CHAPTER II.

ARCH HINGED AT THREE POINTS.

22. **Three-hinged Arch.** — Before taking up for treatment any arches of special curves, we will notice the simple case of a rib, of any form, hinged at both ends and the middle, or, as it is sometimes called, the "three-hinged arch." The three hinges or joints may be located anywhere, and two of them may be placed near together at one abutment, reducing the portion of arch between them to a short link or strut, which necessarily lies in the direction of the thrust at that abutment. For the ribs of this chapter it has been stated that the equilibrium polygon or curve is at once definitely located. If a single load is placed at K, on the arch A D B of Fig. 4, hinged at A, D, and B, one of the two straight lines composing the polygon must, starting from A, pass through D, while the other, starting from B, must meet the former on the vertical line drawn through K, as required by the principle of the funicular polygon: A C B, therefore, is the polygon. If 2–1 represents the weight at K, and 2–0 and 1–0 are drawn parallel to C B and A C, 0–3, drawn horizontally, will give the horizontal thrust, while 1–3 and 3–2 will be the vertical components of the reactions at A and B. Let it be remembered that the total reaction of the abutment at A is, and is in the direction of, 1–0, although it is often convenient to decompose it into P_1 and H.

A load vertically below E will, similarly, have for its equilibrium polygon A E B. For different positions of the weight

between D and B, all of the vertices of the polygons will be found on the straight line D L, and the portion A D does not change for any movement of the weight on the right half of the arch. A weight on the left half will simply reverse the diagram. The dotted lines show the equilibrium polygons for a weight at such successive points as divide the half-span into five equal horizontal parts, and the corresponding changes in the value of H will be seen in the stress diagram on the left.

23. **Formula for H.** — If F D, the height or rise of the arch, is denoted by k, the half-span A F, = F B, by c, and the horizontal distance F G, from the weight to the middle of the span, by b, we shall have $A G = c + b$, and $G B = c - b$. From the similarity of triangles A D F and 0 1 3, we may write,

$$3\text{–}0 : 3\text{–}1 = c : k, \quad \text{or} \quad H : P_1 = c : k.$$

By the usual rule,

$$P_1 = \frac{c - b}{2c} W;$$

therefore

$$H = \frac{c - b}{2k} W.$$

The quantity $c - b$ is to be understood to mean the horizontal distance from the weight to the nearer abutment. H is seen to decrease regularly as the weight moves from the middle of the span.

24. **Stone Arches.** — In the treatment of stone arches it has often been assumed by writers that the equilibrium curve passed through either the middle of the depth of the keystone or some other arbitrary point within the middle third of its depth; and a similar assumption would then be made for the springing-points. Such a treatment immediately reduces the stone arch to this case, and the equilibrium curve can at once be drawn. As such an assumption does not seem to be warranted, it is not thought expedient to go into the case of the stone arch until later (Chap. IX.); but the reader who desires to look up such a mode of handling the problem is referred to a paper by William Bell, in the Transactions of the Institute of Civil Engineers of

Great Britain, vol. xxxiii., reprinted in Van Nostrand's "Engineering Magazine," vol. viii., March to May, 1873.

25. **Example.** — We will, as an example, show how to draw an equilibrium curve for an arch which is loaded uniformly along its rib. Such a distribution will conform quite well to that of the steady load on an arched roof. For definiteness, let the pointed arch of Fig. 5 be of 80 feet span, 40 feet rise, the two arcs having a radius of 60 feet, and let it be loaded with 500 pounds per foot of the rib. We may, if we please, divide the rib into a convenient number of equal portions, which divisions will give us a number of equal weights to be laid off on the load line. Otherwise we may space off a number of equal horizontal distances. In either case, the load of each space will be considered as concentrated at its centre of gravity; and, if the spaces are small enough, the centre of gravity may, without sensible error, be taken as coinciding with the middle of each space. For the sake of reducing the number of lines, so as to avoid confusion in a small figure, we have divided the half-span into four parts, of ten feet each, measured horizontally; and their centres of gravity will be assumed to be at five feet, fifteen feet, &c., from the point of support. Draw verticals through these centres of gravity, D, E, F, and G.

To find the weight on each division: The lengths of the several portions of arc may, with sufficient exactness, be considered the same as the lengths of their chords, which chords are perpendicular to the radii which pass through D, E, &c. If, then, the load on ten feet is 5,000 lbs., draw $a\,b$ horizontally and equal, by any scale, to this amount; then will $b\,g$, $b\,f$, $b\,e$, and $b\,d$, drawn parallel to the respective chords, give the amount of load on each division, at the successive points G, F, E, &c. Upon scaling these amounts we will lay them off upon a vertical line, from 1 to 5. In order to cause the equilibrium polygon to separate from the rib sufficiently to be easily seen in this small figure, we have taken the liberty of doubling the load on D, thus making it 4–6, in place of 4–5. The loads will therefore be, successively, about 5,400 lbs., 5,900 lbs., 7,000 lbs., and

2 × 10,000 lbs., or 20,000 lbs., from G to D, and from 1 to 6. Since $H = \Sigma \frac{c - b}{2\,k} W$, we have for its value

$$\tfrac{1}{2}\,H = \frac{35 \times 5{,}400 + 25 \times 5{,}900 + 15 \times 7{,}000 + 5 \times 20{,}000}{80} = 6{,}769 \text{ lbs.}$$

If the given load were unsymmetrical with regard to a vertical through C, it would be necessary to calculate the two vertical components of the reactions at A and B, or P_1 and P_2, the reaction at B being laid off from that end of the load line from which was measured the load nearest to B, and then to draw a horizontal line from the point of division between P_1 and P_2, on which to lay off the value of H. But, if both sides of the roof are loaded alike, half a diagram and half an equilibrium polygon will be sufficient. The load on the half-arch being 1–6, 6–1 will be the vertical component of the reaction at B, and H will be laid off in the direction 1–0. Since we have calculated H for only one-half of the entire load, the above quantity must be doubled, and the total horizontal thrust will be 13,538 lbs., = 1–0. The reaction at B is therefore 6–0.

Nothing remains but to draw, first a line from B to the vertical through D, parallel to 6–0, then one, parallel to 4–0, from the end of the last line to the vertical through E, and so on, the last line, parallel to 1–0, passing through the hinge at C, as required. The polygon on the side C A will be exactly similar. It is well to have the points of division quite numerous. The maximum ordinate between the rib and the equilibrium polygon, multiplied by H, gives the maximum bending moment.

26. **Caution.** — As this is the first example, it may be well to pause here, and renew the caution to the draughtsman to lay off the polygon of external forces in the order in which the forces are found in going round the arch or truss; otherwise he will fail to make his equilibrium polygon close on the desired point. Thus, beginning at G, he should have the weights at G, F, E, &c., or 1–2, 2–3, 3–4, &c., plotted, one after the other, down the vertical load line in the direction of their action, until the point

B is reached, for which he draws 6–0, from 6 to 0. Then the point A gives a similar line from 0, slanting upwards toward the right; and the remaining loads on the left half of the arch come down a vertical line, and close on 1, the starting-point. The decomposition of 6–0 into 6–1 and 1–0 does not alter the case. If we had gone round the arch in the opposite direction, this stress diagram would have been reversed, or turned 180°.

27. **Relation between Equilibrium Polygon and Curve.** — The true equilibrium curve, for the load uniformly distributed along the rib, is a curve which will be *tangent* to the sides of the funicular or equilibrium polygon just drawn. The closer together the points D, E, &c., are taken, the nearer the two will come together. If the points at which the loads are concentrated divide the span into equal portions, that is, if the end distances are the same as the others, so that the portions of load near B and C are concentrated on those points, or, even with unequal spacing, when the load between each two assumed points is carried by those points as required by the principle of the lever, the true equilibrium curve will pass through the *vertices* of the equilibrium polygon. Such a distribution of load is made in roofs and bridge trusses, when a half panel weight is thrown on each abutment. Compare Part II., "Bridges," § 58.

The curve assumed by a rope or chain, of uniform weight per foot, when suspended between two points, is called a *catenary.* Since the equilibrium curve in Fig. 5, if we had not placed the extra weight on D, would have come quite near to the rib, it would have been a close approximation to a catenary. As we expect to make some use of this curve later, we will show how to draw one at that time.

28. **The Parabola the Equilibrium Curve for a Load Uniform horizontally.** — If the load on this arch were distributed uniformly horizontally, the curve of equilibrium would be a parabola. In case the whole arch were a parabola, with the vertex at the crown, and the load extended over the entire span, the two curves, coinciding at the springing-points and crown,

would be identical throughout, and the rib itself would be in perfect equilibrium. This same point was brought out in reference to the parabolic girder, Part II., "Bridges," § 73. That the parabola is the equilibrium curve for a continuous load, distributed uniformly horizontally, may be shown as follows: —

Let A B, Fig. 6, be a portion of a cord, horizontal at A, which is in equilibrium under such a uniform load, represented by A C, suspended from the cord. The tension at A will be in the line of the tangent A C; the resultant of the load A C will be vertical, and must pass through its middle point D. As the cord A B is in *equilibrium* under its load and the reactions or tensions of the other portions of the cord at A and B, the tension along the tangent at B must, by the principle of the triangle of forces, also pass through D. As B C, drawn vertically, is parallel to the resultant of the load, the sides of the triangle B C D will be proportional to the three external forces; and, if $AC = x$, $BC = y$, $W =$ total load on A B, $= wx$ (where $w =$ load per unit of length), and $H =$ tension at A, we have

$$W : H = BC : DC = y : \tfrac{1}{2}x,$$

or

$$y = \frac{Wx}{2H} = \frac{w}{2H}x^2,$$

the equation of a parabola with vertex at A.

Therefore an arched rib of parabolic form, when loaded uniformly horizontally, has no tendency to change its shape, that is, experiences no bending moment, at any point.

29. **Suspension Bridge.** — A B of Fig. 6 may represent a suspension bridge cable, A C being the half-span, and C B the height of the tower: hence, if $AC = c$ and $CB = k$, we have for the tension in the cable at the mid-span, § 28,

$$H = \frac{wx^2}{2y} = \frac{wc^2}{2k}.$$

The tension T at the tower will then be proportioned to H, as B D to D C, or as $\sqrt{k^2 + \frac{1}{4}c^2}$ to $\frac{1}{2}c$; therefore

$$T = \frac{wc}{2k}\sqrt{4k^2 + c^2}.$$

Each suspending rod must carry the greatest weight that can come at its foot. The pressure on the top of the tower from the half-span will be the weight of the half-span, or $w c$; to this must be added the vertical component of the tension on the anchorage side of the tower. If the cable has the same inclination *both ways*, at the top of the tower, the pressure is $2 w c$.

The manner of stiffening a suspension bridge to resist the tendency to distortion under a partial load is treated in Chap. X.

30. **Equilibrium Curve for Partial Load.** — If the load extends over a portion only of the span of the arch, and is uniformly distributed horizontally, the curve for the loaded portion is parabolic, while that for an unloaded portion is a straight line: thus, if the load extends from one abutment to the middle, we shall have, on the unloaded half, a straight line from the abutment to the crown, and, on the loaded half, a parabola from the crown to the springing. As it was proved in Part II., "Bridges," § 10, that any two sides of the funicular polygon, when prolonged, meet on the vertical drawn through the centre of gravity of so much of the weight as is included between these sides, the equilibrium curves for any cases where the rib is hinged at three points can be drawn without previously determining the value of H. Thus, in the case just supposed, of a load over the half-span, from B to F in Fig. 4, the centre of gravity will be at G. Then, if G C is the vertical drawn from G, the side of the funicular polygon, or, more properly, the tangent to the equilibrium curve, at B, must pass through C, where C G meets A D, and the required parabola will be drawn from D to B on D C and B C as tangents. As one point of the curve we have the middle point of a line from C to the middle of the chord D B. We can then find H by drawing 1–0 and 2–0, parallel to A C and C B. Henck's "Field Book for Railroad Engineers" gives methods for constructing parabolas; two constructions are given in Part II., "Bridges," §§ 20 and 28, one of them applying when two tangents are given.

31. **Suggested Examples.** — We would suggest the following examples for practice: 1st, Given a semicircular rib, loaded uniformly horizontally over the whole span, and pivoted at the crown and springings: find that the maximum bending moment occurs at 30° from the springing, and is equal to one-sixteenth of the total load multiplied by the radius of the arch, while H is equal to one-fourth of the total load. 2d, Given a parabolic arch similarly pivoted, and in equilibrium under a steady load distributed as above; add a similar travelling load from one abutment to the middle of the span: prove that the maximum bending moment is found at one-fourth of the span from either abutment, is of opposite signs at these two places, and is equal to one thirty-second of the travelling load then on the arch multiplied by the span, while H for the travelling load equals the same product divided by one-fourth the rise of the arch, and for the steady load is twice as much.

32. **Extent of Load to produce Maximum Bending Moment.** — It may be desired, when designing an arch of this type, to find the extent of load which will produce the maximum bending moment at each point, and the value of that moment. Suppose the point N, Fig. 4, to be examined: prolong B N until it meets A D at E; it is then manifest that any load in the vertical through E will cause no bending moment at N; that the equilibrium polygon for any load on the right of E will pass outside of the arch at N, while the equilibrium polygon for any load to the left of E will pass inside of N. Therefore the maximum bending moment at N of one kind will be found when all possible loads are put on the arch from B to the vertical through E, and the maximum moment of the other kind occurs when the load extends from A to E. As the arch tends to move away from the equilibrium polygon, the kind of moment is easily distinguished. H can then be found, the equilibrium curve drawn, the ordinate scaled and multiplied by H.

33. **Braced Arch.** — For the reason that the equilibrium curve is at once definitely located by introducing three hinges or pivots, no matter what form the arch may have, that type which used to be known as the braced arch, having a horizontal

upper and a curved lower member, the spandrel being filled with bracing, has usually been treated as free to turn at both crown and springings; in that case a diagram may be drawn by Clerk Maxwell's method, as set forth in Part I., "Roofs," or the stresses may be found from the equilibrium curve. A braced arch, hinged at crown and springings, with an elliptical lower and a straight upper member, carries a track of the Pennsylvania Railroad over Thirtieth Street, Philadelphia. (See "Engineering," July 22, 1870.) While a diagram only gives the stresses in the various members for one position of load at a time, one can determine all the maximum stresses by two diagrams and a tabulation, not difficult to one familiar with such methods. The way to be pursued will be found in Du Bois' "Graphical Statics," appendix, § 7, p. 350. We will explain another treatment in Chap. XII.

34. **Shear; Temperature.** — Since it is not practicable to draw a shear diagram until the form of the rib is defined, we can only, at present, refer the reader to § 14. After we have discussed the parabolic and circular ribs, the reader can doubtless work up any special design of the present class for himself.

One advantage possessed by this type of arch is that changes of temperature have no straining effect, for the crown rises and falls without affecting the two halves of the arch injuriously. If the crown sinks a little, the value of H will be seen from Fig. 4 to be very slightly increased, while the equilibrium polygon will practically go with the arch.

CHAPTER III.

INTRODUCTORY TO PARABOLIC ARCHES.

35. **Parabolic Arch.** — We propose to apply the facts which have been developed thus far to the arch whose centre line is a parabola. This curve is chosen as one form; because it is, as proved in § 28, in perfect equilibrium under a load distributed uniformly horizontally over the entire span. As in the case of a suspension bridge, so in some arches of iron, most of the steady load consists of a platform and such other parts as are distributed in accordance with this requirement (the arch itself and the vertical posts which carry the platform giving a somewhat greater intensity per horizontal foot as we approach the springings), so that, for the former portion, as well as for the travelling load over the whole span, the arch will be subjected to no bending moments, and no shear; hence there will be no stress in the bracing. Then, again, the parabola for a given rise and span is easily plotted and designed; and, lastly, the determination of the equilibrium curves, for the cases taken up, will be simpler than for circular arcs, and will naturally prepare the way by rendering the reader familiar with the steps of the analysis. It may be well to add here that a circular segmental rib, whose rise is not more than one-tenth of its span, is so nearly coincident with a parabolic arch of the same span and rise, that the investigations which follow will apply with sufficient accuracy to such flat segmental ribs.

36. **Vertical Deflection of an Inclined Beam.** — Let us