$$0\text{–}2 : 0\text{–}5 = EF : EN;$$

or, if T denotes the tension 0–2 in the part of the cord which passes though F, we get, upon multiplying extremes and means,

$$H . EF = T . EN; \quad (1.)$$

so that the bending moment at each point is also equal to the product of the tension in the cord by the perpendicular let fall on the cord from the given point; and this is the measure of a *moment*, as shown in mechanics. The discussion of the bending moment might have been approached in this way.

11. **Combined Effect of Bending Moment and Direct Force.**—If a force T acts in the line A K, which, when we consider the curved rib, is an imaginary line, its moment with respect to the rib at E is, then, T . E N. Now, from mechanics, if we analyze the effect of a force T, Fig. 2, at any distance laterally from a point E, we may apply two equal and opposite forces, +T and −T, at this point, which is here the middle of the rib, or what would be, for flexure only, the neutral axis, without destroying the equilibrium. Hence we have at E the direct force +T, producing tension, and the couple T . E N, producing flexure. The enlarged sketches will represent the condition of the rib. The small arrows at E′ denote the magnitude or intensities of the stresses which form the moment of resistance to balance the bending moment, these intensities being taken as uniformly varying, a supposition which is satisfied within the elastic limit; at E″ are shown the stresses on the particles of the section from the direct force; and the combination of the moment and force is represented at E‴, it being understood that these several views represent one and the same section E.

The point of no stress, or the position of the neutral axis, is seen to be shifted from the middle of the section at E′ to one side at E‴; and it will disappear altogether when the arm of the couple or moment becomes sufficiently small, so that the entire section may be under one kind of stress of varying intensity. If we know the form of cross-section of the rib, we

can tell from the location of the equilibrium polygon, by simple inspection, where we shall find both tension and compression, and where only one kind of stress. This matter will be touched upon later: §§ 106–108.

12. **Reversal of Figure; Movement of Rib from Equilibrium Polygon.** — When an arch is under analysis, the figures thus far given will be inverted. Imagine them to be so. All of the forces will then be *reversed*. The polygon which was under tension will be compressed, and its sides will represent struts. It will be in unstable equilibrium, and its relation to vertical forces is not, perhaps, so readily apprehended, by one not acquainted with this subject, as is that of the funicular polygon. For this reason it was thought best to take an inverted arch first. Hereafter the arches will be drawn above the springing line; H becomes the *horizontal thrust* of the rib against its abutments.

The curved rib, between the points A and C, Fig. 1, so long as there is tension along the straight line A C, tends to move towards that line, just as the cord, if drawn towards the arch, returns to its former position; but as soon as the figure is inverted, and C is forced by compression towards A, the arch tends to *move away from the equilibrium polygon.* This fact is true of all points of the rib, and, being borne in mind, will enable one to tell at a glance the kind of moment at each point of the rib. All the bending moments are therefore reversed. Those bending moments which tend to make the arch flatter, or of less curvature, at any point, are called positive; those which tend to make it more convex are called negative.

It may aid in fixing the ideas, to take a piece of small steel wire, bend it into the arc of a circle, and, placing the two ends in two notches upon a board, notice the change of shape arising from a pressure or load imposed on any portion. The movement of the wire will indicate, in a general way, where the equilibrium curve lies in reference to the rib.

13. **Equilibrium Polygon for a Single Load.** — It is now readily seen that the equilibrium polygon for a single, concen-

trated load on an arch is composed of two straight lines which meet on the vertical drawn through the point where the load is imposed. In the case just treated, these lines will start from the two springing points of the arch. The only quantity needful to fix their position will be the distance of their point of intersection vertically from the rib; and the single condition of (1.) § 7, that Σ E F . D E = 0, will determine the unknown quantity. It will be easier to find the effect of a single load at successive points on the arch, and to combine these effects for any possible arrangements and intensities of load, than to treat at once several loads. We shall pursue this method.

14. Direct Force and Shear at a Right Section. — Since an arched rib is often composed of two flanges, and a web or connecting bracing, similar to a girder or truss, we desire, after we have found the bending moments at all points, to find that portion of the vertical force or the shear at each section which must be resisted by the web members. *Shear* was explained in Part II., "Bridges," § 4. In a horizontal beam, carried on two supports, we should have, in Fig. 1, P_2 for the supporting force, and shear on the right of any section between B and W_1; $P_2 - W_1$, or (1–5) — (3–1), for the shear anywhere between W_1 and W_2; $P_2 - W_1 - W_2$, or (3–5) — (4–3), that is — (5–4), between W_2 and W_3; and so on, subtracting each weight from the previous shear or resultant. But in a beam, or a truss with horizontal chords, the other forces, those which oppose the bending moment, are horizontal: here they are not. Supposing the rib to be inverted, the direct thrust, being in the direction of a tangent at the centre line of the rib, has a vertical component which affects the amount of shear to be resisted by the web. In short, the inclined flanges or chords act as braces; and we have, at any section, these chords as well as the web members, among which to distribute the shearing force. The action corresponds with that of the bow in a bowstring girder.

It is not probable that the thrust in the side of the equilibrium polygon will be parallel to the tangent to the curve of the centre line of the rib at a particular section, but this thrust

will be the resultant force at the section. It may then properly be resolved into two rectangular components, one perpendicular to the section, representing the direct force, and the other parallel to the plane of the section, representing the shear. The direct stress, combined with the tension and compression due to bending moment, will be resisted by the flanges or chords, and the shear by the web members, if the rib is so constructed. If the rib is of solid section, like a beam, the separate consideration of shear is generally unnecessary. It will at once be seen that the direct stress at any point of the rib is obtained by projecting the force in that side of the equilibrium polygon which passes near the point upon the tangent to the rib. Thus, in Fig. 1, 0–3 is the tensile force in the side I G of the equilibrium polygon, and 0–6 is drawn parallel to the tangent at U: if a perpendicular were drawn from 3 upon 0–6 prolonged, the distance from 0 to the foot of the perpendicular would be the direct stress, and the perpendicular itself would be the shear on a right section at U. Or, again, if 0–2 is the force in A K, and 0–7 is parallel to the tangent at Q, a perpendicular from 2 on 0–7 will cut off the direct stress, and be itself the shear at Q.

15. Sign of Shear; Maximum Bending Moment at Point of Zero Shear. — The above points may be made more clear, if necessary, by reference to the sketch above and on the left of Fig. 8. Let A C represent a portion of an arch, and A R′ a portion of the equilibrium polygon which exerts a thrust R at A. The components of the abutment reaction will be H, the horizontal thrust, and P_1, the vertical force. But R may also be decomposed, on a right section of the rib *near* A, into T direct thrust and F shear at the section. The little sketch adjoining shows, that, as these components act on the left of the section, we must have the opposite shear on the right of the section, giving what we have been accustomed to call negative shear (see Part II., "Bridges"). When, at any right section, a line parallel to the side of the equilibrium polygon lies above the tangent to the rib, the forces being taken on the left of the section, as is the case at C, where T′ and F′ are the components of R′, the shear will be positive. Where the side of the equilibrium polygon is parallel to the tangent to the rib, as for instance near *d*, at that point there will be no shear, and the shear will be of opposite signs on each side of such point. The direct stress there will be H multiplied by the secant of the inclination of the tangent to the horizon.

As the maximum ordinate between the side of the equilibrium polygon and the arch occurs where the side of the polygon is parallel to the rib, the maximum bending moments in the arch, as in a beam or truss, are found at points of no shear.

16. **Treatment of Arch with Fixed Ends requires Three Conditions.** — If the arched rib is fixed in direction at the ends (in place of being free to turn as previously supposed), by being firmly bolted to the abutments, or by having square ends accurately bedded upon the skewbacks, a bending moment will generally exist at the points of support when the arch is loaded. By taking the piece of easily flexible wire before mentioned, clamping the ends firmly, so as to fix the wire in the position of an arch, and then applying a load or the pressure of the finger, one can easily verify this statement for himself; and he will see that, for many positions of the load, the bending moment at one abutment is of the opposite kind to that at the other. The points at which the equilibrium polygon begins and ends will no longer be A and B of Fig. 1, and some new conditions must be imposed in order to determine these points.

Consider the effect of a single load upon the arched rib A C B of Fig. 3, which rib is fixed in direction at its ends. The equilibrium polygon will be two straight lines, such as I N and N L; and, as there may be bending at both points of support, it will be necessary to find the magnitudes of A I and B L, as well as of N G, three unknown quantities. Three conditions must therefore be satisfied. Such writers as, in treating the arch either graphically or mathematically, require but two conditions to be fulfilled for an arch with fixed ends, err in their

assumptions, and hence in their results. If two conditions only are imposed, where three are necessary, many polygons can be drawn, and the problem is left undetermined.

17. **First Condition.** — One condition which must be satisfied is plainly the one already used, §§ 6 and 7, that the change of span A B shall equal zero, or that

$$\Sigma\, EF.\, DE = 0.$$

18. **Second Condition: Change of Inclination between Abutments equals Zero.** — As the *change of inclination* between any two contiguous points is directly proportional, in direction and magnitude, to the bending moment (for the elongation and compression of the fibres on the two sides, upper and lower, of the rib, result from this bending moment, and cause whatever change of direction or inclination the rib may take on), and as the bending moment has been proved to be proportional simply to the ordinate E F, the change of inclination at any point is proportional to the ordinate E F from that point of the rib to the equilibrium polygon.

The reader must distinguish between the change of inclination produced by flexure, and the original inclination of the rib to the horizon at each point due to the curve to which the rib is constructed. If an arch is loaded, it assumes a form slightly different from its shape when unloaded. The angle, at any particular point, between the two tangents to the curve of the rib, before and after it is loaded, is the *change of inclination* at that point.

Starting from A, Fig. 3, the total change of inclination at C will be proportional to the sum of all the ordinates between A and C. On the other side of C, where the straight line crosses the rib, the bending moment being of the opposite kind, the changes of inclination will be in the opposite direction, and, in any summation of ordinates, for instance from A to E, must be subtracted. Then, as both A and B are fixed in their original directions, if we sum up all of the ordinates E F, from A to B, the total change of inclination between abutments is zero, and this sum must be zero. Therefore the second condition to be realized is that

$$\Sigma \text{E F} = 0;$$

or that *the sum of all the ordinates between the arch and the equilibrium polygon on the inside of the arch must equal the similar sum outside.*

19. **Third Condition: Deflection between Abutments equals Zero.** — Fig. 3 shows that, since the displacement B R of B, relatively to the point E, in case B could move, has been proved, by § 7, to be proportional to H . E F . E B, the vertical component of this displacement varies as H . E F . D B; for, by a similar proportion to the one used in that section,

$$\text{S R} = \text{B R} \frac{\text{D B}}{\text{E B}}; \text{ therefore,}$$

$$\text{S R varies as } \frac{\text{EB . H . EF . DB}}{\text{EB}}, \text{ or as H . E F . D B.}$$

If the products E F . D B should be summed up for all points from A to Q, for example, we should get a quantity proportional to the vertical displacement of Q, arising from the separate minute displacements between A and Q. If we pass beyond C, we have products of an opposite sign; and it then appears, that, since the ends at A and B are fixed both in position and direction, the sum of all the products between A and B must equal zero, or, since H cannot equal zero,

$$\Sigma \text{E F . D B} = 0. \quad (1.)$$

Therefore the third and last condition is, that the *sum of the products of each ordinate, between the arch and the equilibrium polygon on the inside of the arch, by its distance from one springing point, must equal the similar sum on the outside.* It is immaterial which springing is chosen, but all the D B's must be measured to the same abutment.

20. **This Condition not applicable to Hinged Rib.** — It may be expedient to dwell upon this equation a little longer; for the question will apparently arise, why this condition is not also properly applicable to an arch which is jointed or hinged at the ends. Let a tangent A K be drawn to the rib at the point A, and a vertical line be dropped from it to the point Q. If the arch is now bent at the point E′, by a bending moment which is proportional to E′ F, the point Q is moved a distance proportional to E′ F multiplied by the distance from E′ to Q; but the distance which Q moves in the vertical line Q K will be proportional to E′ F multiplied by the horizontal projection of E′ Q, or D T, and similarly for moments at all other points between A and Q. As the tangent at A is fixed in direction in this case, the movement of Q away from the extremity of K Q, or its movement in relation to the tangent at A, will be proportional to the summation of the E F's multiplied by the D T's; and as the abutment B is fixed, the distance of B from a tangent at A must be unchanged by any load, or its displacement must be zero, as above. In the case of the rib hinged at the ends, while the above area moments give the deflection from the tangent at A, this tangent is not fixed, but changes in direction upon the imposition of a load, and this condition cannot be applied. If, however, one should treat an arch which was fixed at A and hinged at B, this condition would be necessary, and all the distances D B would be measured to the hinged end; while the second condition would not apply, and would not be needed.

This third condition was first applied to the determination of the bending moments in continuous bridges and pivot draw spans, in the first edition of Part II. of this work.

21. **Remarks: Abutment Reactions; Shear,** &c. — The arch of Fig. 3 is cut by the equilibrium polygon in three places, and it may be cut in four points, giving as many places of contraflexure. The areas on opposite sides of the rib represent bending moments of opposite kinds, and of which kind is readily known if one remembers that the arch under thrust always moves from the equilibrium polygon. The amount of the weight, not being contained in any of the equations of condition, does not affect the diagram; for H is constant for all points of the arch for any given vertical load, and, not being

equal to zero, is thrown out of the equations. But the weight W does affect the value of H.

If 1–2 represents W in the stress diagram of Fig. 3, and 1–0 and 2–0 are drawn parallel to N I and N L, 0–3 drawn horizontally will determine the horizontal thrust H, while the load-line will be divided at 3 into the two vertical components P_1 and P_2 of the reactions as marked. These vertical forces are not the same as would be obtained for the case previously considered, nor for a beam only supported at the ends. Such forces would be equal to the divisions of 1–2 made by a line drawn through 0, parallel to a line from I to L. If we notice the arrows drawn at the abutment A, we see that, supposing P_1 were at first the fraction of W due to the position of G, or $\frac{G\,B}{A\,B}$ W, we have also at A, besides the horizontal thrust H, a couple H . A I. There is another couple at the other abutment, which may be of the same or opposite kind; their algebraic sum can only be balanced by vertical forces at the two abutments acting with a lever arm of the span; and these vertical forces must be added to one reaction, and subtracted from the other, bringing P_1 and P_2 to the amounts found by the stress diagram. The effect of the couple is the same as if P_1 had been calculated for the point where N I would meet the horizontal line. This is another example of the principle in mechanics cited in § 11.

The remarks on shear in §§ 14, 15, apply equally well here. The direct compression in the rib at any point is obtained, as before, by drawing a line through 0 parallel to the tangent to the rib at the point in question, and dropping a perpendicular upon it from the extremity of the line which represents the stress in the adjacent side of the equilibrium polygon. Thus the compression at E will be the distance from 0 along 0–4 produced to the foot of a perpendicular from 2. Recalling the three conditions just stated, it will be evident, that, while it will be possible to adjust the two lines of the equilibrium polygon to their proper position by successive trials, it will not, as in the former case, be easy. The three ordinates, A I, G N, and B L,

can, however, be computed quite readily, and the remainder of the process is very simple. The statements so far made apply to a structure of any outline, so long as it acts as an arch, although some modification will be called for when the cross-section and the depth vary very much, or when what is known as the moment of inertia is not practically constant; but, for forms other than regular curves, the application of these conditions must probably be made by trial.

21*a*. **Shear at a Vertical Section.**—The relation of the equilibrium polygon to the arch which was pointed out in § 5, Fig. 14, shows how the shear at any vertical section of a loaded rib is affected by the curvature of the arch. In the same way that the ordinates of the rib may be superimposed on those of the triangle which represents the equilibrium polygon for a single load, the two shear diagrams may be placed on one another. One will have the form of *a i m n l*, Fig. 8, conforming to the load which gives the curve of Fig. 14, and found from the amount of vertical reaction which, combined with H, will give a direct thrust at the springing; the other will resemble *a d e f g l*, Fig. 8, the usual shear diagram for a single load, which load produces the triangle of Fig. 14. The flanges of the arch take up at each point an amount equal to the ordinates from *a l* to *i n*, and the web or bracing carries the remainder, which will be positive at some points and negative at others, as marked in the Figure. Thus we see that, through the direct thrust, the arch is relieved of a portion of the *truss stresses* due to both bending moment and shear.