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# CHAPTER II.

## THE RIGHT LINE.

E.T. **172.** *To find the equation of a right line in space.*

Since a line in space is known when two of its *projections* are known (see Church's Desc. Geom., Art. 12), we need only find the equations of the *projections* of the line upon two of the co-ordinate planes.

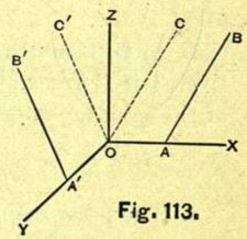



Fig. 113.

Let AB and A′B′ be the projections of a right line on the co-ordinate planes $xz$ and $yz$. Draw through the origin OC and OC′, parallel respectively to AB and A′B′. Let $(x, z)$ be any point in AB, and $(y, z)$ be any point in A′B′; let $a =$ tangent of COZ, and $b =$ tangent of C′OZ; and let $\alpha$ and $\beta$ be the intercepts OA and OA′ respectively. Then we have

$$x = az + \alpha, \qquad (1)$$

and

$$y = bz + \beta, \qquad (2)$$

for the equations of the projections of a right line on the co-ordinate planes $xz$ and $yz$.

Now, since the $x$ and $z$ of *any point in the given line in space* are equal and parallel to the $x$ and $z$ of the *projection* of the same point on the plane $xz$, it follows that (1) expresses the relation between the $x$ and $z$ of every point of the given line. Also, since the $y$ and $z$ of *any point in the given line in space* are equal and parallel to the $y$ and $z$ of the *projection* of the same point on the plane $yz$, it follows that (2) expresses the relation between the $y$ and $z$ of every



point of the given line. Hence, making (1) and (2) *simultaneous*, that is, making the co-ordinates the same in both equations, they together will express the relation between the co-ordinates $x$, $y$, $z$ of every point of the given line; *therefore (1) and (2) are the equations required.*

Cor. 1.—Combining (1) and (2), and eliminating $z$, we obtain

$$y - \beta = \frac{b}{a}(x - \alpha), \qquad (3)$$

which expresses the relation between the $x$ and $y$ of every point of the given line; hence it is the equation of the projection of the line on the plane $xy$.

Cor. 2.—If $y = 0$, we get

$$z = -\frac{\beta}{b}, \qquad x = \frac{b\alpha - a\beta}{b};$$

hence the line pierces the plane $xz$ in the point

$$\left(\frac{b\alpha - a\beta}{b}, \quad 0, \quad -\frac{\beta}{b}\right).$$

Similarly, we find it pierces the plane $yz$ in the point

$$\left(0, \quad \frac{a\beta - b\alpha}{a}, \quad -\frac{\alpha}{a}\right),$$

and the plane $xy$ is the point

$$(\alpha, \beta, 0).$$

Cor. 3.—If the line passes through the origin, we have $\alpha$ and $\beta$ equal to 0; therefore (1) and (2) become

$$x = az, \qquad y = bz, \qquad (4)$$

which are *the equations of a line in space passing through the origin.*

**173.** *To find the equations of a right line in space.*

I. *Passing through a given point;*

II. *Passing through two given points; and*

III. *Passing through a given point, and making the angles $\alpha, \beta, \gamma$ with the co-ordinate axes.*

I. Let $(x', y', z')$ be a given point, and let the equations of the right line be

$$x = az + \alpha, \tag{1}$$

$$y = bz + \beta. \tag{2}$$

Since the point $(x', y', z')$ is to be on the line, it must satisfy its equations, giving us

$$x' = az' + \alpha, \tag{3}$$

$$y' = bz' + \beta. \tag{4}$$

Eliminating $\alpha$ and $\beta$ by subtracting (3) from (1), and (4) from (2), we get

$$x - x' = a\,(z - z'), \tag{5}$$

$$y - y' = b\,(z - z'), \tag{6}$$

for *the equations of a right line passing through a given point in space.*

II. Let $(x'', y'', z'')$ be the second given point. Since this point is to be on the line, it must satisfy its equations, giving us

$$x'' = az'' + \alpha, \tag{7}$$

$$y'' = bz'' + \beta. \tag{8}$$

Eliminating $\alpha$ and $\beta$ by subtracting (7) from (3) and (8) from (4), we get

$$x' - x'' = a\,(z' - z''), \quad \text{or} \quad a = \frac{x' - x''}{z' - z''}; \tag{9}$$

$$y' - y'' = b\,(z' - z''), \quad \text{or} \quad b = \frac{y' - y''}{z' - z''}. \tag{10}$$

Substituting these values of $a$ and $b$ in (5) and (6), we get



$$x - x' = \frac{x' - x''}{z' - z''}\,(z - z'), \tag{11}$$

$$y - y' = \frac{y' - y''}{z' - z''}\,(z - z'), \tag{12}$$

which are *the equations of a right line passing through two given points in space;* or, as they may be more symmetrically written,

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''} = \frac{z - z'}{z' - z''}.$$

III. Let $(x, y, z)$ be any variable point on the line. By Art. 169, $x - x'$, $y - y'$, $z - z'$ are the projections of the distance between the points $(x', y', z')$ and $(x, y, z)$ on the axes; and since this distance is equal to its projection on either of the axes divided by the corresponding direction-cosine, we have

$$\frac{x - x'}{\cos\alpha} = \frac{y - y'}{\cos\beta} = \frac{z - z'}{\cos\gamma}, \tag{13}$$

which are the equations required, and are known as the *symmetrical equations of a right line in space.* (See Art. 22, II).

**174.** *To find the angle between two right lines in space in terms of the angles which they make with the co-ordinate axes.*

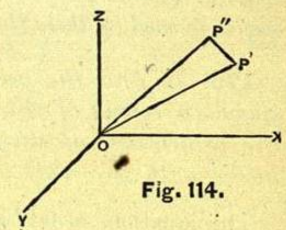

Fig. 114.

The angle between any two right lines in space is equal to the angle between two lines drawn through any given point, and parallel respectively to the given lines. Therefore, let OP′ and OP″ be drawn through the origin and parallel to the given lines; the angle between OP′ and OP″ will be equal to the required angle.

Let $(x', y', z')$ and $(x'', y'', z'')$ be the points P′ and P″ respectively, and OP′ $= r'$, OP″ $= r''$, P′P″ $= d$; also, let the angles which OP′ and OP″ make with the co-ordinate axes be $\alpha, \beta, \gamma$, and $\alpha', \beta', \gamma'$, respectively; and denote the angle P′OP″ by $v$.

Then, by Trigonometry, we have

$$\cos v = \frac{r'^2 + r''^2 - d^2}{2r'r''}. \tag{1}$$

But (Art. 169) we have

$$d^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2. \tag{2}$$

Substituting (2) in (1), and remembering that

$$x'^2 + y'^2 + z'^2 = r'^2, \qquad x''^2 + y''^2 + z''^2 = r''^2,$$

we get

$$\cos v = \frac{x'x'' + y'y'' + z'z''}{r'r''}. \tag{3}$$

But (Art. 170) we have

$$x' = r' \cos\alpha, \quad y' = r' \cos\beta, \quad z' = r' \cos\gamma; \tag{4}$$

$$x'' = r'' \cos\alpha', \quad y'' = r'' \cos\beta', \quad z'' = r'' \cos\gamma', \tag{5}$$

which in (3) give

$$\cos v = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'. \tag{6}$$

That is, *the cosine of the angle between two right lines in space is equal to the sum of the products of the cosines of the angles formed by these lines with the co-ordinate axes.*

end Tuesday

**175.** *To find the angle between two right lines in space in terms of the tangents of the angles which the projections of the lines on the planes* $xz$ *and* $yz$ *make with the axis of* $z$.

Notes 47–48

The equations of OP′ and OP″ (Art. 172, Cor. 3) are,

(OP′), $\qquad x = az, \quad y = bz, \qquad (1)$

and (OP″), $\qquad x = a'z, \quad y = b'z. \qquad (2)$



Since $(x', y', z')$ is on OP′, it must satisfy (1), giving us

$$x' = az', \qquad y' = bz'. \tag{3}$$

Since $(x'', y'', z'')$ is on OP″, it must satisfy (2), giving

$$x'' = a'z'', \qquad y'' = b'z''. \tag{4}$$

Substituting these values of $x'$ and $y'$ given in (3) in

$$x'^2 + y'^2 + z'^2 = r'^2,$$

we get

$$a^2z'^2 + b^2z'^2 + z'^2 = r'^2,$$

or

$$z' = \frac{r'}{\sqrt{a^2 + b^2 + 1}};$$

which in (3) gives us

$$x' = \frac{ar'}{\sqrt{a^2 + b^2 + 1}},$$

$$y' = \frac{br'}{\sqrt{a^2 + b^2 + 1}}.$$

Now these values of $x', y', z'$ in (4) of Art. 174 give us

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + 1}}, \tag{5}$$

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + 1}}, \tag{6}$$

$$\cos\gamma = \frac{1}{\sqrt{a^2 + b^2 + 1}}, \tag{7}$$

In like manner, we find

$$\cos\alpha' = \frac{a'}{\sqrt{a'^2 + b'^2 + 1}},$$

$$\cos\beta' = \frac{b'}{\sqrt{a'^2 + b'^2 + 1}},$$

$$\cos\gamma' = \frac{1}{\sqrt{a'^2 + b'^2 + 1}}.$$

note 47 & 48



Substituting these values of the cosines in (6) of Art. 174, we get

$$\cos v = \frac{aa' + bb' + 1}{\sqrt{a^2 + b^2 + 1}\,\sqrt{a'^2 + b'^2 + 1}}. \tag{8}$$

Cor. 1.—If the lines are parallel to each other, $v = 0$, and $\cos v = 1$; hence, clearing (8) of fractions and squaring, it becomes

$$(a^2 + b^2 + 1)(a'^2 + b'^2 + 1) = (aa' + bb' + 1)^2;$$

transposing and uniting, we obtain

$$(a - a')^2 + (b - b')^2 + (ab' - a'b)^2 = 0.$$

Each term being a square, and therefore positive, this equation can be satisfied only when the terms are separately equal to 0, giving us

$$a = a', \qquad b = b', \qquad ab' = a'b.$$

But the third term follows directly from the other two; hence,

$$a = a' \quad \text{and} \quad b = b' \tag{9}$$

are *the equations of condition* that two lines in space shall be parallel to each other; that is, *if two right lines in space are parallel, their projections on the co-ordinate planes are parallel.* [Art. 172, Eqs. (1) (2), (3); also Art. 27, Cor. 1.]

Cor. 2.—If the lines are perpendicular to each other, $\cos v = 0$, and hence (8) becomes

$$aa' + bb' + 1 = 0, \tag{10}$$

which is *the equation of condition* that makes two right lines in space perpendicular to each other.

**176.** *To find the condition that two right lines in space may intersect, and the position of the point of intersection.*

Let
$$\begin{cases} x = az + \alpha, & (1) \\ y = bz + \beta, & (2) \end{cases}$$



and
$$\begin{cases} x = a'z + \alpha', & (3) \\ y = b'z + \beta', & (4) \end{cases}$$

be the equations of two right lines in space which intersect. If these lines do intersect, the co-ordinates of the point of intersection must satisfy all the equations. But as there are *four* equations, containing only *three* unknown quantities, the equations cannot all be satisfied by the same set of values of $x$, $y$, $z$, if they are independent of each other. That is, there must be such a relation between the known quantities as to make one equation depend upon the other three; and the equation expressing this relation will be the required condition of intersection.

We form this condition, of course, by eliminating $x$, $y$, $z$ from the four equations. Solving (1) and (3), and also (2) and (4) for $z$, we get

$$z = \frac{\alpha' - \alpha}{a - a'}, \tag{5}$$

and

$$z = \frac{\beta' - \beta}{b - b'}. \tag{6}$$

Equating the two values of $z$ in (5) and (6), we get

$$\frac{\alpha' - \alpha}{a - a'} = \frac{\beta' - \beta}{b - b'}; \quad (\ell) \tag{7}$$

which is *the required condition that two lines in space shall intersect.*

Substituting (5) in (1), and (6) in (2), we get

$$x = \frac{a\alpha' - a'\alpha}{a - a'};$$

$$y = \frac{b\beta' - b'\beta}{b - b'}.$$

These values of $x$ and $y$, with the value of $z$ from either (5) or (6), will give *the point of intersection* when (7) is satisfied.

## EXAMPLES.

1. Find the distance between the points (3, 2, 1) and (4, 5, 3). *Ans.* $d = \sqrt{14}$.

2. Find the distance between the points $(-5, 5, -3)$ and (1, 0, 5). *Ans.* $d = 11.18$.

3. Find the equations of a right line passing through the point (2, 3, 4).

*Ans.* $x - 2 = a(z - 4)$; $y - 3 = b(z - 4)$.

4. Find the equations of the right line passing through the two points (3, 4, 2) and (4, 1, 5).

*Ans.* $3x = z + 7$; $3y = -3z + 18$.

5. Find the points in which the line last found pierces the co-ordinate planes.

*Ans.* $(2\frac{1}{3}, 6, 0)$, $(4\frac{1}{3}, 0, 6)$, and $(0, 13, -7)$.

6. Find the equation of the projection of the line in Ex. 4, on the plane $xy$. *Ans.* $3x = -y + 13$.

7. The equations of the projections of a right line on $zx$, $yz$, are

$$x = z + 1, \qquad y = \tfrac{1}{2}z - 2;$$

required its equation on the plane $xy$.

*Ans.* $2y = x - 5$.

8. Find the equations of the three projections of a right line which passes through the two points (2, 1, 0) and $(-3, 0, -1)$.

*Ans.* $x = 5z + 2$; $y = z + 1$; $5y = x + 3$.

9. Find the angle between the right lines

$$x = 3z + 5, \qquad y = 5z + 3;$$

and

$$x = z + 1, \qquad y = 2z.$$

*Ans.* $14° \, 58'$.

10. Find the equations of a right line through the origin and perpendicular to both the lines in Ex. 9.

*Ans.* $x = 3z$; $y = -2z$.



11. Find the cosine of the angle between the lines

$$x = 2z + 1, \qquad y = 2z + 2;$$

and

$$x = z + 5, \qquad y = 4z + 1.$$

*Ans.* $\operatorname{Cos} v = \frac{11}{9\sqrt{2}}$.

12. Find the point of intersection of the two lines

$$x = -2z + 3, \qquad y = z - 2;$$

and

$$x = 3z - 1, \qquad 5y = -10z + 2;$$

and the cosine of the angle between them.

*Ans.* $(\frac{7}{5}, -\frac{6}{5}, \frac{4}{5})$, $\cos v = \mp\sqrt{\frac{7}{12}}$.

13. Find whether the two lines

$$x = 2z + 1, \qquad y = 3z + 4;$$

and

$$x = -2z + 3, \quad y = z - 2;$$

are parallel or perpendicular to each other.

*Ans.* Perpendicular.

14. Find the equations of the line which passes through the point $(-3, 2, -1)$ and is parallel to the line

$$x = -3z - 1, \qquad y = 4z + 3;$$

(see Art. 175, Cor. 1), also of the line through the same point and perpendicular to the same line. (See Art. 175, Cor. 2, and Art. 176.)

*Ans.* To first, $x = -3z - 6$, $y = 4z + 6$;
" To second, $27x = 49z - 32$, $9y = 10z + 28$.

15. Find the direction-cosines of

$$x = 4z + 3, \qquad y = 3z - 2.$$

*Ans.* $\operatorname{Cos} \alpha = \frac{4}{\sqrt{26}}$; $\operatorname{Cos} \beta = \frac{3}{\sqrt{26}}$; $\operatorname{Cos} \gamma = \frac{1}{\sqrt{26}}$.

16. Find the equation of a right line through the point (4, 5, 7), its direction-cosines being $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$.

*Ans.* $\frac{x-4}{2} = \frac{y-5}{1} = \frac{z-7}{2}$; or $\left\{\begin{array}{l} x = z - 3 \\ 2y = z + 3 \end{array}\right\}$.

17. A right line makes an angle of 60° with one axis and 45° with another. What angle does it make with the third axis? (Art. 170.) *Ans.* 60°.

18. Find the angles which the line $x = -2z + 1$, $y = z + 3$, makes with the co-ordinate axes.
*Ans.* $\alpha = 144° 44'$; $\beta = 65° 54'$; $\gamma = 65° 54'$. (Art. 175.)

19. The equations of two lines are

$$x = 2z + 1, \qquad y = 2z + 2;$$

and

$$x = z + 5, \qquad y = 4z + \beta';$$

find the value of $\beta'$ so that the lines shall intersect each other, and also the point of intersection. (Art. 176.)
*Ans.* $\beta' = -6$; the point of intersection is (9, 10, 4).

20. Find the angle between the lines

$$x = z\sqrt{2}, \qquad y = z\sqrt{\tfrac{3}{2}};$$

and

$$x = y\sqrt{3}, \qquad z = 0.$$

[Here $b' = \infty$ and $a' = \infty\sqrt{3}$. See Art. 172.]
*Ans.* 30°.

21. Show that the lines $4x = 3y = -z$, and $3x = -y = -4z$ are at right angles to each other.

NOTE.—The equations are here written in their symmetric form (Art. 173).

22. Find the angle between the lines $\frac{x}{1} = \frac{y}{1} = \frac{z}{0}$, and $\frac{x}{3} = -\frac{y}{4} = \frac{z}{5}$. *Ans.* $\cos^{-1}\frac{1}{10}$.

23. Find the acute angle between the lines whose direction-cosines are $\frac{1}{4}\sqrt{3}$, $\frac{1}{4}$, $\frac{1}{2}\sqrt{3}$, and $\frac{1}{4}\sqrt{3}$, $\frac{1}{4}$, $-\frac{1}{2}\sqrt{3}$.
*Ans.* 60°.

24. Find the equation of the right line through the point (2, 3, 4), which is equally inclined to the axes.
*Ans.* $x - 2 = y - 3 = z - 4$.

# CHAPTER III.

## THE PLANE.

**177. The Equation of a Plane** is the equation which expresses the relation between the co-ordinates of every point of the plane.

*To find the equation of a plane.*

A plane may be generated by revolving a right line about its intersection with another right line, to which it is perpendicular. The revolving line is called the **Generator**, and the line to which it is perpendicular is called the **Director.***

Let

$$x = az + \alpha, \qquad y = bz + \beta, \tag{1}$$

be the equation of a given line which we take for the director. If the director passes through the point $(x', y', z')$ its equations will be

$$x - x' = a(z - z'); \quad y - y' = b(z - z'). \tag{2}$$

The equations of a line through the same point $(x', y', z')$ and perpendicular to the director are

$$x - x' = a'(z - z'); \quad y - y' = b'(z - z'). \tag{3}$$

The equation of condition that makes (3) perpendicular to (2) is (Art. 175, Cor. 2)

$$aa' + bb' + 1 = 0. \tag{4}$$

* In using these words I follow Gregory and Salmon, instead of giving them a feminine termination, and calling them "*generatrix*" and "*directrix*." Also, the word "directrix" has already been used in a different sense (see Art. 51) from the present, and it is well to distinguish between the two.