# ARCHES.

## CHAPTER I.

### GENERAL PRINCIPLES.

1. **Arches.** — An arch may be considered to be any structure which, under the action of vertical forces, exerts horizontal or inclined forces against its supports or abutments. Such a definition will include not only the roof of two simple rafters, but also the suspension bridge; and we see no objection to so including them. The case of two rafters we need not touch upon: the suspension bridge only comes incidentally within the scope of this part, until we take up the means of stiffening such a structure under a moving and partial load.

2. **Funicular Polygon applied to a Curved Rib.** — Suppose that a curved rib A C E B, Fig. 1, of any material which possesses stiffness, for instance iron, is attached by a pin, on which it can turn freely, to each of the points of support A and B, and has suspended from it certain known weights, represented by $W_1$, $W_2$, &c., at known points. The weight of the curved rib itself is not at present considered. The rib, if flexible, as a cord or chain is flexible, will tend to assume the shape of the funicular, or equilibrium polygon, proper to these weights in their respective positions. If we lay off the load line 2–1, to any scale, space off on it the weights in succession, assume any convenient point 0, draw radiating lines from that point to the

points of division and to the extremities of the load line, and then, starting from A, or any other point in the vertical through that point of support, draw lines, successively parallel to the lines radiating from 0, and limited by the verticals through the weights, one such equilibrium polygon will be found.

This polygon was discussed in Part II., "Bridges," § 2. By moving the point 0 of the stress diagram, the place where the equilibrium polygon strikes the vertical drawn through B will be changed; and, if 0 is horizontally opposite the point which divides the load line into the two supporting forces, the polygon, drawn from A as a point of beginning, will strike B. But 0 may move on a horizontal line, and H will then have any value we please. H is therefore, at present, an unknown quantity; but we will suppose that A K I B is the desired equilibrium polygon for this given case, — an imaginary line, the weights being attached to the arch.

3. **Relation between Equilibrium Polygon and Bending Moments.** — If the rib is made of a rigid material, the tendency to take a shape other than the one to which it was first formed will cause a bending action or moment at different points. Thus, between A and C the rib will flatten somewhat, moving towards the straight line A C, and from C to B it will become slightly more convex. At C, where the rib coincides with the equilibrium polygon, there will be no tendency to bend. The bending moments on either side of a point where the equilibrium polygon crosses the rib will therefore be of contrary kinds or signs. It is necessary to know the value of the bending moments at all points, in order to so design the cross-section of the rib that it shall be able to resist them. The point C is not necessarily the crown of the arch: it happens to come near it in our figure. If the arched rib is free to turn at its supporting points, no bending moments can exist there; if it is jointed or hinged at any place, as, for example, the middle or crown, no bending moment will be found there: the equilibrium polygon must therefore pass through all such points. The rib may be so fastened at A and B that it cannot turn in a vertical plane; and there will then be bending moments at those points, as in the analogous case of a beam fixed at both ends, except for such a distribution of the load as makes the equilibrium polygon coincide with the arch at its ends.



If the rib is hinged at three points, that is, at the ends and middle, the equilibrium polygon is immediately fixed in position by the necessity of passing through these three points, and the problem of finding the stresses in the rib becomes very simple, as will be seen later.

4. **Value of Bending Moment.** — Let us suppose, at first, that the rib of Fig. 1 is jointed, and free to turn at its ends only. The stress diagram, 0 1 2, and the imaginary equilibrium polygon, having been constructed, and the horizontal line H from 0 drawn, it will be seen that this line will divide the load line into two forces, the vertical components of the abutment reactions, as proved in Part II., § 6. The arrows in the figure denote these components; and we will call the vertical ones, analogous to the supporting forces of a beam, $P_1$ and $P_2$, as marked. We have here the usual closed polygon of external forces.

Let an imaginary vertical section be made at D F: from the theorem of moments, as equilibrium exists in this loaded arch, the moments of all the external forces must balance around any point, for instance the point E, where the plane of section cuts the rib. If the sum of the moments around E equals zero, the moments on one side of the plane of section must equal those on the other; and, as E is in the section of the rib, these moments can only neutralize one another through the moment of resistance of the section: consequently, the sum of the moments on either side must equal the bending moment at E. Then at E, if $P_2$ and H are the rectangular components of the reaction at B, and $\Sigma W.L$ denotes the sum of the products of each weight by its horizontal distance L from E, the bending moment will be

$$M = P_2 . DB - \Sigma W . L - H . DE. \quad (1.)$$

If the weights had been attached to the cord, or equilibrium polygon, we should have had, for moments on the right of and about F,

$$P_2 . DB - \Sigma W.L - H.DF. \quad (2.)$$

But a cord, being flexible, can resist no bending moment. As this cord is the equilibrium polygon, there can be no tendency to move or no bending moment at any point of it, and expression (2.) must reduce to zero, or

$$P_2 . DB - \Sigma W.L = H.DF.$$

Substitute this value in (1.), and it becomes

$$M = H.DF - H.DE = H.EF; \quad (3.)$$

which signifies that the bending moment at any point of an arched rib, under any vertical load, is equal to the product of the vertical ordinate from that point to the *proper equilibrium polygon*, multiplied by H from the stress diagram.

5. **Remarks.** — It will be noticed that, to the left of C, D F − D E will change sign, becoming negative, and therefore that the bending moment will change in direction, as stated before. If the rib becomes straight and horizontal, the point E moves up to D, and the bending moment becomes equal to H . D F, which is its value for a beam supported at both ends.

The relation of the equilibrium polygon to the arch, or the fact that the bending moment equals H . E F, as just proved, may be readily explained in another way. Suppose that the arch A′ B′ of Fig. 14 has a single weight placed upon it in a certain position: it will thrust horizontally against the abutments an amount H. Let the equilibrium polygon for this weight, and having the same H, be A F B. The ordinates to this equilibrium polygon will be proportional to the bending moments due to the weight on a beam or truss of span A B; the moments will all be positive, and equal to H . D F. But the thrust H of the arch, which actually carries the weight, acting in the line A′ B′, will exert negative bending moments equal to H . D E at all sections of the arch. The resultant bending moment at any point, when the equilibrium polygon is superimposed on the arch, will be the product of H by the



difference of these two ordinates, or H (D F − D E) = H . E F, at some places negative, and at others positive. Thus we see that, while we have for a given system of weights an equilibrium polygon exactly similar to those treated in Part II., "Bridges," the arch, by reason of its horizontal thrust which causes negative bending moments as above, annuls or cuts off a portion of the area of the equilibrium polygon, and the portion of the ordinate in excess or deficient at any point measures the existing bending moment. It is only necessary that the arch and polygon should have the same value of H. The arch, in its capacity of frame, as it were, carries a portion, more or less, of the forces which would otherwise cause bending moments and shears.

Such an arrangement of weights might be devised, continuously distributed along the rib, that there would be no tendency to change the shape of the arch at any point. The equilibrium polygon, becoming a curve for a continuous load, would then coincide with the centre line of the arch, and we should have what is termed an equilibrated rib. And, on the other hand, a rib can be designed for any given distribution of load, of such a shape as to be in equilibrium. This fact can sometimes be made use of when the load is definite, that is, not a moving load, and we shall refer to it again in the sequel.

6. **Condition to determine H; Invariability of Span.** — It may be noticed that in § 4 we used the term *proper equilibrium polygon*. It has been stated that it is easy to draw, between A and B, any number of funicular polygons, which have their angles on the verticals let fall from the weights, by simply moving the point 0 horizontally in the stress diagram, and thus altering the value of H, the horizontal component of the tension. But the actual rib, under a given system of weights, must have a fixed value of H, and definite bending moments at all points: there is therefore but one funicular polygon which will be the *proper* equilibrium polygon. Some condition must be imposed; and a sufficient one is, that, supposing the points A and B to be fixed in position relatively to one

another, the distance A B, or *the span of the rib, shall be unchanged.* An arch between two unyielding abutments satisfies this condition. If the curve A C is flattened by the pull upon it, or by the bending moments by which it is urged towards the straight line A C, the point C will move a little to the right, while the portion between C and B will become slightly more convex. The movement of the point B, however, with reference to A, must be zero.

7. **Formula for this Condition.** — Consider the arched rib as disconnected from its fixed points of support, but suspended in the air by the forces which were but now the reactions at those points. Equilibrium will still exist. The bending moment H . E F at E, from its effect on the particles at that section, causing an elongation of the fibres on one side and a compression of the fibres on the other side, produces what may be called an exceedingly small angle in the rib, or, better, a *change of inclination*, at E, moving the free end B, so far as this change alone is concerned, a very small distance in a direction perpendicular to a straight line from E to B. The amount of this displacement will depend upon the distance E B, and upon the change of inclination at E, which change has just been shown to depend upon the bending moment H . E F. The amount, B R, of this movement, is greatly exaggerated in the figure. But the horizontal component, or projection, B S, of the displacement, which alone affects the horizontal distance of B from A, will manifestly, from the proportionality of the sides of the two right-angled triangles B R S and E B D, be to B R as D E is to E B, or B S will be proportional to D E.

Perhaps this point may be brought out more plainly if stated algebraically, thus: —

$$BR \text{ varies as } EB \,.\, H \,.\, EF;$$

$$BS = BR \,.\, \frac{DE}{EB}; \text{ therefore,}$$

$$BS \text{ varies as } \frac{EB \,.\, H \,.\, EF \,.\, DE}{EB}, \text{ or as } H \,.\, EF \,.\, DE.$$

Taking all the points in the rib into consideration, we see



that the total horizontal displacement of B from A will be proportional to H . $\Sigma$ E F . D E, if $\Sigma$ is the sign of summation of all of the products E F . D E. As the span A B is to be unchanged, the above quantity must equal zero, and therefore, as H cannot be zero, we have the desired condition reduced to

$$\Sigma EF \,.\, DE = 0. \quad (1.)$$

8. **The Equilibrium Polygon determinate.** — As E F changes sign when the equilibrium polygon crosses the rib, as at C, we arrive at this result for a rib free to turn, or hinged, at its ends, that the *summation of the products* E F . D E *for every point where the equilibrium polygon lies on one side of the rib must equal the summation of the similar products for every point where the polygon lies on the other side.* Only one polygon, manifestly, will satisfy this condition; for, if we draw a new polygon between A and B, we immediately increase one set of E F's and diminish the other. An equilibrium polygon may first be drawn tentatively, ordinates be measured at intervals, and the above products computed. It will then be readily seen whether the polygon should be moved up or down; to move it, change H, and draw again. We can deal thus with a rib of any outline; but, for the regular forms of arches commonly in use, we will show presently how to determine the exact equilibrium polygon without experimental trial.

9. **Deflection of the Rib.** — The vertical component R S, of the displacement B R, manifests itself, since B cannot move, by a slight movement of the rib at E vertically, corresponding to the deflection of a beam under transverse forces.

10. **Another Value for Bending Moment.** — It has been shown that the bending moment at E equals H . E F. If we draw from E a perpendicular, E N, to that side of the equilibrium polygon which passes through F, the side being prolonged if necessary, we shall form a right-angled triangle, similar to one formed in the stress diagram by H, the line parallel to the side of the polygon, and the vertical line. Thus, in Fig. 1, the triangle E F N will be similar to 0 2 5, and we may write the proportion