b, -c)$.

COR.—Any point in the plane xy evidently has its $z = 0$; hence, equations of a point in this plane are $x = a$, $y = b$, $z = 0$, or the point is $(a, b, 0)$; and there are similar equations for points in each of the other co-ordinate planes.

Any point on the axis of x has its y and z each $= 0$; hence the equations of a point on this axis are $x = a$, $y = 0$, $z = 0$, or the point is $(a, 0, 0)$; and there are similar equations for points on each of the other co-ordinate axes.

At the origin we evidently have $x = 0$, $y = 0$, $z = 0$, which are the three equations of *the origin of co-ordinates.*

168. The **Orthogonal Projection** of a point on a line or a plane is the foot of a perpendicular from the point to the line or plane. In the present work, when we use the term *projection*, we shall always mean an *orthogonal* projection, since the axes are rectangular. The points M, N, R, are the projections of the point P on the three co-ordinate axes, and the points A, B, C, are the projections of the point P on the three co-ordinate planes.

The projection of a *given right line* upon another right line, or upon a plane, is the line which joins the projections of the extremities of the given line. Thus, OM, ON, OR, are the projections of OP upon the co-ordinate axes x, y, and z respectively; and the lines OA, OB, OC are the projections of OP upon the co-ordinate planes yz, zx, xy, respectively.

The angle which any right line makes with a *plane* is the angle which the line makes with its projection on that plane; the angle which it makes with a given *line* is the angle which it makes with a line drawn through any point of it and parallel to the given line.

It is clear that the projection of a finite right line upon another right line or upon a plane is equal to the first line multiplied by the cosine of the angle which it makes with the second line or with the plane. Hence, it is also evident that the projections of any area of a plane upon another plane is equal to the original area multiplied by the cosine of the angle between the planes.

The line that determines the projection of a point upon a line or plane is called the **Projecting Line** of that point. The projection of any *curve* upon a plane is the curve formed by projecting all of its points. The projecting lines of the different points form a *cylindrical surface* which is called the **Projecting Cylinder** of the curve. When the curve projected is a right line, the projecting cylinder becomes a plane called the **Projecting Plane** of the line.

169. *To find a formula for the distance between two points in space whose co-ordinates are known.*

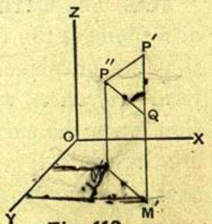

Fig. 112.

Let (x', y', z') and (x'', y'', z'') be the two points P′ and P″. Let the projection of P′P″ on the plane xy be M′M″; draw P″Q parallel to M″M′; represent the distance P′P″ by d. Then we have

$$\overline{P'P''}^2 = \overline{P'Q}^2 + \overline{QP''}^2.$$

But $\qquad P'Q = z' - z''$;

and $\qquad \overline{QP''}^2 = \overline{M'M''}^2$

$$= (x'-x'')^2 + (y'-y'')^2. \quad \text{(Art. 9.)}$$

Therefore, $\qquad d^2 = (x'-x'')^2 + (y'-y'')^2 + (z'-z'')^2$,

or $$d = \sqrt{(x'-x'')^2+(y'-y'')^2+(z'-z'')^2}. \qquad (1)$$

The quantities $(x'-x'')$, $(y'-y'')$, $(z'-z'')$ are equal to the projections of d on the co-ordinate axes x, y, and z, respectively. Hence, *the square of the length of any right line in space is equal to the sum of the squares of its projections on any three rectangular axes.*

COR.—If one of the points, as P″, were the origin, we would have, from (1),

$$d = \sqrt{x'^2 + y'^2 + z'^2}, \qquad (2)$$

which is the distance of any point (x', y', z') from the origin. Hence, *the square of the radius-vector of any point is equal to the sum of the squares of the co-ordinates of the point.*

170. The position of a point is sometimes expressed by its radius-vector and its **Direction Cosines**; that is, the cosines of the three angles which the radius-vector makes with the three co-ordinate axes (see Art. 22, III, Sch. 1); the angles themselves are called the **Direction Angles.** Let these three angles be α, β, γ; then, since the co-ordinates x, y, z of the point are the projections of its radius-vector on the three axes (Art. 168), we have

$$x = \rho\cos\alpha, \qquad y = \rho\cos\beta, \qquad z = \rho\cos\gamma. \qquad (1)$$

Squaring and adding these equations, and remembering that $\rho^2 = x^2 + y^2 + z^2$ (Art. 169, Cor.), we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \qquad (2)$$

which expresses the relation between the direction-cosines of the radius-vector. That is, *the sum of the squares of the direction-cosines of any line is* $= 1$.

171. The position of any point P (Fig. 111*a*), may be expressed by its *polar co-ordinates;* viz., the *radius-vector* OP $(= \rho)$; the angle POR $(= \gamma)$, which the radius-vector makes with the axis of z; and the angle XOC $(= \phi)$, which OC, the projection of the radius-vector on the plane xy, makes with the axis of x. These angles are called the **Vectorial Angles**, and O is called the **Pole.**

From the figure we have

$$x = \rho \sin \gamma \cos \phi,$$
$$y = \rho \sin \gamma \sin \phi,$$
$$z = \rho \cos \gamma,$$

which are the formulæ for transforming from rectangular to polar co-ordinates.

We easily obtain from the above

$$\rho^2 = x^2 + y^2 + z^2; \ \tan \gamma = \frac{\sqrt{x^2 + y^2}}{z}; \ \tan \phi = \frac{y}{x};$$

which are the formulæ for transforming from polar to rectangular co-ordinates.

EXAMPLES.

1. Find the distances between each pair of the points (1, 2, 3), (2, 3, 4), (3, 4, 5), respectively.

Ans. $\sqrt{3}$, $2\sqrt{3}$, $\sqrt{3}$.

2. Prove that the triangle formed by joining the three points (1, 2, 3), (2, 3, 1), (3, 1, 2) respectively is an equilateral triangle.

3. The direction-cosines of a right line are proportional to 2, 3, 6: find their values. *Ans.* $\frac{2}{7}$, $\frac{3}{7}$, $\frac{6}{7}$.

4. The direction-cosines of a right line are proportional to 1, 2, 3: find their values.

Ans. $\frac{1}{\sqrt{14}}$, $\frac{2}{\sqrt{14}}$, $\frac{3}{\sqrt{14}}$.

5. Find (1) the length of the radius vector of (1, 2, 3), (7, -3, -5), and (2) the direction-cosines of the radius vector of each point.

Ans. $\begin{cases} (1)\ \sqrt{14},\ \sqrt{83};\ (2)\ \frac{1}{14}\sqrt{14},\ \frac{1}{7}\sqrt{14},\ \frac{3}{14}\sqrt{14}; \\ \frac{7}{83}\sqrt{83},\ -\frac{3}{83}\sqrt{83},\ -\frac{5}{83}\sqrt{83}. \end{cases}$

6. A right line makes an angle of 60° with one axis and 30° with another: what angle does it make with the third axis? *Ans.* 90°.

7. A, B, C are three points on the axes of x, y, z respectively; if $OA = a$, $OB = b$, $OC = c$, find the co-ordinates of the middle points of AB, BC, and CA respectively.

Ans. $\left(\frac{a}{2}, \frac{b}{2}, 0\right)$, $\left(0, \frac{b}{2}, \frac{c}{2}\right)$, $\left(\frac{a}{2}, 0, \frac{c}{2}\right)$.

8. The polar co-ordinates of a point are

$$\rho = 4, \gamma = \frac{\pi}{6}, \phi = \frac{\pi}{3}:$$

find its rectangular co-ordinates.

Ans. $(1, \sqrt{3}, 2\sqrt{3})$.

9. The rectangular co-ordinates of a point are

$$(\sqrt{3}, 1, 2\sqrt{3}):$$

find its polar co-ordinates.

Ans. $\left(4, \frac{\pi}{6}, \frac{\pi}{6}\right)$.

10. Find the locus of points which are equidistant from the points (1, 2, 3) and (3, 2, -1).

Ans. $x - 2z = 0$.

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