and *second differences*. To illustrate, take the values of H in § 41. If we place these in a column as below, find the amount

b.	H.	D_1.	D_2.
0	.3906		
		—.0186	
.2c	.3720		—.0358
		—.0544	
.4c	.3176		—.0312
		—.0856	
.6c	.2320		—.0238
		—.1094	
.8c	.1226		

of *increase* from quantity to quantity, and then subtract these differences from one another, marking each + if it is an increment, and *vice versa*, we obtain the columns of first and second differences as marked. Now suppose that we wish to determine a value of H at $b = .5\,c$; a will be .3176, $f = \frac{1}{2}$, $D_1 = -.0856$, and D_2 for an average value between .0312 and .0238, $= -.0275$. If we substitute in the formula, it then becomes

$$\begin{aligned} H\ (\text{for } .5\,c) &= .3176 + \tfrac{1}{2}\,[-.0856 - \tfrac{1}{2} \cdot \tfrac{1}{2}\,(-.0275)] \\ &= .3176 + \tfrac{1}{2}\,(-.0856 + .0069) = .2783. \end{aligned}$$

The factor for y_0, at one-third of the interval between .4 c and .6 c, will, in the same way, be

$$1.3223 + \tfrac{1}{3}\,[.0570 - \tfrac{1}{2} \cdot \tfrac{2}{3}\,(.0283)] = 1.3382.$$

Careful heed must be paid to the signs.

46. **Examples.** — It will help to fix the ideas, if we draw an equilibrium polygon for some combination of weights. We shall take but a few loads, in order to have the diagram clear; but the reader may vary the example by taking other amounts in other places. The values of the two vertical components of the abutment reactions will be the sums of the components for each load, and the amount of H for the whole load will be the sum of the separate H's. Multiply each numerical factor which belongs to H by the number of units of weight which are

placed on the point to which the factor refers, add up the products, and plot the resulting value of H horizontally from the point of division on the load line between the two vertical components of the reactions.

For example: Let us draw the equilibrium polygon for an arch of 100 feet span, 20 feet rise, whose weight is at present, for simplicity's sake, neglected, when it is loaded with weights of 3 tons, 2 tons, 4 tons, and 2 tons, at the end of the 3d, 6th, 8th, and 9th division from the left, of ten equal horizontal divisions, as shown in Fig. 9, where the numbers denote the weights and the points of division above mentioned. The supporting force on the left will be

$$P_1 = \frac{2 \times 1 + 4 \times 2 + 2 \times 4 + 3 \times 7}{10} = 3.9 \text{ tons.}$$

$$\therefore \quad P_2 = 7.1 \text{ tons.}$$

From the table for H,

$$\begin{aligned} H &= (0.3176 \times 3 + 0.372 \times 2 + 0.232 \times 4 + 0.1226 \times 2)\,\tfrac{50}{20} \\ &= 2.87 \times \tfrac{5}{2} = 7.175 \text{ tons.} \end{aligned}$$

These quantities are plotted in the stress diagram, as seen in the figure, and the equilibrium polygon is then drawn. The reader who reproduces this figure, or draws another, can be assured of the accuracy of the construction by the closing of the equilibrium polygon on the point of support. The weight of the arch itself may be accounted for by concentrating the proper amount at each point of division. Such amounts will increase towards the springing in proportion to the square of the secant of inclination to the horizon; for we recall the fact that the parabolic rib is to increase in breadth from crown to springing, and the amount in length projected into a horizontal foot increases in the same way. The weight of each division of the arch can be obtained with sufficient accuracy from a moderately large figure.

Another good construction is the curve for a uniform load over one-half of the span. The equilibrium curve for such a load, on the left half of Fig. 8, is represented in that figure; the work may be carried out in detail by the reader, and compared with the same curve for the three-hinged rib.

47. **Numerical Value of M.** — It will be seen that the polygon and rib of Fig. 9 approach quite nearly at 3. We can find the distance between them vertically, if we wish, from the table of M. The bending moment will be, taking the column 3,

$$M = 50\,(+.153 \times 3 - .073 \times 2 - .075 \times 4 - .043 \times 2) = -3.650 \text{ ft. tons.}$$

$$\frac{M}{H} = \frac{-3.65}{7.2} = -0.5 \text{ ft.} = y - z.$$

A similar operation may be performed at any other point.

48. **Shear Diagram.** — This investigation of shear is intended to apply to ribs of an I-section or to those framed with open-work or skeleton webs, and not to those of solid section, rectangular, circular, or otherwise, nor to stone arches: in these latter classes the shearing forces need seldom be taken into account.

Adhering still to the case of a single weight W, at a distance b from the middle of the span, we found that the vertical component, P_2, of the reaction at the end nearest to the weight, would be $\frac{c+b}{2c}$ W, and at the other end $\frac{c-b}{2c}$ W. As seen in Fig. 8, the diagram for shear on a beam will be, if we take the shear on the *left* of any section, $a\,d = P_1, = 3\text{–}1$, on the left of the weight, and $l\,g = -P_2, = 3\text{–}2$, on the right of the weight, giving the two rectangles included between $a\,l$ and the broken line $d\,e\,f\,g$. As the parabola is in equilibrium under a load of uniform intensity horizontally (§ 28), in which case there will be no bracing required, — no shear for any bracing to resist, — it is manifest that the diagram for that portion of the shear which is here carried, at each *vertical* section, by the flanges or chords, must be similar to the shear diagram for a uniform load on a beam supported at both ends; that is, to such a figure as $a\,i\,m\,n\,l$. If, then, we can determine the value of $a\,i$, or of the equal ordinate $l\,n$, we can draw this portion of the figure.

It is a well-known property of the parabola, that a tangent at

the springing of the arch will intersect the middle ordinate at a distance k above the crown, equal to the rise of the arch. If, then, we draw a line 0–4 in the stress diagram, parallel to the tangent A L, drawn as just described, the distance 3–4, intercepted on the vertical line, will be the amount of vertical force necessarily combined with H to give a thrust coinciding with the rib at the springing point. Lay off, therefore, 3–4 at $a\,i$, and an equal amount at $l\,n$; then draw the straight line $i\,n$, cutting $a\,l$ at its middle point m: the ordinates to this line from $a\,l$,

Parabolic Rib, Hinged at Ends.

§ 44. $M = m\,c\,W$. Values of m.

	1	2	3	4	5	6	7	8	9	H
W on 9	−.024	−.039	−.043	−.038	−.023	+.002	+.037	+.082	+.136	$.123\frac{c}{k}$W.
" 8	−.044	−.068	−.075	−.063	−.032	+.017	+.085	+.171	+.076	.232 "
" 7	−.054	−.083	−.087	−.065	−.018	+.055	+.153	+.076	+.025	.318 "
" 6	−.054	−.078	−.073	−.037	+.028	+.123	+.047	+.002	−.014	.372 "
" 5	−.041	−.050	−.028	+.025	+.109	+.025	−.028	−.050	−.041	.391 "
" 4	−.014	+.002	+.047	+.123	+.028	−.037	−.073	−.078	−.054	.372 "
" 3	+.025	+.076	+.153	+.055	−.018	−.065	−.087	−.083	−.054	.318 "
" 2	+.076	+.171	+.085	+.017	−.032	−.063	−.075	−.068	−.044	.232 "
" 1	+.136	+.082	+.037	+.002	−.023	−.038	−.043	−.039	−.024	.123 "

§ 53. $V = n\,W$. Values of n.

		1	2	3	4	5	6	7	8	9
W on 9	−.121	−.072	−.023	+.026	+.075	+.125	+.173	+.223	+.272	−.678
" 8	−.218	−.125	−.032	+.061	+.153	+.247	+.339	+.432	−.475	−.382
" 7	−.272	−.145	−.018	+.109	+.236	+.364	+.491	−.382	−.255	−.128
" 6	−.270	−.121	+.028	+.177	+.325	+.475	−.377	−.228	−.079	+.069
" 5	−.204	−.047	+.109	+.265	+.422	−.422	−.265	−.109	+.047	+.204
" 4	−.069	+.079	+.228	+.377	−.475	−.325	−.177	−.028	+.121	+.[illegible]0
" 3	+.128	+.255	+.382	−.491	−.364	−.236	−.109	+.018	+.145	+.272
" 2	+.382	+.475	−.432	−.339	−.247	−.153	−.061	+.032	+.125	+.218
" 1	+.678	−.272	−.223	−.173	−.125	−.075	−.026	+.023	+.072	+.121

at all points, will represent the amount of vertical force to be combined with the horizontal thrust to put the rib in equilibrium. The remaining ordinates are drawn at the middle of each division; and, where the amount subtracted is greater than the original shear, the remainder will be of the opposite sign. The signs are placed in the areas of this figure; and it will be apparent that the ordinates are reckoned from the inclined line *i n*, all *above that line* in our figure representing *positive* or upward shear on the *left* of a *vertical* plane of section, while those *below i n* will be *negative*. See p. 31.

49. **Shear on a Normal Section.** — To obtain the shear on a right or normal section, as at Q, we must draw a line *q s* parallel to the normal section at Q, and project *r q* upon it, thus finding *s q* as the shear at Q. A similar construction will determine the shear at any other point. The property of the parabola before alluded to makes it easy to find the direction of *q s*, which will be perpendicular to a tangent at Q; a tangent at Q will strike K L at S, a distance above the crown equal to that of the extremity R of the horizontal line Q R below it. What has been done by the above steps may also be easily seen from the sketch above Fig. 8. At A, P_1 will be *a d* or 3–1, and the whole vertical force to be combined with H will be *a i* or 3–4, which when subtracted from *a d* leaves *i d* or 4–1 as the negative shear on a vertical plane, and F, *t d*, or 6–1, as the shear on a right section at A.

In treating any arched rib, we shall desire to find the maximum shear at any section produced by a combination of weights at several points. It will be easier to find the sum of the several shears on a vertical section from single weights, and then find the normal component once for all, than to resolve each vertical shear separately; hence the shear diagram of Fig. 8 and of subsequent figures will simply show the shears on the several vertical sections before they are projected on the normal sections.

50. **Formula for Vertical Shear.** — A formula for this vertical shear may be deduced without difficulty. If Y is the ordinate to *i n* from any point of *a l*, and Y_1 its value at the springing, we have from the statement of the last section,

$$Y_1 : H = 2k : c, \text{ or } Y_1 = \frac{2k}{c} H.$$

The vertical shear V in the web, at the abutment on the left, will then be,

$$V = P_1 - Y_1 = \frac{c \pm b}{2c} W - \frac{2k}{c} H. \quad (1.)$$

For successive points, P_1 will remain the value of the original shear until we pass the weight, when it will become $P_1 - W$ or $-P_2$. Y will diminish at a constant rate; and, if we deduct at each point the ordinate from *a l* to the inclined line, we shall get the desired results.

51. **Computation of Shear.** — As an example we will find the vertical shear *midway* between the points of division of the arch of Fig. 8 with the load there shown.

$$P_1 = 0.3\,W; \; P_2 = 0.7\,W; \; H = .3176 \frac{c}{k} W; \; Y_1 = .6352\,W.$$

This value of Y_1 is applicable to any parabolic arch with hinged ends, since it involves neither *c* nor *k*. Y at the middle of the first space $= \left(.635 - \frac{.635}{10}\right) W = .572\,W$; for every succeeding ordinate it diminishes $\frac{.635}{5}$ W.

VALUES OF V.

Space.	1	2	3	4	5	6	7	8	9	10	
P_1	.3	.3	.3	.3	.3	.3	.3	–.7	–.7	–.7	$-P_2$
Y	.572	.445	.318	.191	+.064	–.064	–.191	–.318	–.445	–.572	
P – Y	–.272	–.145	–.018	+.109	+.236	+.364	+.491	–.382	–.255	–.128	W.

Three decimal places here will be as exact as four in the values of M. It will be seen by the ordinates in the shear diagram of Fig. 8, how the signs change.

52. **Remarks on Shear.** — We repeat that, as P_1 was taken as positive, the signs of the shears apply to the left side of each vertical or each normal section. In Fig. 10 the sketch marked R is an instance of positive shear, which acts up or outward on the left of the imaginary section and inward on the right of the same section. From the way in which the two parts of the arch will tend to slide at the section, we see that at R a tie will be required sloping down from the upper chord to the right (or a strut in the opposite direction), while negative shear, as represented in the sketch marked S, calls for a tie in the reverse direction.

53. **Table of Shears.** — A table has been computed by the preceding process, for shears at the middle points of ten equal spaces, into which the span is divided. It is intended to supplement the previous table of bending moments, and will serve as a guide for the calculation of any table with a greater or less number of spaces. It will be found on p. 53. A shear at a joint can be found, if desired, by taking the mean of two adjacent shears just obtained. It is easy to select from this table that combination of loads which will give on any parabolic arch, hinged at the ends only, the maximum shear of either kind in any one division, one arrangement being the complement of the other. These shears, as should be the case, foot up very nearly to zero for an equal load on every joint. It is only necessary to calculate one-half of the table; the other half will contain the same numbers in the reverse order, with the opposite signs. A table for an arch of twenty divisions was printed in "Engineering News," vol. iv., p. 124.

54. **Extent of Load to Produce Maximum Bending Moments and Shears.** — In single-span trusses the maximum bending moments, and consequently the maximum stresses in the chords, occur when the bridge is entirely covered with the live load; and the greatest shear at any section, or the greatest stress in any brace, exists when the bridge is covered with live load over one or the other, usually the longer, of the two segments into which the section divides the span. A simple inspection of the tables for M and V, lately given, will show that such rules are not true for an arch. Why this is so, will be seen, if we consider the fact that the portion of the arch, Fig. 8, between B and the point where C A crosses the rib, is under a bending moment of the positive kind, when there is a single weight at I, while from that point to A bending moments of the negative kind exist; and that an addition of another load near I will increase in amount most of the positive and negative moments, while one placed on the left half of the arch will have an opposite effect. The shearing forces for the braces, depending upon the change of stress in the flanges, will also be affected in the same way.

While an inspection of Fig. 8 will show, as was pointed out with regard to Fig. 4, in § 32, the extent of load to produce the maximum bending moment at any one point, and while the

load to produce maximum shear at the same point can also be ascertained by inspection, § 15, an attempt has been made to represent, by the horizontal lines in the diagram, Fig. 11, those positions of the live load, or the extent of the loaded portion, which will give the maximum moments of both kinds at each of nineteen points of division represented in the figure, and also that arrangement of the live load which gives the maximum shear of either kind at the middle of each division. The full line denotes the loaded portion of the span when the maximum positive moment occurs at that point whose number is placed at the end of the line, positive being understood to mean that kind of moment which would make a previously straight beam concave on the upper side; and the remaining portion of the span must alone be covered with the live load to produce the maximum negative moment at the same point. Thus the maximum positive bending moment at 2, and at 3 also, is found when the load is on all points from the left to 7 inclusive. A load from 8 to the right abutment gives the maximum $-M$. The maximum $+M$ at 11 occurs when the arch is loaded from 9 to 14 inclusive.

The extent of live load required to produce the greatest upward, or positive, shear on the left of a section through the web or brace in any division, is indicated by the broken line drawn in its proper space; and a load over the complementary blank portion will give the maximum shear of the opposite kind in the same division. Thus the maximum $+F$, at the middle of 3–4, is found when the load extends from 4 to 9 inclusive; and the maximum $-F$, at the same place, when the load reaches from 1 to 3 and 10 to 19 inclusive. As a partial load, not extending to either abutment, will give the greatest M at some points, and as the same thing is true of the values of F, those writers who determine the greatest stresses by the usual test for maximum applied to an algebraic equation, which contains the expression for load as continuous from one abutment, must err in their results.

55. **Resultant Maximum Stresses.** — The steady or fixed

load, unless distributed uniformly horizontally, gives some definite bending moment and shear, of one sign or the other, at each point; and these amounts must first be obtained from the tables or by diagram. If, at a given point, the bending moment from fixed weight is $+$, the arrangement of rolling load which gives the maximum $+M$ at that point will conspire with the steady load, and give an actual maximum $+M$; while that arrangement of rolling load which, in itself, gives a maximum $-M$, will reduce the moment from steady load. If large enough to prevail against the $+M$, the rolling load will produce an actual maximum $-M$; but, if not, it will only cause a minimum $+M$. Similar remarks might be made concerning shear.

An absolute maximum M of either kind, for a uniform load, will be found, if we sum up the quantities in the table, to occur at the middle of the half-span. The loads to produce these values are seen in Fig. 11. The absolute maximum $\pm F$ is found at the abutments, while another value, nearly equal in amount, occurs at the crown. These absolute maxima are found by comparing footings of the several columns, p. 53.

If Fig. 10 is supposed to represent a portion of the rib of Fig. 8 or Fig. 12, the web system being of any type or a continuous plate, we shall find that, when the chords or flanges lie on the opposite sides of any equilibrium polygon, they will be in compression from the weight which belongs to that polygon. When they both lie on the same side, the nearer chord or flange will be in compression and the farther one in tension. Hence the extent and amount of load to produce maximum stress of either kind in any chord piece can be found by inspection.

The actual stress is found by taking moments about the proper joint in the opposite chord, as is done in bridge trusses, using either H multiplied by the vertical ordinate, or the thrust in the side of the equilibrium polygon multiplied by the length of the perpendicular, drawn from the joint to that side, as may be preferred, and dividing by the length of the perpendicular from the same joint to the chord piece in question, considered as straight between its two joints. In this way the stress result-

ing from the direct thrust combined with the bending moment is at once determined.

Again, imagine a right section made in Fig. 8, through any panel like Fig. 10, and arrow-heads placed on the equilibrium polygons on the left of, and thrusting against the section. If the forces represented by such arrows have components acting up or outward along the section, they will cause positive shear in the web at that section; if such components act inward, they will cause negative shear. Hence the extent of load to produce maximum shear of either sign in a particular panel can also be found by inspection, and the amount of that shear can then be determined.

56. **Example of Flange Stresses.**—It may be instructive to make a little numerical calculation for the rib of Fig. 9, 100 feet span and 20 feet rise, supposing it to be loaded with the four weights only which are shown in the figure. The maximum positive moment is plainly at 8. If the rib is made of a web and two flanges $2\frac{1}{2}$ feet from centre to centre, what will be, with this load, the stress in each flange at 8? If our figure were larger, we could scale the ordinate above 8, and get the bending moment directly; but, as the sketch is small, we will refer to the table. We thus find that

$$M = (.082 \times 2 + .171 \times 4 + .002 \times 2 - .083 \times 3)\,50 = 30.15 \text{ foot tons.}$$

From the same table we find that

$$H = (.123 \times 2 + .232 \times 4 + .372 \times 2 + .318 \times 3)\,\tfrac{50}{20} = 7.18 \text{ tons.}$$

Then $30.15 \div 7.18 = 4.2$ feet, ordinate at 8. If we call the vertical depth of the rib at 8, three feet, the whole ordinate to the lower flange will be $4.2 + 1.5 = 5.7$ feet, and to the upper flange $4.2 - 1.5 = 2.7$ feet. The compression in the upper flange will be $7.18 \times 5.7 \div 2.5 = 16.37$ tons; and the tension in the lower flange $7.18 \times 2.7 \div 2.5 = 7.75$ tons.

Draw 0–5 parallel to the tangent at 8. Drop perpendiculars 3–6 and 4–7 on it from 3 and 4. On a right section close to, but on the left of 8, there will be positive shear 4–7, equal to 2.1 tons. On the right of 8 will be found 3–6, or 1.5 tons negative shear, to be resisted by the web.