upper and a curved lower member, the spandrel being filled with bracing, has usually been treated as free to turn at both crown and springings; in that case a diagram may be drawn by Clerk Maxwell's method, as set forth in Part I., "Roofs," or the stresses may be found from the equilibrium curve. A braced arch, hinged at crown and springings, with an elliptical lower and a straight upper member, carries a track of the Pennsylvania Railroad over Thirtieth Street, Philadelphia. (See "Engineering," July 22, 1870.) While a diagram only gives the stresses in the various members for one position of load at a time, one can determine all the maximum stresses by two diagrams and a tabulation, not difficult to one familiar with such methods. The way to be pursued will be found in Du Bois' "Graphical Statics," appendix, § 7, p. 350. We will explain another treatment in Chap. XII.

34. **Shear; Temperature.** — Since it is not practicable to draw a shear diagram until the form of the rib is defined, we can only, at present, refer the reader to § 14. After we have discussed the parabolic and circular ribs, the reader can doubtless work up any special design of the present class for himself.

One advantage possessed by this type of arch is that changes of temperature have no straining effect, for the crown rises and falls without affecting the two halves of the arch injuriously. If the crown sinks a little, the value of H will be seen from Fig. 4 to be very slightly increased, while the equilibrium polygon will practically go with the arch.

CHAPTER III.

INTRODUCTORY TO PARABOLIC ARCHES.

35. **Parabolic Arch.** — We propose to apply the facts which have been developed thus far to the arch whose centre line is a parabola. This curve is chosen as one form; because it is, as proved in § 28, in perfect equilibrium under a load distributed uniformly horizontally over the entire span. As in the case of a suspension bridge, so in some arches of iron, most of the steady load consists of a platform and such other parts as are distributed in accordance with this requirement (the arch itself and the vertical posts which carry the platform giving a somewhat greater intensity per horizontal foot as we approach the springings), so that, for the former portion, as well as for the travelling load over the whole span, the arch will be subjected to no bending moments, and no shear; hence there will be no stress in the bracing. Then, again, the parabola for a given rise and span is easily plotted and designed; and, lastly, the determination of the equilibrium curves, for the cases taken up, will be simpler than for circular arcs, and will naturally prepare the way by rendering the reader familiar with the steps of the analysis. It may be well to add here that a circular segmental rib, whose rise is not more than one-tenth of its span, is so nearly coincident with a parabolic arch of the same span and rise, that the investigations which follow will apply with sufficient accuracy to such flat segmental ribs.

36. **Vertical Deflection of an Inclined Beam.** — Let us

consider the two cases of a horizontal beam and of one inclined to the horizon at an angle i; it is known from the usual formulæ for deflection, Part II., "Bridges," Chap. VI., that, other things being equal, the deflection of a beam is directly proportional to the load and the cube of the length. If, then, the inclined beam is of a length l, and the horizontal one of a length $l \cos i$, as shown in Fig. 7, the deflection of each, measured *perpendicularly* to the respective beams, will, as regards length only, be in the ratio of l^3 to $l^3 \cos^3 i$. But, if each carries the same load W, the *transverse* component of W, which alone causes flexure of the inclined beam, the longitudinal component producing direct compression, will be $W \cos i$; whence the deflection perpendicular to each beam will, for similar points, be proportioned as 1 to $\cos^2 i$. And, again, the vertical component of the deflection of the inclined beam will be to the perpendicular amount as $\cos i$ to 1; whence the *vertical* deflection of the inclined beam will be to that of the horizontal beam of the same cross-section as 1 to $\cos i$. As the stiffness of a beam is directly proportioned to its breadth, should the inclined beam be made broader in its horizontal dimension than is the horizontal beam, in the ratio of 1 to $\cos i$, the depth being unchanged, the *vertical* deflections of the two beams for the same load would be exactly the same.

37. **Application to Arches.** — Any very small portion of an arch, taken within such narrow limits as to be considered straight, behaves like the inclined beam, as regards its flexure under a load; and therefore it follows, that if an arch has the dimension perpendicular to its face increased, from the crown to the springing, in the ratio of the secant of the inclination to the horizon, it may be discussed as if it were a beam of uniform cross-section, of the same span, similarly supported, and carrying the same load which produces flexure. In the arch some of the load does not produce flexure; in the parabolic rib, for instance, before cited, a uniform horizontal load gives equilibrium. We propose, in our analysis of the parabolic rib, to make this supposition, that the rib is broader at the abutments than at the crown in the ratio just mentioned, and thus to simplify the work of investigation. Iron arches whose flanges or chords are thicker, as we approach the springing, in the above ratio, while the perpendicular depth between the two flanges is constant, practically satisfy this case. In this class of ribs the intensity of the direct thrust on the square inch for a complete uniform load will be the same at all cross-sections.

As we desire the reader to reproduce, on a much larger scale, the figures and problems for himself, we remind him that points on the curve of a parabolic rib are easily found by the construction of Fig. 8, Part II., "Bridges."

PARABOLIC RIB, HINGED AT ENDS.

38. **Equilibrium Polygon for Single Load.** — Taking up the case of the parabolic rib, hinged at the ends only, let us place a single weight at the point I, Fig. 8. If the lines A C B fulfil the condition of § 7, that the sum of the products of the ordinates D E and E F for all points of the arch equals zero or

$$\Sigma EF \cdot DE = 0,$$

A C B will be the required equilibrium polygon. From the reasoning of § 37, it will be proper to divide the areas above the springing line A B by *equidistant* vertical lines, moderately near together, scale off the quantities corresponding to E F and D E, and find the proper position of A C B by one or two trials. It can thus be located with all desirable accuracy, as a slight movement of the point C vertically alters the quantities to be computed very materially. The reader who is not familiar with the higher mathematics can thus verify the results we are about to obtain.

Since C G may be considered the unknown quantity by which to locate A C and B C, its value may be deduced from the above equation. Let the half-span A K, = K B, = c; the height or rise of the arch at the crown = k; the distance K G, from mid-span to the position of the single weight, = b;

and the required maximum ordinate C G $= y_0$. Then will the value of C G be

$$y_0 = \frac{32}{5} k \, \frac{c^2}{5c^2 - b^2},$$

which becomes, if $b = nc$, where $n =$ a fraction of the half-span,

$$y_0 = \frac{32}{5(5 - n^2)} k, \quad (1.)$$

a quantity independent of the span of the arch.

39. **Proof of Formula.** — Let A D, the distance from the abutment A to any ordinate D E, between A and G, $= x$. A G $= c + b$; G B $= c - b$. Since the ordinates to a parabola from the line A B are proportional to the product of the segments into which they divide the span, we have

$$\text{D E} : k = x\,(2c - x) : c^2, \text{ or D E} = \frac{k}{c^2}(2cx - x^2).$$

Also,

$$\text{D F} : y_0 = x : c + b, \qquad \text{or D F} = \frac{y_0}{c + b} x.$$

The required condition is that

$$\Sigma \text{ E F . D E} = 0, \text{ or } \Sigma \text{ (D E} - \text{D F) D E} = 0;$$

therefore,

$$\Sigma \text{ D E}^2 = \Sigma \text{ D F. D E.} \quad (1.)$$

(From the above expressions we see, that, if the area included between the rib and A B is considered positive, the area of the triangle A C B, superimposed upon it, will be deemed negative as before explained in Fig. 14.)

Substituting the values of the lines from above in (1.), multiplying by dx, and writing the sign of integration, we get for the left-hand member,

$$\int_0^{2c} \frac{k^2}{c^4}(2cx - x^2)^2 \, dx = \frac{k^2}{c^4} \cdot \int_0^{2c} (4c^2x^2 - 4cx^3 + x^4)\, dx$$

$$= \frac{k^2}{c^4}\left(\tfrac{4}{3}c^2x^3 - cx^4 + \tfrac{1}{5}x^5\right)_0^{2c} = \tfrac{16}{15}k^2 c. \quad (2.)$$

For the right-hand member, between A and G, we get

$$\int_0^{c+b} \frac{y_0}{c+b} x \cdot \frac{k}{c^2}(2cx - x^2)\, dx = \frac{k y_0}{c^2(c+b)} \cdot \int_0^{c+b} (2cx^2 - x^3)\, dx$$

$$= \frac{k y_0}{c^2(c+b)}\left(\tfrac{2}{3}cx^3 - \tfrac{1}{4}x^4\right)_0^{c+b} = \frac{k y_0}{c^2}\left[\tfrac{2}{3}c\,(c+b)^2 - \tfrac{1}{4}(c+b)^3\right]. \quad (3.)$$

For the portion between G and B, if we write $c - b$ for $c + b$, and reckon x from B to the left, D F will equal $\frac{y_0}{c - b}x$, while D E will be unchanged;

so that the integration for the right-hand member of (1.), between G and B, and between the limits $x = 0$ and $x = c - b$, will give, simply by writing $- b$ for $+ b$,

$$\frac{k y_0}{c^2}\left[\tfrac{2}{3}c\,(c - b)^2 - \tfrac{1}{4}(c - b)^3\right]. \quad (4.)$$

These two portions (3.) and (4.), for the right-hand member of (1.), being added together, will produce, when the terms with the odd powers of b are cancelled,

$$\frac{k y_0}{c^2}\left(\tfrac{5}{6}c^3 - \tfrac{1}{6}cb^2\right).$$

Finally equate this value with (2.) to satisfy (1.), and

$$\frac{k y_0}{6c}(5c^2 - b^2) = \tfrac{16}{15}k^2 c; \text{ or } y_0 = \tfrac{32}{5} k \, \frac{c^2}{5c^2 - b^2}, \quad (5.)$$

which is the desired value of C G in terms of the constant quantities, and the variable distance K G. This expression is plainly applicable to points on either side of K.

40. **Formula for Horizontal Thrust.** — For any position of the weight, plot the value of y_0, and draw the equilibrium polygon. Then draw two lines from the extremities of the load line W, parallel to the sides of the polygon, and thus determine H, and the two vertical components of the reactions, which vertical components will be the same as for a beam supported at its ends. But, from the simple relations of the similar triangles A G C and 0 3 1, Fig. 8, as also B G C and 0 3 2, we may write a general formula for H, if desired. Thus we have

$$y_0 : c - b = \text{P}_2 : \text{H}, \qquad \text{or P}_2 = \frac{y_0}{c - b}\text{H};$$

$$y_0 : c + b = \text{W} - \text{P}_2 : \text{H, or W} - \text{P}_2 = \frac{y_0}{c + b}\text{H}.$$

Eliminating P_2 in the second equation, by substituting its value from the first one, we get

$$\text{W} - \frac{y_0}{c - b}\text{H} = \frac{y_0}{c + b}\text{H}; \text{ or } (c^2 - b^2)\,\text{W} = 2cy_0\text{H};$$

$$\text{H} = \frac{c^2 - b^2}{2cy_0}\text{W} = \frac{1 - n^2}{2} \cdot \frac{5(5 - n^2)}{32} \cdot \frac{c}{k}\text{W}.$$

This value also will apply to a load on either side of the centre.

It will be observed that, to obtain this value of H, we have simply to divide $\frac{1}{2}(1-n^2)$ by the factor which multiplies k in (1.), § 38, to obtain the variable factor here.

41. **Computation of y_0 and H.** — The numerical values of these factors are worth obtaining, as, the computations once made, the results apply to every parabolic rib with pivoted ends. Let the span of the arch be divided into any convenient number of equal parts, and, for illustration, suppose that the number is ten, as shown in the figure; let a weight W be placed successively over each point of division, being supported by the rib. The calculation may conveniently proceed in the following manner: —

Find the different values of y_0 for different positions of W, by equation (1.), § 38. Then compute H by § 40. The calculation and results are given below; the equilibrium polygons and values of H for one-half of the arch are represented in Fig. 8. As n^2 is positive, whether n is + or —, the values of y_0 and H will be symmetrical on each side of the centre.

VALUES OF y_0 AND H.

$n=\frac{b}{c}$ =	0	0.2	0.4	0.6	0.8
$5-n^2$ =	5.00	4.96	4.84	4.64	4.36
$5(5-n^2)$ =	25.00	24.80	24.20	23.20	21.80
$\frac{32}{5(5-n^2)}$ =	1.280	1 2903	1.3223	1 3793	1.4679.

Multiply these factors by k to give y_0.

$\frac{1}{2}(1-n^2)$ =	0.50	0.48	0.42	0.32	0.18
$\frac{1}{2}(1-n^2)\div\frac{32}{5(5-n^2)}$ =	0.3906	0.3720	0.3176	0.2320	0.1226.

Multiply these factors by $\frac{c}{k}$ W to give H.

For any other desired division of the span, proceed in a similar way.

42. **Remarks.** — If every point of division were loaded with W at the same time, the value of the horizontal thrust would be equal to the sum of the H's for each load, that is, the factor in column 0 plus twice each of the others, and the sum multiplied by the factor $\frac{c}{k}$ W; we thus obtain $2.479\frac{c}{k}\,W = H$. If a *truss* were uniformly loaded horizontally, the bending moment at the middle would be one-eighth of the total load multiplied by the span, or, for a truss of ten panels, with W = one panel load,

$$M=\frac{10\,W.2c}{8}=2\tfrac{1}{2}\,c\,W;$$

and the tension in the lower chord, or the compression in the upper chord, would be found by dividing this quantity by the height of the truss, k. If the span of the arch just treated had been divided into twenty equal parts, the value of H, for loads at all the points of division, would have been $4.990\frac{c}{k}$ W. The truss, as before, would give $\frac{20\,W.2c}{8k}=5\frac{c}{k}W$.

We thus see that the equilibrium polygon, for a number of equal loads, equidistant horizontally, on a parabolic rib, gives a value of H approximating closely to that for a uniform load on a truss of height k, coming nearer as the loads increase in number, and agreeing when the load is continuous. Then the equilibrium polygon becomes a curve, coinciding perfectly with the parabolic rib, and gives the horizontal thrust to which we are accustomed in the bowstring girder under a maximum load.

43. **Computation of Bending Moments.** — While the ordinates can be readily scaled from a diagram, one who wishes may compute values of the bending moment M for numerous points, when W is placed on any one point. If y denotes the ordinate from A B to the inclined line, and z the ordinate of the parabola from any point D, the bending moment may be written, —

$$M = H\,(y-z).$$

If put in this form, it will be seen, that, in the neighborhood of y_0, M will be positive, coinciding with the moments for a beam supported at its two ends. As this is the most familiar flexure of a beam or truss, we have chosen to consider it as positive: § 12. The ordinates y and z can be readily calculated from the figure. Thus, if the weight is at 0.4 c from the middle of the span, we have found y_0 to be 1.3223 k. If the span is divided into ten parts, the number of divisions on one side of the weight being seven, y will be successively $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, &c., of y_0; on the other side y will be $\frac{1}{3}$ and $\frac{2}{3}$ of y_0. The sum of the denominators always equals the number of divisions, and the fractions increase from both ends up to unity. After finding the first y at each end, we get the others by simple addition, and the row is checked by obtaining y_0 at the proper point. As stated in § 39, the ordinate z is proportional to the product of the segments into which it divides the span; or, if it is at a distance $n\,c$ from the middle, we have,

$$z = (1 + n)\, c\, (1 - n)\, c \frac{k}{c^2} = (1 - n^2)\, k.$$

The factors by which k is to be multiplied can therefore be at once obtained by taking the *decimals* which are found in the second line of the table for y_0, § 41.

The computations may then be set down in the following shape, viz.: —

VALUES OF M.

Point of Division.	1	2	3	4	5	6	y_0 7	8	9	
$\frac{n}{7} y_0 =$	.1889	.3778	.5667	.7556	.9445	1.1334	1.3223	.8815	.4408	$= \frac{n}{3} y_0$.
$z =$	.36	.64	.84	.96	1.00	.96	.84	.64	.36	k
$y - z =$	−.1711	−.2622	−.2733	−.2044	−.0555	+.1734	+.4823	+.2415	+.0808	k
Multiply by $H = 0.3176 \frac{c}{k}$ W.										
M $=$	−.0543	−.0833	−.0868	−.0649	−.0176	+.0551	+.1532	+.0767	+.0257	c W

With the explanation already given, this table will be understood. The letter y_0 is placed over 7 as a convenience, to show that the value y_0 occurs at this point of division. If the load is on the right of the centre, these numbers run from the left abutment; if the load is on the left of the centre, they must be reckoned from the right abutment.

44. **Table of Bending Moments.** — We have carried out this computation for a load at each joint successively, the span being divided into ten equal parts, and have prepared a table given on p. 53. A table for a span divided into twenty parts may be found in "Engineering News," Vol. IV. p. 108. As a load on either side of the middle gives the same set of values in the reverse order, it is necessary to calculate but one-half of the table.

As many decimals may be taken as will give sufficiently accurate results. By the use of logarithms the labor of preparing another table for a different number of divisions is very little. Each column belongs to the point of division whose number stands at its top, the numbers commencing at the left abutment. Each horizontal line contains the factor for bending moment at each point of division for a load W on the point marked at the beginning of the line. The values of H are placed for convenience in the last column.

It is worthy of notice, that, while the value of y_0 is independent of the span of the arch, M is independent of the height of the arch. As it was proved, in § 28, that the parabola is the equilibrium curve for a load distributed uniformly horizontally, this arch ought to be very nearly in equilibrium when we place at once on each one of the nine points a load W: by footing up the vertical columns of the table we shall find but a very small residual moment at each joint.

45. **Interpolation.** — In the solution of a particular example, it may happen that the points at which the weight will be concentrated will not coincide with the points of division which we have taken. It will then be necessary to determine new values of y_0 and H, which may be done by the original formulæ or by interpolation. A new table of M may then be calculated, values may be interpolated in the one given here, or, if preferred, from the value of H, and the vertical components of the reactions, we may draw an equilibrium curve for any combination of loads. The table here given, if not directly applicable in all cases, serves two purposes; one to show how a similar table can be made, and the other to indicate, by inspection, what arrangement of loads on any arch will produce the maximum bending moments.

If the successive values of any quantity increase at a tolerably uniform rate, any intermediate value between two given ones may be found by simple proportion. Otherwise we may use the formula for interpolation, —

$$\text{Desired quantity} = a + f\,[D_1 - \tfrac{1}{2}(1 - f)\, D_2],$$

in which a denotes the first given quantity, f the fraction of a division from a to the desired quantity, and D_1 and D_2 the *first*