

109. **Determination of the Axial Ratio, etc.**—The following relations serve to connect the axial ratio, that is, the length of the vertical axis  $c$ , when  $a = 1$ , with the fundamental angles  $(001 \wedge 101)$  and  $(001 \wedge 111)$ :

$$\tan(001 \wedge 101) = c; \quad \tan(001 \wedge 111) \times \frac{1}{2}\sqrt{2} = c.$$

For faces in the same rectangular zone the tangent principle applies. The most important cases (cf. Fig. 171) are:

$$\frac{\tan(001 \wedge h0l)}{\tan(001 \wedge 101)} = \frac{h}{l};$$

$$\frac{\tan(001 \wedge 0kl)}{\tan(001 \wedge 011)} = \frac{k}{l};$$

$$\frac{\tan(001 \wedge hhl)}{\tan(001 \wedge 111)} = \frac{h}{l}.$$

For the prisms

$$\tan(010 \wedge hk0) = \frac{h}{k}, \quad \text{or} \quad \tan(100 \wedge hk0) = \frac{k}{h}.$$

110. **Other Calculations.**—It will be noted that in the spherical projection (Fig. 171) all those spherical triangles are right-angled which are formed by great circles (diameters) which meet the prismatic zone-circle 100, 010, 100, 010. Again, all those formed by great circles drawn between 100 and 100, or 010 and 010, and crossing respectively the zone-circles 100, 001, 100, or 010, 001, 010. Also, all those formed by great circles drawn between 110 and 110 and crossing the zone-circle 110, 001, 110, or between 110 and 110 and crossing the zone-circle 110, 001, 110.

These spherical triangles may hence be readily used to calculate any angles desired; for example, the angles between the pole of any face, as  $hkl$  (say 321), and the pinacoids 100, 010, 001. The terminal angles ( $x$  and  $z$ , Fig. 166) of the ditetragonal pyramid,  $212 \wedge 212$  (or  $313 \wedge 313$ , etc.), and  $212 \wedge 122$  (or  $313 \wedge 133$ , etc.), can also be obtained in the same way. The zonal relations give the symbols of the poles on the zones 001, 100 and 001, 110 for the given case. For example, the zone-circle  $110, 313, 133, 110$  meets  $110, 001, 110$  at the pole 223, and the calculated angle  $313 \wedge 223$  is half the angle  $313 \wedge 133$ . If a large number of similar angles are to be calculated, it is more convenient to use a formula, as that given below.

111. **Formulas.**—It is sometimes convenient to have the normal interfacial angles expressed directly in terms of the axis  $c$  and the indices  $h, k$ , and  $l$ . Thus:

(1) The distances of the pole of any face  $P(hkl)$  from the pinacoids  $a(100) = Pa$ ,  $b(010) = Pb$ ,  $c(001) = Pc$  are given by the following equations:

$$\cos^2 Pa = \frac{h^2 c^2}{h^2 c^2 + k^2 c^2 + l^2}; \quad \cos^2 Pb = \frac{k^2 c^2}{h^2 c^2 + k^2 c^2 + l^2}; \quad \cos^2 Pc = \frac{l^2}{h^2 c^2 + k^2 c^2 + l^2}.$$

These may also be expressed in the form

$$\tan^2 Pa = \frac{k^2 c^2 + l^2}{h^2 c^2}; \quad \tan^2 Pb = \frac{h^2 c^2 + l^2}{k^2 c^2}; \quad \tan^2 Pc = \frac{h^2 c^2 + k^2 c^2}{l^2}.$$

(2) For the distance between the poles of any two faces  $(hkl), (pqr)$ , we have in general

$$\cos PQ = \frac{hpc^2 + kqc^2 + lr}{\sqrt{[(h^2 + k^2)c^2 + l^2][(p^2 + q^2)c^2 + r^2]}}.$$

The above equations take a simpler form for special cases often occurring; for example, for  $hkl$  and the angle of the edge  $y$  of Fig. 166.

112. **Prismatic Angles.**—The angles for the commonly occurring ditetragonal prisms are as follows:

Angle on $a(100, i-i)$	Angle on $m(110, I)$	Angle on $a(100, i-i)$	Angle on $m(110, I)$
410, $i-4$	14° 24'	530, $i-\frac{5}{2}$	30° 57½'
310, $i-3$	18 26	320, $i-\frac{3}{2}$	33 41¼
210, $i-2$	26 34	430, $i-\frac{1}{2}$	36 52¼
	18 26		8 7¾

### III. HEXAGONAL SYSTEM.

113. The **HEXAGONAL SYSTEM** includes all the forms which are referred to four axes, three equal lateral axes in a common plane intersecting at angles of 60°, and a fourth, vertical axis, at right angles to them.

Two sections are here included, each embracing a number of distinct groups related among themselves. They are called the *Hexagonal Division* and the *Trigonal* (or *Rhombohedral*) *Division*. The symmetry of the former, about the vertical axis, belongs to the hexagonal type, that of the latter to the trigonal type.

Miller (1852) referred all the forms of the hexagonal system to three equal axes parallel to the faces of the fundamental rhombohedron, and hence intersecting at equal angles, not 90°. This method (further explained in Art. 163) has the disadvantage of failing to bring out the relationship between the normal hexagonal and tetragonal types, both characterized by a principal axis of symmetry, which (on the system here adopted) is the vertical crystallographic axis. It further gives different symbols to faces which are crystallographically identical. It is more natural to employ the three rhombohedral axes for trigonal forms only, as done by Groth (1894), who includes these groups in a *Trigonal System*; but this also has some disadvantages.

114. **Groups.**—There are five possible groups in the Hexagonal Division. Of these the normal group is much the most important, and two others are also of importance among crystallized minerals.

In the Trigonal Division there are seven groups; of these the rhombohedral group or that of the Calcite Type is by far the most common, and three others are also of importance.

115. **Axes and Symbols.**—The position of the four axes taken is shown in Fig. 193; the three lateral axes are called  $a$ , and the vertical axis is  $c$ . Further, when it is desirable to distinguish between the lateral axes they may be designated  $a_1, a_2, a_3$ . The general position of any plane on the method of Bravais (who adapted the system of Miller to this system) may be expressed in a manner analogous to that applicable in the other systems, viz.:

$$\frac{1}{h}a_1 : \frac{1}{k}a_2 : \frac{1}{i}a_3 : \frac{1}{l}c.$$

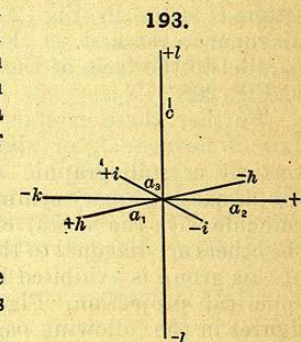
The corresponding indices for a given plane are then  $h, k, i, l$ ; these always refer to the axes named in the above scheme:

It is found convenient to consider the axis  $a_3$  as negative in front and positive behind, hence the general symbol is  $hki\bar{l}$ . Further, as following from the angular relation of the three lateral axes, it can be readily shown to be always true that the algebraic sum of the indices  $h, k, i$ , is equal to zero:

$$h + k + i = 0.$$

The general expression for any plane in accordance with the system of Naumann is

$$na : pa : -a : ma.$$





Here it is always true that  $p = \frac{n}{n-1}$ . The shortened form for the above expression as adopted by Naumann is  $mPn$ .

The relation of Miller's indices to those of Naumann is obvious if for a given plane with the symbol, say,  $2\bar{1}\bar{3}1$  the parameters are given in full, namely:

$$(1) \quad \frac{1}{2}a_1 : 1a_2 : -\frac{1}{3}a_3 : 1c.$$

This is equivalent (after multiplying by 3) to

$$(2) \quad \frac{3}{2}a_1 : 3a_2 : -1a_3 : 3c.$$

Here  $m = 3$ ,  $n = \frac{3}{2}$ , and the value of  $p$  is 3. The symbol is hence written

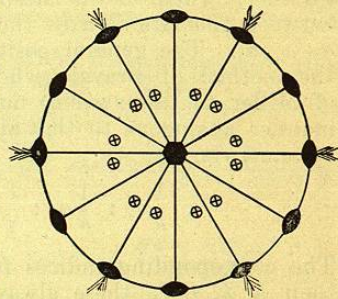
$$3P\frac{3}{2}, \text{ or } 3\frac{3}{2}.$$

**A. Hexagonal Division.**

**1. NORMAL GROUP (13). BERYL TYPE.**

**116. Symmetry.**—Crystals belonging to the normal group of the Hexagonal Division have one principal plane of symmetry, the plane of the lateral axes; also, normal to this and meeting in the vertical axis at angles of 30°, six other planes of symmetry, like three-and-three. Those of one set pass through the lateral axes  $a, a, a$  (Figs. 195, 198) and are hence parallel to the faces of the form  $(10\bar{1}0)$  lettered  $m$ , as in Fig. 208. The others are diagonal to the first set (Figs. 196, 198) and are parallel to the faces of the form  $(11\bar{2}0)$  lettered  $a$ , Fig. 208.

Further, these crystals have one principal axis of hexagonal, or sixfold, symmetry, the vertical crystallographic axis; also six horizontal axes of binary symmetry; three of these coincide with the lateral crystallographic axes, the others are diagonal to them. The symmetry of this group is exhibited in the accompanying spherical projection, Fig. 194, and by the figures in the following pages from 195 to 209.



The analogy between this group and the normal group of the tetragonal system is obvious at once and will be better appreciated as greater familiarity is gained with the individual forms and their combinations.

**117. Forms.**—The possible forms in this group are as follows:

	Miller-Bravais.	Naumann.
1. Base.....	(0001) $\infty a : \infty a : \infty a : c$	$OP$ or $O, c$
2. Unit prism, or prism of the first order	$\left\{ \dots (10\bar{1}0) \right.$ $a : \infty a : -a : \infty c$	$\infty P$ or $I, m$
3. Diagonal prism, or prism of the second order	$\left\{ (11\bar{2}0) \right.$ $2a : 2a : -a : \infty c$	$\infty P2$ or $i-2, a$

	Miller-Bravais.	Naumann.
4. Dihexagonal prism.....	$(hk\bar{i}0)$ $na : pa : -a : \infty c$	$\infty Pn$ or $i-n$
as, $(21\bar{3}0)$	$\frac{2}{3}a : 3a : -a : \infty c$	$\infty P\frac{2}{3}$ or $i-\frac{2}{3}$
5. Pyramids of the unit, or first order	$\left\{ \dots (h0\bar{h}l) \right.$ $a : \infty a : -a : mc$	$mP$ or $m$
as, $(10\bar{1}1)$	$a : \infty a : -a : c$	$P$ or 1; also $20\bar{2}1$ $(a : \infty a : -a : 2c)$ $2P$ or 2
6. Diagonal pyramids, or pyramids of the second order	$\left\{ (h\bar{h} \cdot 2\bar{h} \cdot l) \right.$ $2a : 2a : -a : mc$	$mP2$ or $m-2$
as, $(11\bar{2}2)$	$2a : 2a : -a : c$	$P2$ or 1-2
7. Dihexagonal pyramids, or berylloids	$\left\{ \dots (hk\bar{i}l) \right.$ $na : pa : -a : mc$	$mPn$ or $m-n$
as, $(21\bar{3}1)$	$\frac{2}{3}a : 3a : -a : 3c$	$3P\frac{2}{3}$ or $3-\frac{2}{3}$

In the above  $h > k$ , and  $h + k = -i$ .

**118. Base**—The *base*, or *basal pinacoid*, includes the two faces,  $0001$  and  $000\bar{1}$ , parallel to the plane of the lateral axes. It is uniformly designated by the letter  $c$ ; see Figs. 195 *et seq.*

**119. Prisms. Unit Prism.**—There are three types of prisms, or forms in which the faces are parallel to the vertical axis.

The *unit prism*, or prism of the first order, Fig. 195, includes six faces, each one of which is parallel to the vertical axis and meets two adjacent lateral axes at equal distances, while it is parallel to the third lateral axis. It has hence the general symbol  $(10\bar{1}0)$  and is uniformly designated by the letter  $m$ ; its six faces, taken in order (see Figs. 195 and 209), are:

$$10\bar{1}0, 01\bar{1}0, \bar{1}100, \bar{1}010, 0\bar{1}10, 1\bar{1}00.$$

**120. Diagonal Prism.**—The *diagonal prism*, or prism of the second order, Fig. 196, has six faces, each one of which is parallel to the vertical axis, and meets the three lateral axes, the two alternate at the unit distance, the other at one-half this distance; or, which is the same thing, it meets the last-named axis at the unit distance, the others at double this distance.\* The general symbol is  $(11\bar{2}0)$  and it is uniformly designated by the letter  $a$ ; the six faces (see Figs. 196 and 209) in order are:

$$11\bar{2}0, \bar{1}2\bar{1}0, 2\bar{1}10, \bar{1}\bar{1}20, 1\bar{2}10, 2\bar{1}\bar{1}0.$$

The unit prism and the diagonal prism are not to be distinguished geometrically, each being a regular hexagonal prism with normal interfacial angles of 60°. They are related to each other in the same way as the two square prisms  $m$  (110) and  $a$  (100) of the tetragonal system.

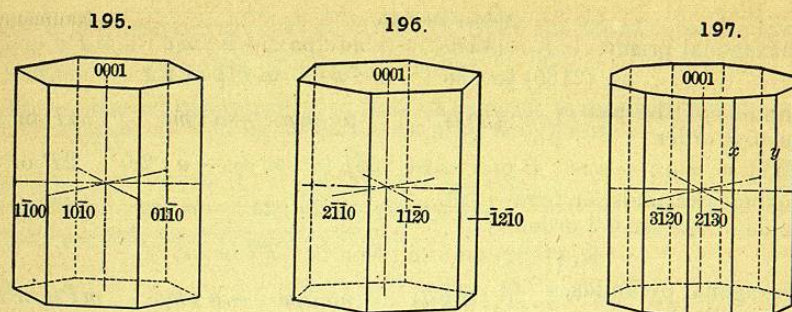
The relation in position between the unit prism (and pyramids) on the one hand and the diagonal prism (and pyramids) on the other will be understood better from Fig. 198, representing a cross-section parallel to the base  $c$ .

**121. Dihexagonal Prism.**—The *dihexagonal prism*, Fig. 197, is a twelve-sided prism bounded by twelve faces, each one of which is parallel to the vertical axis, and also meets two adjacent lateral axes at unequal distances the ratio of which always lies between 1 : 1 and 1 . 2 (see 2 p. 66). This prism has two unlike edges, lettered  $x$  and  $y$ , as shown in Fig. 197. The general symbol is  $(hk\bar{i}0)$ , and the faces of a given form, as  $(21\bar{3}0)$ , are:

$$21\bar{3}0, 12\bar{3}0, \bar{1}3\bar{2}0, 2\bar{3}10, 3\bar{2}10, 3\bar{1}20, \\ \bar{2}130, \bar{1}\bar{2}30, 1\bar{3}20, 2\bar{3}10, 3\bar{2}\bar{1}1, 3\bar{1}\bar{2}0.$$

\* Since  $1a_1 : 1a_2 : -\frac{1}{2}a_3 : \infty c$  is equivalent to  $2a_1 : 2a_2 : -1a_3 : \infty c$ .



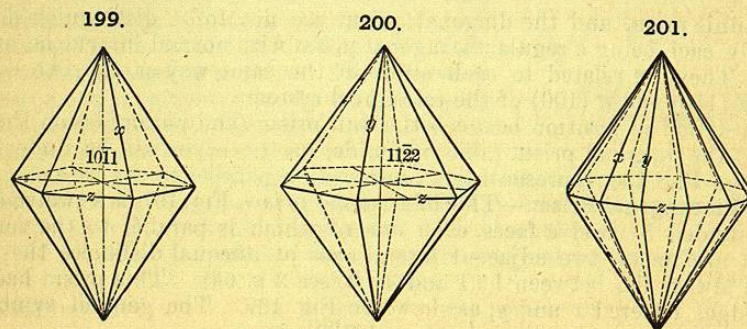


122. **Pyramids. Unit Pyramids.**—Corresponding to the three types of prisms just mentioned, there are three types of pyramids.

A *unit pyramid*, or pyramid of the first order, Fig. 199, is a form bounded by twelve similar triangular faces—six above and six below—which have the same position relative to the lateral axes as the faces of the unit prism, while they also intersect the vertical axis. The general symbol is hence  $(h\ 0\ h\ l)$ . The faces of a given form, as  $(10\bar{1}\ 1)$ , are:

Above  $10\bar{1}\ 1$ ,  $01\bar{1}\ 1$ ,  $\bar{1}10\ 1$ ,  $\bar{1}01\ 1$ ,  $0\bar{1}1\ 1$ ,  $1\bar{1}0\ 1$ .  
Below  $10\bar{1}\ \bar{1}$ ,  $01\bar{1}\ \bar{1}$ ,  $\bar{1}10\ \bar{1}$ ,  $\bar{1}01\ \bar{1}$ ,  $0\bar{1}1\ \bar{1}$ ,  $1\bar{1}0\ \bar{1}$ .

On a given species there may be a number of unit pyramids, differing in the ratio of the lateral to the vertical axis, and thus forming a zone between the base  $(0001)$  and the faces of the unit prism  $(10\bar{1}\ 0)$ . Their symbols, passing from the base  $(0001)$  to the unit prism  $(10\bar{1}\ 0)$ , would be, for example,  $10\bar{1}\ 4$ ,  $10\bar{1}\ 2$ ,  $20\bar{2}\ 3$ ,  $10\bar{1}\ 1$ ,  $30\bar{3}\ 2$ ,  $20\bar{2}\ 1$ , etc. In Fig. 202, the faces  $o$  and  $s$  are unit pyramids and they have the symbols respectively  $(10\bar{1}\ 1)$  and  $(20\bar{2}\ 1)$ , here  $l = 1.014$ . In Fig. 205,  $p$  is the unit pyramid  $(10\bar{1}\ 1)$ ;

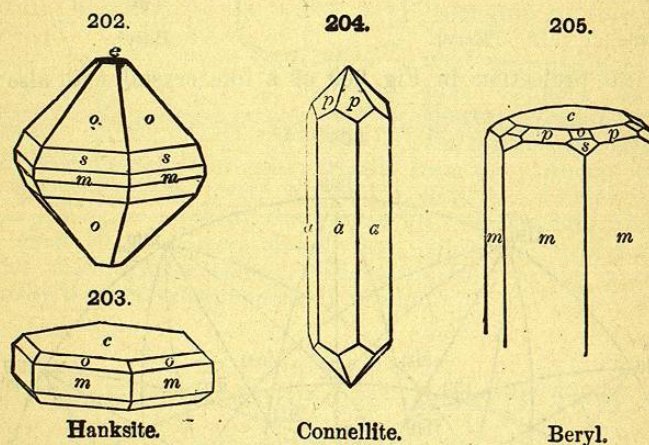


here  $l = 0.50$ . As shown in these cases the faces of the unit pyramids replace the edges of the unit prism. On the other hand, they replace the solid angles of the diagonal prism  $a$   $(1120)$  as shown in Fig. 204.

123. **Diagonal Pyramids.**—The *diagonal pyramid*, or pyramid of the second order (Fig. 200), is a double six-sided pyramid including the twelve similar faces which have the same position relative to the lateral axes as the faces of the diagonal prism, and which also intersect the vertical axis. They have the general symbol  $(h \cdot h \cdot 2h \cdot l)$ . The symbols of the faces of the form  $(11\bar{2}\ 2)$  are:

Above  $11\bar{2}\ 2$ ,  $\bar{1}2\bar{1}\ 2$ ,  $\bar{2}11\ 2$ ,  $\bar{1}\bar{1}2\ 2$ ,  $1\bar{2}1\ 2$ ,  $2\bar{1}\bar{1}\ 2$ .  
Below  $11\bar{2}\ \bar{2}$ ,  $\bar{1}2\bar{1}\ \bar{2}$ ,  $\bar{2}11\ \bar{2}$ ,  $\bar{1}\bar{1}2\ \bar{2}$ ,  $1\bar{2}1\ \bar{2}$ ,  $2\bar{1}\bar{1}\ \bar{2}$ .

The faces of the diagonal pyramid replace the edges between the faces of the diagonal prism and the base. Further, they replace the solid angles of the unit prism  $m$   $(10\bar{1}\ 0)$ . There may be on a single crystal a number of diagonal pyramids forming a zone between the base  $c$   $(0001)$  and the faces of the diagonal prism  $a$   $(11\bar{2}\ 0)$ , as, naming them in order:  $11\bar{2}\ 4$ ,  $11\bar{2}\ 2$ ,  $22\bar{4}\ 3$ ,  $11\bar{2}\ 1$ , etc. In Fig. 205,  $o$ ,  $s$  are the diagonal pyramids  $(11\bar{2}\ 2)$  and  $(11\bar{2}\ 1)$ .



Hanksite.

Connellite.

Beryl.

124. **Dihexagonal Pyramid.**—The *dihexagonal pyramid*, Fig. 201, is a double twelve-sided pyramid, having the twenty-four similar faces embraced under the general symbol  $(h\ k\ i\ l)$ . It is bounded by twenty-four similar faces, each meeting the vertical axis and having a ratio for the intercepts on two adjacent lateral axes between 1:1 and 1:2 (cf. the general symbol (2) given in Art. 115). Thus the form  $(21\bar{3}\ 1)$  includes the following twelve faces in the upper half of the crystal:

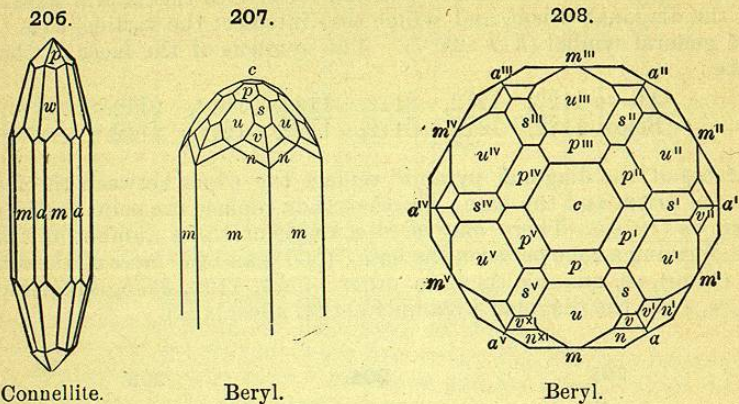
$21\bar{3}\ 1$ ,  $12\bar{3}\ 1$ ,  $\bar{1}3\bar{2}\ 1$ ,  $\bar{2}3\bar{1}\ 1$ ,  $\bar{3}21\ 1$ ,  $\bar{3}12\ 1$ ,  
 $\bar{2}\bar{1}3\ 1$ ,  $\bar{1}\bar{2}3\ 1$ ,  $1\bar{3}2\ 1$ ,  $2\bar{3}1\ 1$ ,  $3\bar{2}1\ 1$ ,  $3\bar{1}2\ 1$ .

And similarly below with  $l$  (here 1) negative,  $21\bar{3}\ \bar{1}$ , etc. The dihexagonal pyramid is often called a *berylloid* because a common form with the species beryl. In Fig. 206,  $w$  is the berylloid  $(11 \cdot 2 \cdot \bar{1}\bar{3} \cdot 3)$ .

125. **Combinations.**—Fig. 207 of beryl shows a combination of the base  $c$   $(0001)$  and prism  $m$   $(10\bar{1}\ 0)$  with the unit pyramids  $p$   $(10\bar{1}\ 1)$  and  $u$   $(20\bar{2}\ 1)$ ; the diagonal pyramid  $s$   $(11\bar{2}\ 1)$  and the berylloids  $v$   $(21\bar{3}\ 1)$  and  $n$   $(31\bar{4}\ 1)$ . Both the last forms lie in a zone between  $m$  and  $s$ , for which it is true that  $k = l$ .



The basal projection of a similar crystal shown in Fig. 208 is very instructive as exhibiting the symmetry of the normal hexagonal group. This is also true

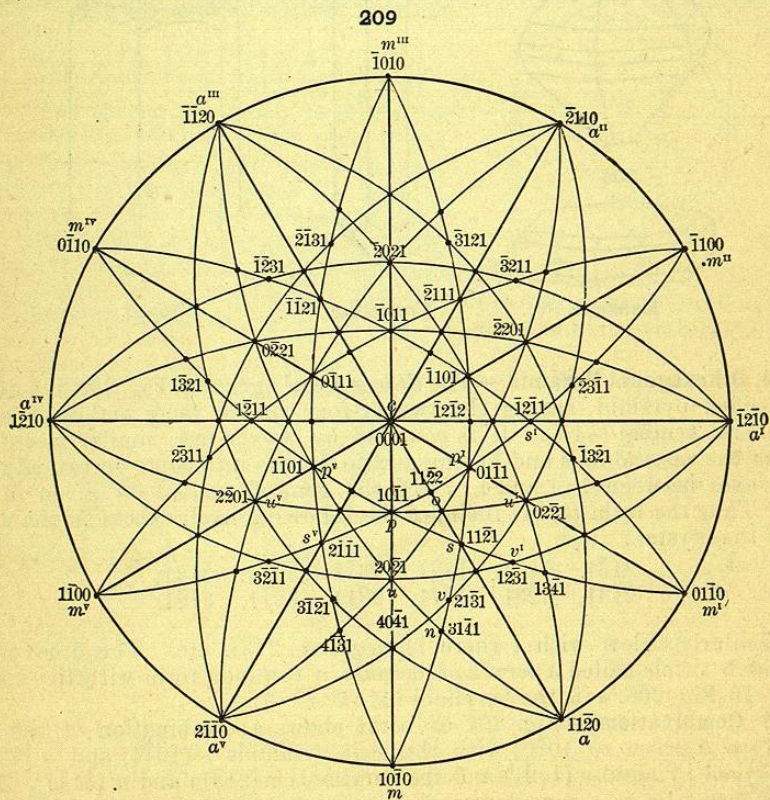


Connellite.

Beryl.

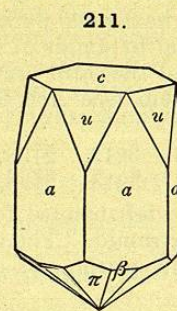
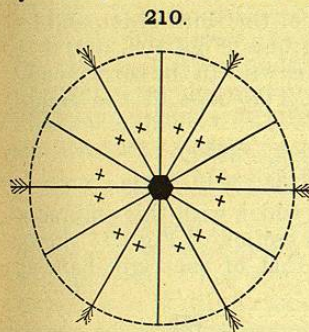
Beryl.

of the spherical projection in Fig. 209 of a like crystal with also the form  $o$  ( $11\bar{2}2$ ).



2. HEMIMORPHIC GROUP (14) IODYRITE TYPE.

126. Symmetry.—This group differs from the normal group only in having no transverse plane of principal symmetry and no horizontal axes of binary symmetry. It has, however, the same two sets of planes of symmetry meeting at angles of  $30^\circ$  in the vertical axis which is an axis of hexagonal symmetry. There is no center of symmetry. The symmetry is exhibited in the spherical projection, Fig. 210.



Iodyrite.

127 Forms.—The forms belonging to this group are the two basal planes,  $0001$  and  $000\bar{1}$ , here distinct forms, the plus (upper) and minus (lower) pyramids of each of the three types; also the three prisms, which last do not differ geometrically from the prisms of the normal group. An example of this group is found in iodyrite, or silver iodide, Fig. 211; here  $u = (40\bar{1}1)$ ,  $\pi = (40\bar{4}5)$ ,  $\beta = (9\cdot9\cdot18\cdot20)$ . Greenockite and wurtzite, also zincite (Fig 50, p. 18) are classed here, but there is some reason for believing that these species belong, with tourmaline, to the corresponding group under the trigonal (rhombohedral) division.

prisms, which last do not differ geometrically from the prisms of the normal group. An example of this group is found in iodyrite, or silver iodide, Fig. 211; here  $u = (40\bar{1}1)$ ,  $\pi = (40\bar{4}5)$ ,  $\beta = (9\cdot9\cdot18\cdot20)$ . Greenockite and wurtzite, also zincite (Fig 50, p. 18) are classed here, but there is some reason for believing that these species belong, with tourmaline, to the corresponding group under the trigonal (rhombohedral) division.

3. PYRAMIDAL GROUP (15). APATITE TYPE.

128. Typical Forms and Symmetry.—This group is important because it includes the common species of the Apatite Group, apatite, pyromorphite, mimetite, vanadinite. The typical form is the hexagonal prism ( $hk\bar{i}0$ ) and the hexagonal pyramid ( $hk\bar{i}l$ ), each designated as of the *third order*. These forms are shown in Figs. 213 and 214. They and the other forms of the group have only one plane of symmetry, the plane of the horizontal axes, and also one axis of hexagonal symmetry (the vertical axis).

The symmetry is exhibited in the spherical projection (Fig. 212). It is seen here, as in the figures of crystals given, that, like the pyramidal group under the tetragonal system, the faces of the general form ( $hk\bar{i}l$ ) present are half those belonging to each sectant, and further that those above and below fall in same vertical zone.

129. Prism and Pyramid of the Third Order.—The prism of the third order (Fig. 213) has six like faces embraced under the general symbol ( $hk\bar{i}0$ ), and the form is a regular hexagonal prism with angles of  $60^\circ$ , not to be distinguished

