Early papers:

Frankenheim. De Crystallorum Cohæsione, 1829; also in Baumgartner's Zeitschrift für Physik, 9, 94, 194, 1831. Die Lehre von der Cohäsion, Breslau, 1835. Ueber die Anordnung der Molecüle im Krystall; Pogg., **97**, 337, 1856.

Hessel. Article 'Krystall' in Gehler's physikal. Wörterbuch, 5, 1830 (see Sohncke, Zs. Kryst., 18, 486).

Bravais. Mémoire sur les systèmes formes par des points distribués régulièrement sur un plan ou dans l'espace, Paris, 1850; and in Études cristallographiques. Paris, 1866.

Gadolin. Act. Soc. Fennicæ, **9**, 1, 1871 (republished in Ostwald's Klass ker d. exakten Wissenschaften, No. 75).

Later works and papers:

Barlow. Nature, **29**, 186, 205, 1883; Zs. Kryst., **23**, 1, 1894; **25**, 86, 1895; Min. Mag., **11**, 119, 1896, and Zs. Kryst., **27**, 449, 468, 1896; R. Dublin Soc., **8**, 527, 1897, and Zs. Kryst., **29**, 433, 1898.

Curie. Bull. Soc. Min., **7**, 89, 418, 1884.

Fedorow. Zs. Kryst., **20**, 25, 259; **24**, 209, 1894; **25**, 113, 1895; **28**, 36, 232, 468, 1897.

Goldschmidt. Zs. Kryst., **28**, 1, 414, 1897.

Kelvin. Proc. R. Soc. Edinb., **16**, 693, 1888; Proc. Roy. Soc., **55**, 1, 1894.

Minnegerode. Nachr. Ges. Göttingen, 1884; Jb. Min. Beil.-Bd., **5**, 145, 1887.

Schönflies. Nachr. Ges. Göttingen, 483, 1888; 239, 1890. Also as a separate work, Krystallsysteme und Krystallstructur, Leipzig, 1891.

Sohncke. Die Gruppirung der Moleküle in den Krystallen, Pogg. Ann., **132**, 75, 1867. Also Wied. Ann., **16**, 489, 1882; Zs. Kryst., **13**, 209, 214, 1887; **14**, 417, 426, 1888; **18**, 486, 1890. Entwickelung einer Theorie der Krystallstructur, Leipzig, 1879.

Viola. Zs. Kryst., **27**, 1, 1896; **28**, 452; **29**, 1, 234, 1897.

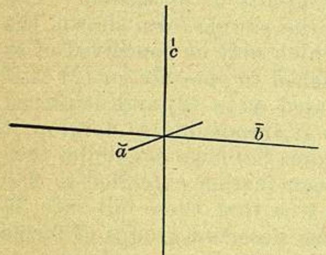
L. Wulff. Zs. Kryst., **13**, 503, 1887; **15**, 366, 1889; **18**, 174, 1890.

Wülfing. For title see p. 2.

GENERAL MATHEMATICAL RELATIONS OF CRYSTALS.

33. Axial Ratio, Axial Plane.—The crystallographic axes have been defined (Art. 22) as certain lines, usually determined by the symmetry, which are used in the description of the faces of crystals, and in the determination of their position and angular inclination. With these objects in view, certain lengths of these axes are assumed as units to which the occurring faces are referred.

55.



The axes are, in general, lettered a, b, c, to correspond to the scheme in Fig. 55. To aid the memory, the letters may be further distinguished; as $\dot{c}$ (vertical axis); $\breve{a}$, $\bar{b}$ (shorter and longer lateral axes), etc.

If two of the axes are equal, they are designated a, a, $\dot{c}$; if three, a, a, a. In one system, the hexagonal, there are four axes, lettered a, a, a, $\dot{c}$.

Further, in the systems other than the isometric, one of the lateral axes is taken as the unit to which the other axes are referred; hence the lengths of the axes express strictly the *axial ratio.* Thus for sulphur (orthorhombic, see Fig. 57) the axial ratio is

$$\breve{a} : \bar{b} : \dot{c} = 0{\cdot}8131 : 1 : 1{\cdot}9034.$$

For rutile (tetragonal) it is

$$a : \dot{c} = 1 : 0{\cdot}64415, \quad \text{or,} \quad \text{simply,} \ \dot{c} = 0{\cdot}64415.$$

The plane of any two of the axes is called an *axial plane,* and the space included by the three axial planes is an *octant,* since the total space about the center is thus divided by the three axes into eight parts. In the hexagonal system, however, where there are three lateral axes, the space about the center is divided into 12 parts, or *sectants.*

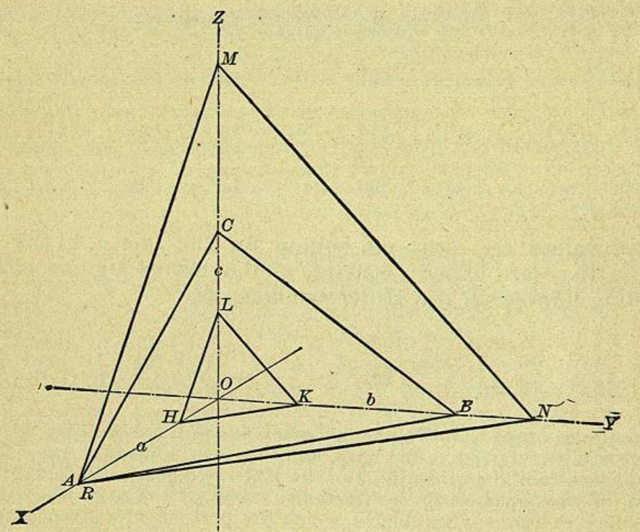
34. Parameters, Symbol.—The *parameters* of a plane are its intercepts on the assumed axes. The *symbol* expresses, often in abbreviated form, the relation of these intercepts to certain lengths of the axes taken as units.

For example, in Fig. 56 let the lines OX, OY, OZ be taken as the directions of the crystallographic axes, and let OA, OB, OC represent the unit lengths, designated (always in the same order) by the letters a, b, c. Then the parameters for the plane (1) HKL are OH, OK, OL; for the plane (2) RNM they are OR, ON, OM. But in terms of the unit lengths these are, respectively,

$$(1) \ \frac{1}{4}a : \frac{1}{3}b : \frac{1}{2}c, \quad \text{or} \quad (2) \ 1a : \frac{4}{3}b : 2c.$$

These two expressions are *identical,* since the two planes HKL, MNR are parallel and hence crystallographically the same. Obviously each of the above expressions may be changed into the other by multiplying (or dividing) by 4.

56.



It will be noted that in (1) the numerators of the fractional numbers expressing the relation to the axes are all unity; while in (2) the number referring to one of the lateral axes (a) is made unity. The significance of this distinction will appear at once.

The general expression for any plane referred to these axes, written after the same method, will be

$$(1) \ \frac{1}{h}a : \frac{1}{k}b : \frac{1}{l}c, \quad \text{or} \quad (2) \ 1a : nb : mc.$$

Here in (1) the numbers, or *indices*, *hkl* (in the case above, 432) constitute the *symbol* after the method of Miller (1839; earlier developed by Whewell and Grassmann).

The second form (2) is the symbol essentially as early written by Weiss. This last was contracted by Naumann to *mPn* (*mOn* in the isometric system), the axes being omitted from the expression and the order reversed; the same with the omission of the *P* (or *O*), *m-n*, is adopted in Dana's System of Mineralogy, in the last edition (1892) of which work, however, the Miller symbols are given the preference.

In the hexagonal system there are assumed four axes, three of them lateral axes. Corresponding to this, in the symbols after the method of Miller as adapted by Bravais, there are four indices, *hkil*. The relation of these to the axes is the same as in the other cases, as explained under the hexagonal system.

The following are other examples of planes with the symbols written after the two methods given. It will be seen that the respective expressions under (1) and (2) are identical.

(1)	Miller's Symbol.		(2)	Naumann's Symbol.
$\frac{1}{2}a : \frac{1}{2}b : \frac{1}{1}c$	221	or	$1a : 1b : 2c$	$2P$ or 2
$\frac{1}{2}a : \frac{1}{1}b : \frac{1}{2}c$	212	"	$1a : 2b : 1c$	$P2$ or 1-2
$\frac{1}{2}a : \frac{1}{0}b : \frac{1}{1}c$	201	"	$1a : \infty b : 2c$	$2P\infty$ or 2-i
$\frac{1}{2}a : \frac{1}{1}b : \frac{1}{0}c$	210	"	$1a : 2b : \infty c$	$\infty P2$ or i-2
$\frac{1}{1}a : \frac{1}{0}b : \frac{1}{0}c$	100	"	$1a : \infty b : \infty c$	$\infty P\infty$ or i-i

If the axial values are measured behind for the axis *a*, to the left for *b*, or below for *c*, they are called negative, and a minus sign is placed *over* the corresponding number of the Miller symbols; as,

	Miller.		Miller.
$-\frac{1}{2}a : -\frac{1}{2}b : \frac{1}{2}c$	$\bar{2}\bar{2}1$	$-\frac{1}{2}a : \frac{1}{0}b : \frac{1}{1}c$	$\bar{2}01$

It is sometimes stated that Naumann's symbols are the more easy of comprehension because more readily referred to the axes, and this is in a measure true. If the student, however, will accustom himself to think of the Miller symbols in the form given above, that is, always as the denominators of the fractional values of the axes whose numerators are unity, he will never have any trouble in seeing the position of a given plane relatively to the axes. He must remember that the order is always that given above, *h*, *k*, and *l* referring respectively to the axes *a*, *b*, and *c*; moreover, he will note that a zero, 0, always means that the given plane is parallel to the axis to which it refers, since $\frac{1}{0} = \infty$.

With the symbols of Naumann, the *m*, written first, always refers to the vertical axis, while the *n*, which follows, and is always greater than unity, refers to one of the lateral axes, the other being made unity. To which lateral axis the *n* belongs is often indicated by a mark over the *n* ($\bar{n}$, or $\breve{n}$, or $\grave{n}$), or attached to the *P* as explained under the different systems. When $m = 1$, it is omitted before the *P* or *O* (but not so when the *P* is dropped); and when $n = 1$, it is omitted in all cases.

Other systems of symbols, besides the two explained, have also been or still are in use, as those of Weiss, of Mohs and Haidinger, Hausmann, Lévy, Goldschmidt, and others.

Of these the symbols of Weiss are essentially those already given (under 2, p. 23) which, abbreviated (and inverted in order), were adopted by Naumann. The symbols of Lévy have been extensively used by the French school of mineralogists. A very full explanation of all the different systems, as of that recently devised by himself, is given in Goldschmidt's Index (1886–1891). Transformation equations for the important cases, are given by Groth (Phys. Kryst.), Mallard (Crist., vol. 1), Liebisch (Kryst.), and others; see p. 2.

35. Law of Rational Indices.—The study of crystals has established the general law that the ratios between the intercepts on the axes for any face on a crystal to those of any other face can always be expressed by rational numbers. These ratios may be $1:2$, $2:1$, $2:3$, $1:0\ (\infty:1)$, etc., but never $1:\sqrt{2}$, etc. Hence the values of *hkl* in the Miller symbols must always be either *whole numbers or zero*, and similarly the *m* and *n* of Naumann's symbols may be whole numbers or fractions, or infinity.

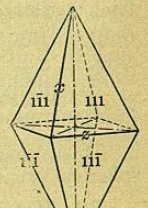
If the form whose intercepts on the axes *a*, *b*, *c* determine their assumed unit lengths—the *unit form* as it is called—is well chosen, these numerical values of the indices are in most cases very simple. In the Miller symbols, 0 and the numbers from 1 to 6 are most common.

The above law, which has been established as the result of experience, in fact follows from the consideration of the molecular structure as hinted at in an earlier paragraph (Art. **31**).

The law of rational indices finds an illustration later under the isometric system. It is stated there that three of the five regular solids of geometry, viz., the cube, octahedron, and the regular triangular pyramid (crystallographically the tetrahedron) all occur among crystals; the regular pentagonal dodecahedron and icosahedron, on the contrary, are impossible forms. This is true because the ratios of their intercepts on the axes for such forms would be irrational; thus for the regular dodecahedron the ratio would be $1 : \frac{1+\sqrt{5}}{2}$. There are, it is true, two forms respectively twelve-sided and twenty-sided which approximate to these regular solids, but their faces in the first case are not all regular pentagons, and in the second they are not all regular triangles. In the latter case it will be seen that the twenty faces in fact belong to two distinct forms, eight of one and twelve of the other.

36. Form.—A *form* in crystallography includes *all* the faces which have a like position relative to the planes, or axes, of symmetry. The full meaning of this will be appreciated after a study of the several systems. It will be seen that in the most general case, that of a form having the symbol (*hkl*), whose planes meet the assumed unit axes at unequal lengths, there must be forty-eight like faces in the isometric system * (see Fig. 101), twenty-four in the hexagonal (Fig. 201), sixteen in the tetragonal (Fig. 166), eight in the orthorhombic (Fig. 57), four in the monoclinic, and two in the triclinic. In the first four systems the faces named yield an enclosed solid, and hence the form is called a *closed form;* in the remaining two systems this is not true, and such forms in these and other cases are called *open forms*. Fig. 275 shows a crystal bounded by three pairs of unlike faces; each pair is hence an open form. Figs. 58–61 show open forms.

57.



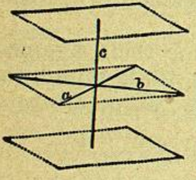
The *unit or fundamental form* is one where parameters correspond to the assumed unit lengths of the axes. Fig. 57 shows the unit pyramid of sulphur whose symbol is (111); it has eight similar faces, the position of which determines the ratio of the axes given in Art. **33**.

* The normal group is referred to in each case.

The forms in the isometric system have special individual names, given later. In the other systems certain general names are employed which may be briefly mentioned here. A form whose faces are parallel to two of the axes* is called a *pinacoid* (from πίναξ, a board, it is shown in Fig. 58. One whose faces are parallel to the vertical axis but meet both the lateral axes is called a *prism*, as Fig. 59. If the faces are parallel to one lateral axis only, it is a *dome* (Figs. 60, 61). If the faces meet all the axes, the form is a *pyramid* (Fig. 57); this name is given even if there is only one face belonging to the form.

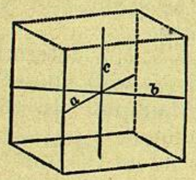
In Fig. 62, *a* (100), *b* (010), *c* (001) are pinacoids; *m* (110), *s* (120) are prisms; *d* (101), also *h* (011), *k* (021) are domes; all these are open forms. Finally, *e* (111), *f* (121) are pyramids, and in this case they are closed forms. The relation existing in each of these cases between the symbol and the position of the faces to the axes should be carefully studied.

58.



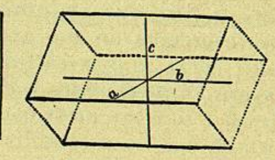
Basal Pinacoid.
(001)

59.



Prism.
(110)
($hk0$)

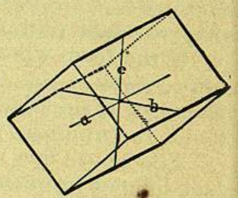
60.



(101)
($h0l$)

Domes.

61.



(011)
($0kl$)

As shown in the above cases, the symbol of a *form* is usually included in parentheses, as (111), (100); or it may be in brackets [111] or {111}. If the symbol is written without parenthesis, as 111, it usually refers to a single face of the form only. Note also that with the Miller symbols, each face of a given form has its own individual symbol.

37. Zone.—A zone includes a series of faces on a crystal whose intersection-lines are mutually parallel to each other and to a common line drawn through the center of the crystal, called the zone-axis. This parallelism means simply that the parameters of the given faces have a constant ratio for two of the axes. Some simple numerical relation exists, in every case, between all the faces in a zone, which is expressed by the *zonal equation*. The faces *a*, *m*, *s*, *b* (Fig. 62) are in a zone; also, *b*, *k*, *h*, *c*, etc.

If a face of a crystal falls simultaneously in two zones, it follows that its symbol is fixed and can be determined from the two zonal equations, without the measurement of angles. Further, it can be proved that the face corresponding to the intersection of two zones is always a possible crystal face, that is, one having rational values for the indices which define its position.

In many cases the zonal relation is obvious at sight, but it can always be determined, as shown in Arts. **43, 44**, by an easy calculation.

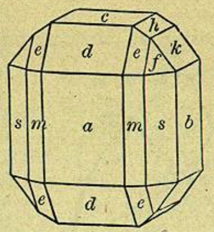
Illustrations will be given after the methods of representing a crystal by horizontal and spherical projections have been explained.

38. Horizontal Projections.—In addition to the usual perspective figures of crystals, projections on the basal plane (or more generally the plane normal to the prismatic zone) are very conveniently used. These give in fact a map of the crystal as viewed from above looking in the direction of the axis of the prismatic zone. Figs. 30, 32, 34 give simple examples; also Fig. 63 a projection of Fig. 62, both repeated from p. 16. In these the successive faces may be indicated by accents, as in Fig. 63, passing around in the

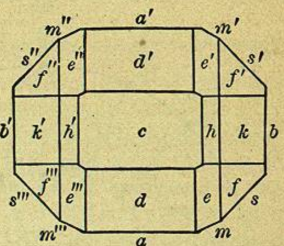
* In the tetragonal system the form (100) is, however, called a prism and (101) a pyramid.

direction of the axes *a*, *b*, *a'*, that is, counter-clockwise. On the construction of these projections see the Appendix A.

62.



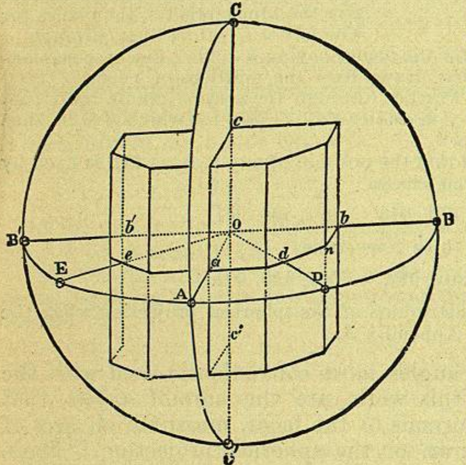
63.



39. Spherical Projections.—The study of actual crystals, particularly as regards the angular and zonal relations of their faces, is much facilitated by the use of the *spherical projection*. In this the position of each face is represented by a point called its *pole*, where a normal drawn to it from the center and produced meets the surface of the sphere. The symbols after Miller are immediately connected with this projection, and by means of it all ordinary calculations can be performed in a very simple manner. Fig. 65 shows a spherical projection of the orthorhombic crystal, Fig. 62.

If the center of a crystal, that is, the point of intersection of the crystallographic axes, be taken as the center of a sphere, and normals be drawn from it to the successive faces of the crystal, the points, where they meet the surface of the sphere, will be, as before defined, the *poles* of the respective faces. For example, in Fig. 64, the common center of the crystal and sphere is at O, the normal to the face *b* meets the surface of the sphere at B, of *b'* at B', of *d* and *e* at D and E respectively, and so on. These poles evidently determine the position of the face in each case.

64.



After Brezina.

It is obvious that the pole of the face *b'* ($0\bar{1}0$) opposite *b* (010) will be at the opposite extremity of the diameter of the sphere, and so in general for (120) and ($1\bar{2}0$), etc. It is seen also that all the poles, or normal points, of faces in the same *zone*, that is, faces whose intersection-lines are parallel, are in the same great circle, for instance *B* (010), *D* (110), *A* (100), *E* (110), and so on.

It is customary in the use of the sphere to regard it as projected upon a horizontal plane, usually that normal to the prismatic zone, so that, as in Fig. 65, the poles of the prismatic faces lie in the circumference of the circle, and those of the other faces within it. The eye being supposed to be situated at the opposite extremity of the diameter of the sphere normal to this plane, the great circles then appear either as arcs of circles, or as straight lines, *i.e.*, diameters.

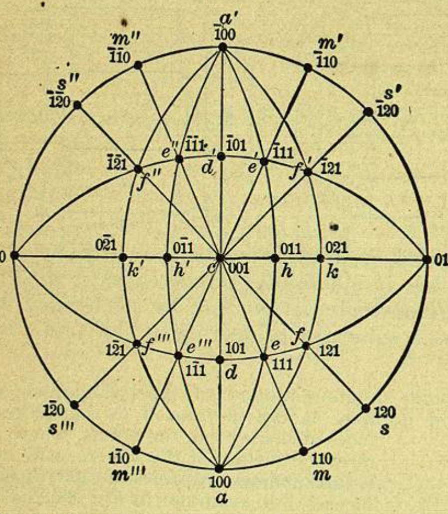
It will be further obvious from Fig. 64 that the arc BD, between the poles of *b* and *d*, measures an angle at the center (BOD), which is the *supplement* of the actual interior angle *bnd* between the two faces; and this is true in general.

40. Construction of the Spherical Projection.—Since in the method ordinarily followed the poles of the prismatic faces lie in the circumference of the circle, their position is fixed at once by the angles laid off, *e.g.* from 100, with a protractor. Further, the distances of the poles of all faces measured from the center of the circle (which, when the vertical axis is at right angles to those in the lateral plane, is the pole of the base 001) are proportional to *the tangents of half the angles*. For example, to construct the spherical projection of Fig. 62, first draw the circle, and lay off on the circumference, from a point taken as 100, the angular distances characteristic of this species (chrysolite):

$$am,\ 100 \wedge 110 = 24°\ 58'; \qquad as,\ 100 \wedge 120 = 42°\ 58'; \qquad ab,\ 100 \wedge 010 = 90°.$$

The position of the poles of the faces *a* (100), *m* (110), *s* (120), *b* (010) are thus fixed. The poles of the other faces of these forms are obviously fixed, *a'*, *m'*, *s'*, *b'*, *m''*, *s''*, *m'''*, *s'''*, by the symmetry. Again, to find the pole of *d* (101), which lies on the great circle, or zone, *a* (100), *c* (001), *a'* ($\bar{1}$00) (for which $k = 0$): since *c* (001) ∧ *d* (101) = 51° 33′, the distance *cd* is proportional to the tangent of ½(51° 33′) or 25° 46½′, that is, 0.483 of the radius *ca*. Similarly for *h* (011) and *k* (021) on the zone-circle *c* (001), *b* (010), since *ch* (001 ∧ 011) = 30° 23½′ and *ck* (001 ∧ 021) = 49° 33′, the distances are proportional to the tangents of half these angles respectively. So also from the angles *ce* (001 ∧ 111) = 54° 15′ and *cf* (001 ∧ 121) = 59° 50½′ the distances on the corresponding zone-circles, *c* (001) *m* (110) and *c* (001) *s* (120), may be determined. In practice, however, these last steps are unnecessary; since if the circular arc through *b* (010), *d* (101), *b'* (0$\bar{1}$0) is drawn, it gives the zone-circle for all the faces for which $h = l$; similarly that through *a* (100), *h* (011), *a'* ($\bar{1}$00) give the zone-circle for the planes for which $k = l$, while that through *a* (100), *k* (021), *a'* ($\bar{1}$00) gives the zone-circle for the planes having $k = 2l$. The intersection-points between these last arcs and that first drawn fixes the positions of *e* (111), *f* (121), each of which satisfies the two relations. Further, through these same points must pass the zone-circle *c* (001), *m* (110), for which $h = k$, and *c* (001), *s* (120), for which $k = 2h$, thus giving a check upon the accuracy of the work.

65.



It is obvious from the above explanation that the position of any face, as 431, is fixed by the intersection of either two of the three zone-circles

100, 031, $\bar{1}$00 010, 401, 0$\bar{1}$0 001, 430, 00$\bar{1}$.

In general any face, *hkl*, must lie in the three zone-circles

100, 0*kl*, $\bar{1}$00, 010, *h*0*l*, 0$\bar{1}$0, 001, *hk*0, 00$\bar{1}$.

Some further points in regard to the construction of the spherical projection when the axial intersections are oblique are added in Appendix A.

41. Angles between Faces.—The angles most conveniently used with the Miller symbols, and those given in this work, are the *normal angles*, that is, the angles between the poles or normals to the faces, measured on arcs of great circles joining the poles as shown on the spherical projection. These normal angles are the supplements of the actual interfacial angles, as has been explained.

The relations between these normal angles, for example in a given zone, is much simpler than those existing between the actual interfacial angles. Thus it is always true that, for a series of faces in the same zone, the normal angle between two end faces is equal to the sum of the angles of faces falling between. Thus (Figs. 62, 65) the normal angle of *ab* (100 ∧ 010) is the sum of *am* (100 ∧ 110), *ms* (110 ∧ 120), and *sb* (120 ∧ 010). This relation holds true in all the systems.

Furthermore, it will be seen that, supposing *aca'* (Fig. 65) a plane of symmetry as in the orthorhombic system, the angle 100 ∧ 110, or *am* (Fig. 62), is half the angle 110 ∧ $\bar{1}$10 (*mm'''*). Similarly 010 ∧ 120 (*bs*) is half the angle 120 ∧ $\bar{1}$20 (*ss'*); again, 100 ∧ 111 (*ae*) is the complement of half the angle 111 ∧ $\bar{1}$11 (*ee'*) and 010 ∧ 111 (*be*) the complement of half the angle 111 ∧ 1$\bar{1}$1 (*ee''*).

Here, as throughout this work, the sign ∧ is used to represent the angle between two faces, usually designated by letters.

42. Use of the Spherical Projection to Exhibit the Symmetry.—The symmetry of any one of the crystalline groups may be readily exhibited by the help of the spherical projection, following the notation introduced by Gadolin (1871, see p. 22).

The axes of binary, trigonal, tetragonal, hexagonal symmetry are represented respectively by the following signs: ⬬▲◆⬢. Further, a plane of symmetry is represented by a full line (zone-circle), while a dotted line indicates that the plane of symmetry is wanting. The position of the crystallographic axes is shown by arrows at the extremities of the lines. The pole of a face in the upper half of the crystal (above the plane of projection) is represented by a cross; one below by a circle. If two like faces fall in a vertical zone a double sign is used, a cross within the circle. Figs. 69, 111, 125, etc., give illustrations.

43. General Relations between Planes in the Same Zone.—It may be demonstrated that if on a crystal two faces P (*hkl*) and R (*pqr*) lie in the same zone, then the following equation must hold good:

$$ua \cos XQ + vb \cos YQ + wc \cos ZQ = 0,$$

where $u = kr - lq$, $v = lp - hr$, $w = hq - kp$.

The letters u, v, w are called the symbol of the zone or great circle PR. Every face (*xyz*) of this zone must satisfy the equation

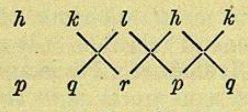
$$ux + vy + wz = 0.$$

If now (uvw) be the symbol of one zone, and (efg) of another intersecting it, then the point of intersection will always be the pole of a possible crystal face. Its indices (*hkl*) must obviously satisfy two equations similar to (1). These indices are hence equal to

$$h = gv - fw, \quad k = ew - gu, \quad l = fu - ev.$$

The application of this principle is extremely simple, and its importance cannot be overestimated.

The zone-symbols can be always obtained by arranging the symbols of the two faces in order, repeating the first two indices and then multiplying according to the following scheme:



Hence $u = kr - lq$; $v = lp - hr$; $w = hq - kp$.

44. Examples of Zones and Zonal Relations.—The following are cases in which the zonal equation is seen at once. In Fig. 62, p. 27, the faces *a* (100), *m* (110), *s* (120), *b* (010), form a vertical zone with mutually parallel intersections, since they are alike in position in so far as this: that they are all parallel to the vertical axis; that is, for all faces in this zone it must be true that $l = 0$.

Again, the faces *a* (100), *d* (101), *c* (001) are in zone, all being parallel to a lateral axis $\bar{b}$; hence for them and all others in this zone $k = 0$. Also *b* (010), *k* (021), *h* (011), *c* (001) are in a zone, all being parallel to the axis $\breve{a}$, so that $h = 0$.

Also the faces *f* (121), *e* (111), *d* (101), *e'''* (1$\bar{1}$1), *f'''* (1$\bar{2}$1) are in a zone, since they have a common ratio for the axes *a* : *c*. With them, obviously, $h = l$.

The faces *c* (001) *e* (111), *m* (110) are also in a zone, and again *c* (001), *f* (121), *s* (120), though intersections do not happen to be made between *c* and *e* in the one case, and *c* and *f* in the other. For each of these zones it is true that there is a common ratio of the lateral axes, that is, of *h* to *k* in the symbols. For the first it may be shown that $h = k$; for the second, that $2h = k$

All the relations named may be obtained at once from the above scheme. For example, for the faces *s* (120) and *f* (121) the scheme gives

$$\begin{array}{ccccc} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 \end{array}$$

$$u = 2, \quad v = \bar{1}, \quad w = 0; \qquad \therefore 2h - k = 0, \text{ or } 2h = k.$$

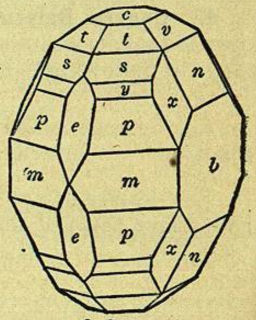
The symbol of a face lying at once in two zones, as stated above, must satisfy the zonal equation of each; these symbols are hence easily obtained either by combining the equations or by a scheme of multiplication like that given above.

For example, in Fig. 66, of sulphur, the face lettered x is in the zone (1) with b (010) and s (113), also in zone (2) with p (111) and n (011). These zones give, respectively:

$$(1)\ \begin{array}{ccccc} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ \hline & 3 & 0 & \bar{1} & \end{array} \qquad (2)\ \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ \hline & 0 & \bar{1} & 1 & \end{array}$$

$$u = 3, \quad v = 0, \quad w = \bar{1}. \qquad e = 0, \quad f = \bar{1}, \quad g = 1.$$

66.



Sulphur.

Hence for (1) the zonal equation is $3h = l$; for (2) $k = l$. Combining these, we obtain $h = 1$, $k = 3$, $l = 3$.

The symbol of the face x is, therefore, 133.

The same result is given by multiplying the zonal indices $0\bar{1}1$, $30\bar{1}$, together after the same method, thus:

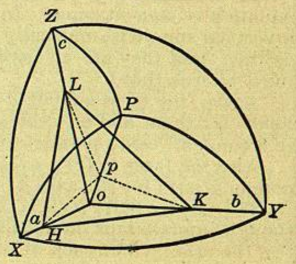
$$\begin{array}{ccccc} 0 & \bar{1} & 1 & 0 & \bar{1} \\ \bar{3} & 0 & 1 & \bar{3} & 0 \\ \hline & 1 & 3 & 3 & \end{array} \qquad \text{Hence, again, } x = 133.$$

This method of calculation belongs to all the different systems. In the hexagonal system, in which there are four indices, one of the three referring to the lateral axes (usually the third) is omitted when the zonal relations are applied. See Art. 160.

45. Methods of Calculation.—In general the angles between the poles can be calculated by the methods of spherical trigonometry from the triangles shown in the sphere of projection (Fig. 65)—which for the most part are right-angled. Certain fundamental relations connect the axes with the elemental angles of the projection; the most important of these are given under the individual systems. Some general relations only are explained here.

46. Relation between the Indices of a Plane and the Angle made by it with the Axes.—When the assumed axes are at right angles to each other they coincide with the normals to the pinacoid faces (100, 010, 001), and consequently meet the spherical surface at their poles. When the axial angles are not 90°, this is no longer true. In all cases, however, the following relation holds good between the cosines of the angles made by a plane, HKL, with the axes:

67.



$$\frac{Op}{OH} = \cos PX; \quad \frac{Op}{OK} = \cos PY; \quad \frac{Op}{OL} = \cos PZ.$$

This is equivalent to

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ.$$

This equation is fundamental, and several of the relations given beyond are deduced from it.

The most useful application is that when the axial angles are 90°; then X, Y, Z are the poles of 100, 010, 001, respectively. Also if the plane HKL is taken as a face of the unit pyramid, that is, if its intercepts on the axes are taken as the unit lengths

$$OH = a, \qquad OK = b, \qquad OL = c.$$

Then the lines HK, HL, KL give also the intersections of the planes 110, 101, 011 on the three axial planes, and their poles are hence at the points fixed by normals to these lines drawn from O. It will be obvious from this figure, then, that the following relations hold true:

$$\tan (100 \wedge 110) = \frac{a}{b};$$

$$\tan (001 \wedge 101) = \frac{c}{a};$$

$$\tan (001 \wedge 011) = \frac{c}{b}.$$

These values are often used later.

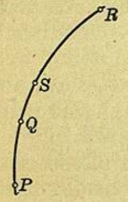
47. Cotangent and Tangent Relations.—If the angles between the poles of three faces in a zone are known, the angle between any one of them and the pole of a fourth face can be calculated by a formula called the *cotangent formula*. Conversely, if the angular position of this fourth face is given, the ratio of its indices can be calculated.

Let P, Q, S, R be the poles of four faces in a zone, taken in such an order * that PQ < PR, and let the indices of these faces be respectively:

P	Q	R	S
hkl	*pqr*	*uvw*	*xyz*

Then it may be proved that

68.



$$\frac{\cot PS - \cot PR}{\cot PQ - \cot PR} = \frac{(P.Q)}{(Q.R)} \cdot \frac{(S.R)}{(P.S)},$$

where

$$\frac{(P.Q)}{(Q.R)} = \frac{kr - lq}{qw - rv} = \frac{lp - hr}{ru - pw} = \frac{hq - kp}{pv - qu},$$

$$\frac{(S.R)}{(P.S)} = \frac{wy - zv}{kz - ly} = \frac{zu - xw}{lx - hz} = \frac{xv - yu}{hy - kx}.$$

If one of these fractions reduces to an indeterminate form, $\frac{0}{0}$, then one of the others must be taken in its place.

This formula is chiefly used in the monoclinic and triclinic systems; and some special cases are referred to under these systems.

The cotangent relation becomes much simplified for a rectangular zone, that is, a zone between a pinacoid and a face in the zone of the other pinacoids at right angles to it. Thus if Pa, Pb, Pc, Qa, Qb, Qc represent respectively the angles between two faces in the same rectangular zone, viz., P (hkl) and Q (pqr) and the pinacoids a (100), b (010), c (010), the following relations hold good:

$$\frac{h}{p} \cdot \frac{\tan Pa}{\tan Qa} = \frac{k}{q} = \frac{l}{r};$$

$$\frac{h}{p} = \frac{k}{q} \cdot \frac{\tan Pb}{\tan Qb} = \frac{l}{r};$$

$$\frac{h}{p} = \frac{k}{q} = \frac{l}{r} \cdot \frac{\tan Pc}{\tan Qc}.$$

* In the application of this principle it is essential that the planes should be taken in the proper order, as shown above; to accomplish this it is often necessary to use the indices and corresponding angles, not of (hkl), but the face opposite ($\bar{h}\bar{k}\bar{l}$), etc.